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GREEK MATHEMATICS

II

SELECTIONS

ILLUSTRATING THE HISTORY OF

GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY

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IN TWO VOLUMES

II

FROM ARISTARCHUS TO PAPPUS

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XVI. ARISTARCHUS OF SAMOS

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(a) General

Aët. i. 15. 5 : *Doxographi Graeci*, ed. Diels 313. 16-18

Ἀρίσταρχος Σάμιος μαθηματικὸς ἀκουστὴς Στράτωνος[a] φῶς εἶναι τὸ χρῶμα τοῖς ὑποκειμένοις ἐπιπῖπτον.

Archim. *Aren.* 1, Archim. ed. Heiberg ii. 218. 7-18

Ἀρίσταρχος δὲ ὁ Σάμιος ὑποθεσίων τινῶν ἐξέδωκεν γραφάς, ἐν αἷς ἐκ τῶν ὑποκειμένων συμβαίνει τὸν κόσμον πολλαπλάσιον εἶμεν τοῦ νῦν εἰρημένου. ὑποτίθεται γὰρ τὰ μὲν ἀπλανέα τῶν ἄστρων καὶ τὸν ἅλιον μένειν ἀκίνητον, τὰν δὲ γᾶν περιφέρεσθαι περὶ τὸν ἅλιον κατὰ κύκλου περιφέρειαν, ὅς ἐστιν ἐν μέσῳ τῷ δρόμῳ κείμενος, τὰν δὲ τῶν ἀπλανέων ἄστρων σφαῖραν περὶ τὸ

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[a] Strato of Lampsacus was head of the Lyceum from 288/287 to 270/269 B.C. The next extract shows that Aristarchus formulated his heliocentric hypothesis before Archimedes wrote the *Sand-Reckoner*, which can be shown to have been written before 216 B.C. From Ptolemy, *Syntaxis* iii. 2, Aristarchus is known to have made an observation of the summer solstice in 281/280 B.C. He is ranked by Vitruvius, *De Architectura* i. 1. 17 among those rare men, such as Philolaus, Archytas, Apollonius, Eratosthenes,

XVI. ARISTARCHUS OF SAMOS

(a) General

Aëtius i. 15. 5 ; *Doxographi Graeci*, ed. Diels 313. 16-18

Aristarchus of Samos, a mathematician and pupil of Strato,[a] held that colour was light impinging on a substratum.

Archimedes, *Sand-Reckoner* 1, Archim. ed. Heiberg ii. 218. 7-18

Aristarchus of Samos produced a book based on certain hypotheses, in which it follows from the premises that the universe is many times greater than the universe now so called. His hypotheses are that the fixed stars and the sun remain motionless, that the earth revolves in the circumference of a circle about the sun, which lies in the middle of the orbit, and that the sphere of the fixed stars, situated

Archimedes and Scopinas of Syracuse, who were equally proficient in all branches of science. Vitruvius, *loc. cit.* ix. 8. 1, is also our authority for believing that he invented a sun-dial with a hemispherical bowl. His greatest achievement, of course, was the hypothesis that the earth moves round the sun, but as that belongs to astronomy it can be mentioned only casually here. A full and admirable discussion will be found in Heath, *Aristarchus of Samos : The Ancient Copernicus*, together with a critical text of Aristarchus's only extant work.

αὐτὸ κέντρον τῷ ἁλίῳ κειμέναν τῷ μεγέθει ταλικαύταν εἶμεν, ὥστε τὸν κύκλον, καθ' ὃν τὰν γᾶν ὑποτίθεται περιφέρεσθαι, τοιαύταν ἔχειν ἀναλογίαν ποτὶ τὰν τῶν ἀπλανέων ἀποστασίαν, οἵαν ἔχει τὸ κέντρον τᾶς σφαίρας ποτὶ τὰν ἐπιφάνειαν.

Plut. *De facie in orbe lunae* 6, 922 F–923 A

Καὶ ὁ Λεύκιος γελάσας, " Μόνον," εἶπεν, " ὦ τάν, μὴ κρίσιν ἡμῖν ἀσεβείας ἐπαγγείλῃς, ὥσπερ Ἀρίσταρχον ᾤετο δεῖν Κλεάνθης τὸν Σάμιον ἀσεβείας προσκαλεῖσθαι τοὺς Ἕλληνας, ὡς κινοῦντα τοῦ κόσμου τὴν ἑστίαν, ὅτι τὰ φαινόμενα σῴζειν ἀνὴρ ἐπειρᾶτο, μένειν τὸν οὐρανὸν ὑποτιθέμενος, ἐξελίττεσθαι δὲ κατὰ λοξοῦ κύκλου τὴν γῆν, ἅμα καὶ περὶ τὸν αὑτῆς ἄξονα δινουμένην."

(*b*) Distances of the Sun and Moon

Aristarch. Sam. *De Mag. et Dist. Solis et Lunae*, ed. Heath (*Aristarchus of Samos : The Ancient Copernicus*) 352. 1–354. 6

⟨Ὑποθέσεις[1]⟩

α'. Τὴν σελήνην παρὰ τοῦ ἡλίου τὸ φῶς λαμβάνειν.

β'. Τὴν γῆν σημείου τε καὶ κέντρου λόγον ἔχειν πρὸς τὴν τῆς σελήνης σφαῖραν.

γ'. Ὅταν ἡ σελήνη διχότομος ἡμῖν φαίνηται,

[1] ὑποθέσεις add. Heath.

[a] Aristarchus's last hypothesis, if taken literally, would mean that the sphere of the fixed stars is infinite. All that he implies, however, is that in relation to the distance of the

about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve has such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.[a]

Plutarch, *On the Face in the Moon* 6, 922 F–923 A

Lucius thereupon laughed and said: "Do not, my good fellow, bring an action against me for impiety after the manner of Cleanthes, who held that the Greeks ought to indict Aristarchus of Samos on a charge of impiety because he set in motion the hearth of the universe; for he tried to save the phenomena by supposing the heaven to remain at rest, and the earth to revolve in an inclined circle, while rotating at the same time about its own axis."[b]

(*b*) Distances of the Sun and Moon

Aristarchus of Samos, *On the Sizes and Distances of the Sun and Moon*, ed. Heath (*Aristarchus of Samos: The Ancient Copernicus*) 352. 1–354. 6

HYPOTHESES

1. The moon receives its light from the sun.
2. The earth has the relation of a point and centre to the sphere in which the moon moves.[c]
3. When the moon appears to us halved, the great

fixed stars the diameter of the earth's orbit may be neglected. The phrase appears to be traditional (*v*. Aristarchus's second hypothesis, *infra*).

[b] Heraclides of Pontus (along with Ecphantus, a Pythagorean) had preceded Aristarchus in making the earth revolve on its own axis, but he did not give the earth a motion of translation as well.

[c] Lit. "sphere of the moon."

νεύειν εἰς τὴν ἡμετέραν ὄψιν τὸν διορίζοντα τό τε σκιερὸν καὶ τὸ λαμπρὸν τῆς σελήνης μέγιστον κύκλον.

δʹ. Ὅταν ἡ σελήνη διχότομος ἡμῖν φαίνηται, τότε αὐτὴν ἀπέχειν τοῦ ἡλίου ἔλασσον τεταρτημορίου τῷ τοῦ τεταρτημορίου τριακοστῷ.

εʹ. Τὸ τῆς σκιᾶς πλάτος σεληνῶν εἶναι δύο.

ϛʹ. Τὴν σελήνην ὑποτείνειν ὑπὸ πεντεκαιδέκατον μέρος ζῳδίου.

Ἐπιλογίζεται οὖν τὸ τοῦ ἡλίου ἀπόστημα ἀπὸ τῆς γῆς τοῦ τῆς σελήνης ἀποστήματος μεῖζον μὲν ἢ ὀκτωκαιδεκαπλάσιον, ἔλασσον δὲ ἢ εἰκοσαπλάσιον, διὰ τῆς περὶ τὴν διχοτομίαν ὑποθέσεως· τὸν αὐτὸν δὲ λόγον ἔχειν τὴν τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς σελήνης διάμετρον. τὴν δὲ τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς γῆς διάμετρον μείζονα μὲν λόγον ἔχειν ἢ ὃν τὰ ιθ πρὸς γ̄, ἐλάσσονα δὲ ἢ ὃν μ̄γ̄ πρὸς ϛ̄, διὰ τοῦ εὑρεθέντος περὶ τὰ ἀποστήματα λόγου, τῆς ⟨τε[1]⟩ περὶ τὴν σκιὰν ὑποθέσεως, καὶ τοῦ τὴν σελήνην ὑπὸ πεντεκαιδέκατον μέρος ζῳδίου ὑποτείνειν.

Ibid., Prop. 7, ed. Heath 376. 1–380. 28

Τὸ ἀπόστημα ὃ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς τοῦ

[1] τε add. Heath.

[a] Lit. "verges towards our eye." For "verging," *v.* vol. i. p. 244 n. *a*. Aristarchus means that the observer's eye lies in the plane of the great circle in question. A great circle is a circle described on the surface of the sphere and having the same centre as the sphere; as the Greek implies, a great circle is the "greatest circle" that can be described on the sphere.

circle dividing the dark and the bright portions of the moon is in the direction of our eye.[a]

4. When the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant.[b]

5. The breadth of the earth's shadow is that of two moons.[c]

6. The moon subtends one-fifteenth part of a sign of the zodiac.[d]

It may now be proved that *the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon*—this follows from the hypothesis about the halved moon; that *the diameter of the sun has the aforesaid ratio to the diameter of the moon*; and that *the diameter of the sun has to the diameter of the earth a ratio which is greater than* 19 : 3 *but less than* 43 : 6—this follows from the ratio discovered about the distances, the hypothesis about the shadow, and the hypothesis that the moon subtends one-fifteenth part of a sign of the zodiac.

Ibid., Prop. 7, ed. Heath 376. 1–380. 28

The distance of the sun from the earth is greater than

[b] *i.e.*, is less than 90° by 3°, and so is 87°. The true value is 89° 50′.

[c] *i.e.*, the breadth of the earth's shadow where the moon traverses it during an eclipse. The figure is presumably based on records of eclipses. Hipparchus made the figure $2\frac{1}{2}$ for the time when the moon is at its mean distance, and Ptolemy a little less than $2\frac{3}{5}$ for the time when the moon is at its greatest distance.

[d] *i.e.*, the angular diameter of the moon is one-fifteenth of 30°, or 2°. The true value is about $\frac{1}{2}$°, and in the *Sand-Reckoner* (Archim. ed. Heiberg ii. 222. 6-8) Archimedes says that Aristarchus "discovered that the sun appeared to be about $\frac{1}{720}$th part of the circle of the Zodiac"; as he believed

ἀποστήματος οὗ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς μεῖζον μέν ἐστιν ἢ ὀκτωκαιδεκαπλάσιον, ἔλασσον δὲ ἢ εἰκοσαπλάσιον.

Ἔστω γὰρ ἡλίου μὲν κέντρον τὸ Α, γῆς δὲ τὸ Β, καὶ ἐπιζευχθεῖσα ἡ ΑΒ ἐκβεβλήσθω, σελήνης δὲ κέντρον διχοτόμου οὔσης τὸ Γ, καὶ ἐκβεβλήσθω διὰ τῆς ΑΒ καὶ τοῦ Γ ἐπίπεδον, καὶ ποιείτω τομὴν ἐν τῇ σφαίρᾳ, καθ' ἧς φέρεται τὸ κέντρον τοῦ ἡλίου, μέγιστον κύκλον τὸν ΑΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΓΒ, καὶ ἐκβεβλήσθω ἡ ΒΓ ἐπὶ τὸ Δ.

Ἔσται δή, διὰ τὸ τὸ Γ σημεῖον κέντρον εἶναι τῆς σελήνης διχοτόμου οὔσης, ὀρθὴ ἡ ὑπὸ τῶν

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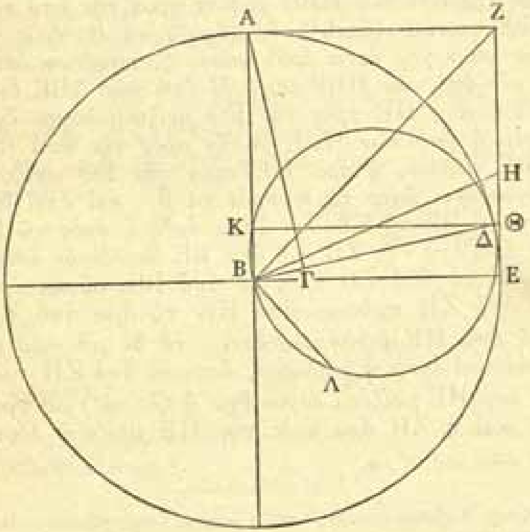
that the sun and moon had the same angular diameter he must, therefore, have found the approximately correct angular diameter of ½° after writing his treatise *On the Sizes and Distances of the Sun and Moon.*

eighteen times, but less than twenty times, the distance of the moon from the earth.

For let A be the centre of the sun, B that of the earth; let AB be joined and produced; let Γ be the centre of the moon when halved; let a plane be drawn through AB and Γ, and let the section made by it in the sphere on which the centre of the sun moves be the great circle AΔE, let AΓ, ΓB be joined, and let BΓ be produced to Δ.

Then, because the point Γ is the centre of the moon when halved, the angle AΓB will be right.



ΑΓΒ. ἤχθω δὴ ἀπὸ τοῦ Β τῇ ΒΑ πρὸς ὀρθὰς ἡ ΒΕ. ἔσται δὴ ἡ ΕΔ περιφέρεια τῆς ΕΔΑ περιφερείας λʹ· ὑπόκειται γάρ, ὅταν ἡ σελήνη διχότομος ἡμῖν φαίνηται, ἀπέχειν ἀπὸ τοῦ ἡλίου ἔλασσον τεταρτημορίου τῷ τοῦ τεταρτημορίου λʹ· ὥστε καὶ ἡ ὑπὸ τῶν ΕΒΓ γωνία ὀρθῆς ἐστι λʹ. συμπεπληρώσθω δὴ τὸ ΑΕ παραλληλόγραμμον, καὶ ἐπεζεύχθω ἡ ΒΖ. ἔσται δὴ ἡ ὑπὸ τῶν ΖΒΕ γωνία ἡμίσεια ὀρθῆς. τετμήσθω ἡ ὑπὸ τῶν ΖΒΕ γωνία δίχα τῇ ΒΗ εὐθείᾳ· ἡ ἄρα ὑπὸ τῶν ΗΒΕ γωνία τέταρτον μέρος ἐστὶν ὀρθῆς. ἀλλὰ καὶ ἡ ὑπὸ τῶν ΔΒΕ γωνία λʹ ἐστι μέρος ὀρθῆς· λόγος ἄρα τῆς ὑπὸ τῶν ΗΒΕ γωνίας πρὸς τὴν ὑπὸ τῶν ΔΒΕ γωνίαν ⟨ἐστὶν[1]⟩ ὃν ⟨ἔχει[2]⟩ τὰ ιε πρὸς τὰ δύο· οἵων γάρ ἐστιν ὀρθὴ γωνία ξ, τοιούτων ἐστὶν ἡ μὲν ὑπὸ τῶν ΗΒΕ ιε, ἡ δὲ ὑπὸ τῶν ΔΒΕ δύο. καὶ ἐπεὶ ἡ ΗΕ πρὸς τὴν ΕΘ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ τῶν ΗΒΕ γωνία πρὸς τὴν ὑπὸ τῶν ΔΒΕ γωνίαν, ἡ ἄρα ΗΕ πρὸς τὴν ΕΘ μείζονα λόγον ἔχει ἤπερ τὰ ιε πρὸς τὰ β. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΖ, καὶ ἔστιν ὀρθὴ ἡ πρὸς τῷ Ε, τὸ ἄρα ἀπὸ τῆς ΖΒ τοῦ ἀπὸ ΒΕ διπλάσιόν ἐστιν· ὡς δὲ τὸ ἀπὸ ΖΒ πρὸς τὸ ἀπὸ ΒΕ, οὕτως ἐστὶ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ ΗΕ· τὸ ἄρα ἀπὸ ΖΗ τοῦ ἀπὸ ΗΕ διπλάσιόν ἐστι. τὰ δὲ μθ τῶν κε ἐλάσσονά ἐστιν ἢ διπλάσια, ὥστε τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ ΗΕ μείζονα λόγον ἔχει ἢ ⟨ὃν τὰ[3]⟩ μθ πρὸς κε· καὶ ἡ ΖΗ ἄρα πρὸς τὴν ΗΕ μείζονα λόγον

[1] ἐστὶν add. Nizze.
[2] ἔχει add. Wallis.
[3] ὃν τὰ add. Wallis.

[a] Lit. "circumference," as in several other places in this proposition.

From B let BE be drawn at right angles to BA. Then the arc[a] EΔ will be one-thirtieth of the arc EΔA; for, by hypothesis, when the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant [Hypothesis 4]. Therefore the angle EBΓ is also one-thirtieth of a right angle. Let the parallelogram AE be completed, and let BZ be joined. Then the angle ZBE will be one-half of a right angle. Let the angle ZBE be bisected by the straight line BH; then the angle HBE is one-fourth part of a right angle. But the angle ΔBE is one-thirtieth part of a right angle; therefore angle HBE : angle ΔBE = 15 : 2; for, of those parts of which a right angle contains 60, the angle HBE contains 15 and the angle ΔBE contains 2.

Now since

$$HE : E\Theta > \text{angle } HBE : \text{angle } \Delta BE,^{b}$$

therefore $HE : E\Theta > 15 : 2$.

And since $BE = EZ$, and the angle at E is right, therefore

$$ZB^2 = 2BE^2.$$

But $\quad ZB^2 : BE^2 = ZH^2 : HE^2.$

Therefore $\quad ZH^2 = 2HE^2.$

Now $\quad 49 < 2 \cdot 25,$

so that $\quad ZH^2 : HE^2 > 49 : 25.$

Therefore $\quad ZH : HE > 7 : 5.$

[a] Aristarchus's assumption is equivalent to the theorem

$$\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta},$$

where $\beta < \alpha \leq \frac{1}{2}\pi$. Euclid's proof in *Optics* 8 is given in vol. i. pp. 502-505.

ἔχει ἢ ⟨ὃν[1]⟩ τὰ ζ̄ πρὸς τὰ ε̄· καὶ συνθέντι ἡ ΖΕ ἄρα πρὸς τὴν ΕΗ μείζονα λόγον ἔχει ἢ ὃν τὰ ιβ̄ πρὸς τὰ ε̄, τουτέστιν, ἢ ὃν ⟨τὰ[2]⟩ λϛ̄ πρὸς τὰ ιε̄. ἐδείχθη δὲ καὶ ἡ ΗΕ πρὸς τὴν ΕΘ μείζονα λόγον ἔχουσα ἢ ὃν τὰ ιε̄ πρὸς τὰ δύο· δι' ἴσου ἄρα ἡ ΖΕ πρὸς τὴν ΕΘ μείζονα λόγον ἔχει ἢ ὃν τὰ λϛ̄ πρὸς τὰ δύο, τουτέστιν, ἢ ὃν τὰ ιη̄ πρὸς ᾱ· ἡ ἄρα ΖΕ τῆς ΕΘ μείζων ἐστὶν ἢ ιη̄. ἡ δὲ ΖΕ ἴση ἐστὶν τῇ ΒΕ· καὶ ἡ ΒΕ ἄρα τῆς ΕΘ μείζων ἐστὶν ἢ ιη̄· πολλῷ ἄρα ἡ ΒΗ τῆς ΘΕ μείζων ἐστὶν ἢ ιη̄. ἀλλ' ὡς ἡ ΒΘ πρὸς τὴν ΘΕ, οὕτως ἐστὶν ἡ ΑΒ πρὸς τὴν ΒΓ, διὰ τὴν ὁμοιότητα τῶν τριγώνων· καὶ ἡ ΑΒ ἄρα τῆς ΒΓ μείζων ἐστὶν ἢ ιη̄. καὶ ἔστιν ἡ μὲν ΑΒ τὸ ἀπόστημα ὃ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς, ἡ δὲ ΓΒ τὸ ἀπόστημα ὃ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἄρα ἀπόστημα ὃ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὗ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, μεῖζόν ἐστιν ἢ ιη̄.

Λέγω δὴ ὅτι καὶ ἔλασσον ἢ κ̄. ἤχθω γὰρ διὰ τοῦ Δ τῇ ΕΒ παράλληλος ἡ ΔΚ, καὶ περὶ τὸ ΔΚΒ τρίγωνον κύκλος γεγράφθω ὁ ΔΚΒ· ἔσται δὴ αὐτοῦ διάμετρος ἡ ΔΒ, διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Κ γωνίαν. καὶ ἐνηρμόσθω ἡ ΒΛ ἑξαγώνου. καὶ ἐπεὶ ἡ ὑπὸ τῶν ΔΒΕ γωνία λ' ἐστιν ὀρθῆς, καὶ ἡ ὑπὸ τῶν ΒΔΚ ἄρα λ' ἐστιν ὀρθῆς· ἡ ἄρα ΒΚ περιφέρεια ξ' ἐστιν τοῦ ὅλου κύκλου. ἔστιν δὲ καὶ ἡ ΒΛ ἕκτον μέρος τοῦ ὅλου κύκλου· ἡ ἄρα ΒΛ περιφέρεια τῆς ΒΚ περιφερείας ι̅ ἐστίν. καὶ ἔχει ἡ ΒΛ περιφέρεια πρὸς τὴν ΒΚ περιφέρειαν μείζονα λόγον ἤπερ ἡ ΒΛ

[1] ὃν add. Wallis.

[2] τὰ add. Wallis.

Therefore, *componendo*, ZE : EH > 12 : 5,

that is, ZE : EH > 36 : 15.

But it was also proved that

HE : EΘ > 15 : 2.

Therefore, *ex aequali*,[a] ZE : EΘ > 36 : 2,

that is, ZE : EΘ > 18 : 1.

Therefore ZE is greater than eighteen times EΘ. And ZE is equal to BE. Therefore BE is also greater than eighteen times EΘ. Therefore BH is much greater than eighteen times ΘE.

But BΘ : ΘE = AB : BΓ,

by similarity of triangles. Therefore AB is also greater than eighteen times BΓ. And AB is the distance of the sun from the earth, while ΓB is the distance of the moon from the earth; therefore the distance of the sun from the earth is greater than eighteen times the distance of the moon from the earth.

I say now that it is less than twenty times. For through Δ let ΔK be drawn parallel to EB, and about the triangle ΔKB let the circle ΔKB be drawn; its diameter will be ΔB, by reason of the angle at K being right. Let BΛ, the side of a hexagon, be fitted into the circle. Then, since the angle ΔBE is one-thirtieth of a right angle, therefore the angle BΔK is also one-thirtieth of a right angle. Therefore the arc BK is one-sixtieth of the whole circle. But BΛ is one-sixth part of the whole circle.

Therefore arc BΛ = 10 . arc BK.

And the arc BΛ has to the arc BK a ratio greater

[a] For the proportion *ex aequali*, *v.* vol. i. pp. 448-451.

εὐθεῖα πρὸς τὴν ΒΚ εὐθεῖαν· ἡ ἄρα ΒΛ εὐθεῖα τῆς ΒΚ εὐθείας ἐλάσσων ἐστὶν ἢ ῑ. καὶ ἔστιν αὐτῆς διπλῆ ἡ ΒΔ· ἡ ἄρα ΒΔ τῆς ΒΚ ἐλάσσων ἐστὶν ἢ κ̄. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΒΚ, ἡ ΑΒ πρὸς ⟨τὴν[1]⟩ ΒΓ, ὥστε καὶ ἡ ΑΒ τῆς ΒΓ ἐλάσσων ἐστὶν ἢ κ̄. καὶ ἔστιν ἡ μὲν ΑΒ τὸ ἀπόστημα ὃ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς, ἡ δὲ ΒΓ τὸ ἀπόστημα ὃ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἄρα ἀπόστημα ὃ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὗ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, ἔλασσόν ἐστιν ἢ κ̄. ἐδείχθη δὲ καὶ μεῖζον ἢ ιη̄.

(c) Continued Fractions (?)

Ibid., Prop. 13, ed. Heath 396. 1-2

Ἔχει δὲ καὶ τὰ $\overline{͵ζϡκα}$ πρὸς $\overline{͵δν}$ μείζονα λόγον ἤπερ τὰ $\overline{πη}$ πρὸς $\overline{με}$.

Ibid., Prop. 15, ed. Heath 406. 23-24

Ἔχει δὲ καὶ ὁ $\overset{͵ζροε}{\mathrm{M}}$ ͵εωοε πρὸς $\overset{͵ϛρογ}{\mathrm{M}}$ ͵εφ μείζονα λόγον ἢ ὃν τὰ $\overline{μγ}$ πρὸς $\overline{λζ}$.

[1] τὴν add. Wallis.

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[a] This is proved in Ptolemy's *Syntaxis* i. 10, *v. infra*, pp. 435-439.

[b] If $\frac{7921}{4050}$ is developed as a continued fraction, we obtain the approximation $1+\frac{1}{1+}\,\frac{1}{21+}\,\frac{1}{2}$, which is $\frac{88}{45}$. Similarly, if $\frac{71755875}{61735500}$ or $\frac{21261}{18292}$ is developed as a continued fraction, we

than that which the straight line BΛ has to the straight line BK.[a]

Therefore $B\Lambda < 10 \,.\, BK$.

And $B\Delta = 2\, B\Lambda$.

Therefore $B\Delta < 20 \,.\, BK$.

But $B\Delta : BK = AB : B\Gamma$.

Therefore $AB < 20 \,.\, B\Gamma$.

And AB is the distance of the sun from the earth, while BΓ is the distance of the moon from the earth; therefore the distance of the sun from the earth is less than twenty times the distance of the moon from the earth. And it was proved to be greater than eighteen times.

(c) Continued Fractions (?)

Ibid., Prop. 13, ed. Heath 396. 1-2

But 7921 has to 4050 a ratio greater than that which 88 has to 45.

Ibid., Prop. 15, ed. Heath 406. 23-24

But 71755875 has to 61735500 a ratio greater than that which 43 has to 37.[b]

obtain the approximation $1 + \frac{1}{6+}\,\frac{1}{6}$ or $\frac{43}{37}$. The latter result was first noticed in 1823 by the Comte de Fortia D'Urban (*Traité d'Aristarque de Samos*, p. 186 n. 1), who added: "Ainsi les Grecs, malgré l'imperfection de leur numération, avaient des méthodes semblables aux nôtres." Though these relations are hardly sufficient to enable us to say that the Greeks knew how to develop continued fractions, they lend some support to the theory developed by D'Arcy W. Thompson in *Mind*, xxxviii. pp. 43-55, 1929.

XVII. ARCHIMEDES

XVII. ARCHIMEDES

(a) General

Tzetzes, *Chil.* ii. 103-144

Ὁ Ἀρχιμήδης ὁ σοφός, μηχανητὴς ἐκεῖνος,
Τῷ γένει Συρακούσιος ἦν, γέρων γεωμέτρης,
Χρόνοός τε ἑβδομήκοντα καὶ πέντε παρελαύνων,
Ὅστις εἰργάσατο πολλὰς μηχανικὰς δυνάμεις,
Καὶ τῇ τρισπάστῳ μηχανῇ χειρὶ λαιᾷ καὶ μόνῃ
Πεντεμυριομέδιμνον καθείλκυσεν ὁλκάδα
Καὶ τοῦ Μαρκέλλου στρατηγοῦ ποτε δὲ τῶν
    Ῥωμαίων
Τῇ Συρακούσῃ κατὰ γῆν προσβάλλοντος καὶ
    πόντον,
Τινὰς μὲν πρῶτον μηχαναῖς ἀνείλκυσεν ὁλκάδας
Καὶ πρὸς τὸ Συρακούσιον τεῖχος μετεωρίσας
Αὐτάνδρους πάλιν τῷ βυθῷ κατεπέμπεν ἀθρόως,
Μαρκέλλου δ' ἀποστήσαντος μικρόν τι τὰς ὁλκάδας
Ὁ γέρων πάλιν ἅπαντας ποιεῖ Συρακουσίους

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[a] A life of Archimedes was written by a certain Heraclides—perhaps the Heraclides who is mentioned by Archimedes himself in the preface to his book *On Spirals* (Archim. ed. Heiberg ii. 2. 3) as having taken his books to Dositheus. We know this from two references by Eutocius (Archim. ed. Heiberg iii. 228. 20, Apollon. ed. Heiberg ii. 168. 5, where, however, the name is given as Ἡράκλειος), but it has not survived. The surviving writings of Archimedes, together with the commentaries of Eutocius of Ascalon (*fl.* A.D. 520), have been edited by J. L. Heiberg in three volumes of the Teubner series (references in this volume are to the 2nd ed., Leipzig, 1910–1915). They have been put into mathematical notation by T. L. Heath, *The Works of Archimedes* (Cam-

XVII. ARCHIMEDES [a]

(a) General

Tzetzes, *Book of Histories* ii. 103-144 [b]

Archimedes the wise, the famous maker of engines, was a Syracusan by race, and worked at geometry till old age, surviving five-and-seventy-years [c]; he reduced to his service many mechanical powers, and with his triple-pulley device, using only his left hand, he drew a vessel of fifty thousand medimni burden. Once, when Marcellus, the Roman general, was assaulting Syracuse by land and sea, first by his engines he drew up some merchant-vessels, lifted them up against the wall of Syracuse, and sent them in a heap again to the bottom, crews and all. When Marcellus had withdrawn his ships a little distance, the old man gave all the Syracusans power to lift

bridge, 1897), supplemented by *The Method of Archimedes* (Cambridge, 1912), and have been translated into French by Paul Ver Eecke, *Les Œuvres complètes d'Archimède* (Brussels, 1921).

[b] The lines which follow are an example of the "political" (πολιτικός, popular) verse which prevailed in Byzantine times. The name is given to verse composed by accent instead of quantity, with an accent on the last syllable but one, especially an iambic verse of fifteen syllables. The twelfth-century Byzantine pedant, John Tzetzes, preserved in his *Book of Histories* a great treasure of literary, historical, theological and scientific detail, but it needs to be used with caution. The work is often called the *Chiliades* from its arbitrary division by its first editor (N. Gerbel, 1546) into books of 1000 lines each—it actually contains 12,674 lines.

[c] As he perished in the sack of Syracuse in 212 B.C., he was therefore born about 287 B.C.

Μετεωρίζειν δύνασθαι λίθους ἀμαξιαίους
Καὶ τὸν καθένα πέμποντας[1] βυθίζειν τὰς ὁλκάδας.
Ὡς Μάρκελλος δ' ἀπέστησε βολὴν ἐκείνας τόξου,
Ἑξάγωνόν τι κάτοπτρον ἐτέκτηνεν ὁ γέρων,
Ἀπὸ δὲ διαστήματος συμμέτρου τοῦ κατόπτρου
Μικρὰ τοιαῦτα κάτοπτρα θεὶς τετραπλᾶ γωνίαις
Κινούμενα λεπίσι τε καί τισι γιγγλυμίοις,
Μέσον ἐκεῖνο τέθεικεν ἀκτίνων τῶν ἡλίου
Μεσημβρινῆς καὶ θερινῆς καὶ χειμεριωτάτης.
Ἀνακλωμένων δὲ λοιπὸν εἰς τοῦτο τῶν ἀκτίνων
Ἔξαψις ἤρθη φοβερὰ πυρώδης ταῖς ὁλκάσι,
Καὶ ταύτας ἀπετέφρωσεν ἐκ μήκους τοξοβόλου.
Οὕτω νικᾷ τὸν Μάρκελλον ταῖς μηχαναῖς ὁ γέρων.
Ἔλεγε δὲ καὶ δωριστί, φωνῇ Συρακουσίᾳ·
"Πᾷ βῶ, καὶ χαριστίωνι τὰν γᾶν κινήσω πᾶσαν."

[1] πέμποντας Cary, πέμποντα codd.

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[a] Unfortunately, the earliest authority for this story is Lucian, *Hipp.* 2: τὸν δὲ (sc. Ἀρχιμήδην) τὰς τῶν πολεμίων τριήρεις καταφλέξαντα τῇ τέχνῃ. It is also found in Galen, Περὶ κρασ. iii. 2, and Zonaras xiv. 3 relates it on the authority of Dion Cassius, but makes Proclus the hero of it.

[b] Further evidence is given by Tzetzes, *Chil.* xii. 995 and Eutocius (Archim. ed. Heiberg iii. 132. 5-6) that Archimedes wrote in the Doric dialect, but the extant text of his best-known works, *On the Sphere and Cylinder* and the *Measurement of a Circle*, retains only one genuine trace of its original Doric—the form τῆνος. Partial losses have occurred in other books, the *Sand-Reckoner* having suffered least. The subject is fully treated by Heiberg, *Quaestiones Archimedeae*, pp. 69-94, and in a preface to the second volume of his edition of Archimedes he indicates the words which he has restored to their Doric form despite the manuscripts; his text is adopted in this selection.

The loss of the original Doric is not the only defect in the

stones large enough to load a waggon and, hurling them one after the other, to sink the ships. When Marcellus withdrew them a bow-shot, the old man constructed a kind of hexagonal mirror, and at an interval proportionate to the size of the mirror he set similar small mirrors with four edges, moved by links and by a form of hinge, and made it the centre of the sun's beams—its noon-tide beam, whether in summer or in mid-winter. Afterwards, when the beams were reflected in the mirror, a fearful kindling of fire was raised in the ships, and at the distance of a bow-shot he turned them into ashes.[a] In this way did the old man prevail over Marcellus with his weapons. In his Doric[b] dialect, and in its Syracusan variant, he declared: "If I have somewhere to stand, I will move the whole earth with my *charistion*."[c]

text. The hand of an interpolater—often not particularly skilful—can be repeatedly detected, and there are many loose expressions which Archimedes would not have used, and occasional omissions of an essential step in his argument. Sometimes the original text can be inferred from the commentaries written by Eutocius, but these extend only to the books *On the Sphere and Cylinder*, the *Measurement of a Circle*, and *On Plane Equilibriums*. A partial loss of Doric forms had already occurred by the time of Eutocius, and it is believed that the works most widely read were completely recast a little later in the school of Isidorus of Miletus to make them more easily intelligible to pupils.

[c] The instrument is otherwise mentioned by Simplicius (*in Aristot. Phys.*, ed. Diels 1110. 2-5) and it is implied that it was used for weighing: *ταύτῃ δὲ τῇ ἀναλογίᾳ τοῦ κινοῦντος καὶ τοῦ κινουμένου καὶ τοῦ διαστήματος τὸ σταθμιστικὸν ὄργανον τὸν καλούμενον χαριστίωνα συστήσας ὁ Ἀρχιμήδης ὡς μέχρι παντὸς τῆς ἀναλογίας προχωρούσης ἐκόμπασεν ἐκεῖνο τὸ "πᾶ βῶ καὶ κινῶ τὰν γᾶν."* As Tzetzes in another place (*Chil.* iii. 61: *ὁ γῆν ἀνασπῶν μηχανῇ τῇ τρισπάστῳ βοῶν "ὅπα βῶ καὶ σαλεύσω τὴν χθόνα"*) writes of a triple-pulley device in the same connexion, it may be presumed to have been of this nature.

Οὗτος, κατὰ Διόδωρον, τῆς Συρακούσης ταύτης
Προδότου πρὸς τὸν Μάρκελλον ἀθρόως γενομένης,
Εἴτε, κατὰ τὸν Δίωνα, Ῥωμαίοις πορθηθείσης,
Ἀρτέμιδι τῶν πολιτῶν τότε παννυχιζόντων,
Τοιουτοτρόπως τέθνηκεν ὑπό τινος Ῥωμαίου.
Ἦν κεκυφώς, διάγραμμα μηχανικόν τι γράφων,
Τὶς δὲ Ῥωμαῖος ἐπιστὰς εἷλκεν αἰχμαλωτίζων.
Ὁ δὲ τοῦ διαγράμματος ὅλος ὑπάρχων τότε,
Τίς ὁ καθέλκων οὐκ εἰδώς, ἔλεγε πρὸς ἐκεῖνον·
"Ἀπόστηθι, ὦ ἄνθρωπε, τοῦ διαγράμματός μου."
Ὡς δ' εἷλκε τοῦτον συστραφεὶς καὶ γνοὺς
Ῥωμαῖον εἶναι,
Ἐβόα, "τὶ μηχάνημα τὶς τῶν ἐμῶν μοι δότω."
Ὁ δὲ Ῥωμαῖος πτοηθεὶς εὐθὺς ἐκεῖνον κτείνει,
Ἄνδρα σαθρὸν καὶ γέροντα, δαιμόνιον τοῖς ἔργοις.

Plut. *Marcellus* xiv. 7–xvii. 7

Καὶ μέντοι καὶ Ἀρχιμήδης, Ἱέρωνι τῷ βασιλεῖ συγγενὴς ὢν καὶ φίλος, ἔγραψεν ὡς τῇ δοθείσῃ δυνάμει τὸ δοθὲν βάρος κινῆσαι δυνατόν ἐστι· καὶ νεανιευσάμενος, ὥς φασι, ῥώμῃ τῆς ἀποδείξεως εἶπεν ὡς, εἰ γῆν εἶχεν ἑτέραν, ἐκίνησεν ἂν ταύτην μεταβὰς εἰς ἐκείνην. θαυμάσαντος δὲ τοῦ Ἱέρωνος, καὶ δεηθέντος εἰς ἔργον ἐξαγαγεῖν τὸ πρόβλημα καὶ δεῖξαί τι τῶν μεγάλων κινούμενον ὑπὸ σμικρᾶς δυνάμεως, ὁλκάδα τριάρμενον τῶν βασιλικῶν πόνῳ μεγάλῳ καὶ χειρὶ πολλῇ νεωλκηθεῖσαν, ἐμβαλὼν ἀνθρώπους τε πολλοὺς καὶ τὸν συνήθη φόρτον, αὐτὸς ἄπωθεν καθήμενος, οὐ μετὰ σπουδῆς, ἀλλὰ

[a] Diod. Sic. Frag. Book xxvi.
[b] The account of Dion Cassius has not survived.
[c] Zonaras ix. 5 adds that when he heard the enemy were

Whether, as Diodorus[a] asserts, Syracuse was betrayed and the citizens went in a body to Marcellus, or, as Dion[b] tells, it was plundered by the Romans, while the citizens were keeping a night festival to Artemis, he died in this fashion at the hands of one of the Romans. He was stooping down, drawing some diagram in mechanics, when a Roman came up and began to drag him away to take him prisoner. But he, being wholly intent at the time on the diagram, and not perceiving who was tugging at him, said to the man : "Stand away, fellow, from my diagram."[c] As the man continued pulling, he turned round and, realizing that he was a Roman, he cried, "Somebody give me one of my engines." But the Roman, scared, straightway slew him, a feeble old man but wonderful in his works.

Plutarch, *Marcellus* xiv. 7–xvii. 7

Archimedes, who was a kinsman and friend of King Hiero, wrote to him that with a given force it was possible to move any given weight ; and emboldened, as it is said, by the strength of the proof, he averred that, if there were another world and he could go to it, he would move this one. Hiero was amazed and besought him to give a practical demonstration of the problem and show some great object moved by a small force ; he thereupon chose a three-masted merchantman among the king's ships which had been hauled ashore with great labour by a large band of men, and after putting on board many men and the usual cargo, sitting some distance away and without any special effort, he pulled gently with his hand at

coming "πὰρ κεφαλάν" ἔφη "καὶ μὴ παρὰ γραμμάν"—"Let them come at my head," he said, "but not at my line."

ἠρέμα τῇ χειρὶ σείων ἀρχήν τινα πολυσπάστου προσηγάγετο λείως καὶ ἀπταίστως καὶ ὥσπερ διὰ θαλάττης ἐπιθέουσαν. ἐκπλαγεὶς οὖν ὁ βασιλεὺς καὶ συννοήσας τῆς τέχνης τὴν δύναμιν, ἔπεισε τὸν Ἀρχιμήδην ὅπως αὐτῷ τὰ μὲν ἀμυνομένῳ, τὰ δ' ἐπιχειροῦντι μηχανήματα κατασκευάσῃ πρὸς πᾶσαν ἰδέαν πολιορκίας, οἷς αὐτὸς μὲν οὐκ ἐχρήσατο, τοῦ βίου τὸ πλεῖστον ἀπόλεμον καὶ πανηγυρικὸν βιώσας, τότε δ' ὑπῆρχε τοῖς Συρακουσίοις εἰς δέον ἡ παρασκευὴ καὶ μετὰ τῆς παρασκευῆς ὁ δημιουργός.

Ὡς οὖν προσέβαλον οἱ Ῥωμαῖοι διχόθεν, ἔκπληξις ἦν τῶν Συρακουσίων καὶ σιγὴ διὰ δέος, μηδὲν ἂν ἀνθέξειν πρὸς βίαν καὶ δύναμιν οἰομένων τοσαύτην. σχάσαντος δὲ τὰς μηχανὰς τοῦ Ἀρχιμήδους ἅμα τοῖς μὲν πεζοῖς ἀπήντα τοξεύματά τε παντοδαπὰ καὶ λίθων ὑπέρογκα μεγέθη, ῥοίζῳ καὶ τάχει καταφερομένων ἀπίστῳ, καὶ μηδενὸς ὅλως τὸ βρῖθος στέγοντος ἀθρόους ἀνατρεπόντων τοὺς ὑποπίπτοντας καὶ τὰς τάξεις συγχεόντων, ταῖς δὲ ναυσὶν ἀπὸ τῶν τειχῶν ἄφνω ὑπεραιωρούμεναι κεραῖαι τὰς μὲν ὑπὸ βρίθους στηρίζοντος ἄνωθεν ὠθοῦσαι κατέδυον εἰς βυθόν, τὰς δὲ χερσὶ σιδηραῖς ἢ στόμασιν εἰκασμένοις γεράνων ἀνασπῶσαι πρῴραθεν ὀρθὰς ἐπὶ πρύμναν ἐβάπτιζον,

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[a] πολύσπαστος. Galen, *in Hipp. De Artic.* iv. 47 uses the same word. Tzetzes (*loc. cit.*) speaks of a triple-pulley device (τῇ τρισπάστῳ μηχανῇ) in the same connexion, and Oribasius, *Coll. med.* xlix. 22 mentions the τρίσπαστος as an invention of Archimedes; he says that it was so called because it had three ropes, but Vitruvius says it was thus named because it had three wheels. Athenaeus v. 207 a-b says that a *helix* was used. Heath, *The Works of Archimedes*,

the end of a compound pulley [a] and drew the vessel smoothly and evenly towards himself as though she were running along the surface of the water. Astonished at this, and understanding the power of his art, the king persuaded Archimedes to construct for him engines to be used in every type of siege warfare, some defensive and some offensive; he had not himself used these engines because he spent the greater part of his life remote from war and amid the rites of peace, but now his apparatus proved of great advantage to the Syracusans, and with the apparatus its inventor.[b]

Accordingly, when the Romans attacked them from two elements, the Syracusans were struck dumb with fear, thinking that nothing would avail against such violence and power. But Archimedes began to work his engines and hurled against the land forces all sorts of missiles and huge masses of stones, which came down with incredible noise and speed; nothing at all could ward off their weight, but they knocked down in heaps those who stood in the way and threw the ranks into disorder. Furthermore, beams were suddenly thrown over the ships from the walls, and some of the ships were sent to the bottom by means of weights fixed to the beams and plunging down from above; others were drawn up by iron claws, or crane-like beaks, attached to the prow and were

p. xx, suggests that the vessel, once started, was kept in motion by the system of pulleys, but the first impulse was given by a machine similar to the κοχλίας described by Pappus viii. ed. Hultsch 1066, 1108 ff., in which a cog-wheel with oblique teeth moves on a cylindrical helix turned by a handle.

[b] Similar stories of Archimedes' part in the defence are told by Polybius viii. 5. 3-5 and Livy xxiv. 34.

ἢ δι' ἀντιτόνων ἔνδον ἐπιστρεφόμεναι καὶ περιαγόμεναι τοῖς ὑπὸ τὸ τεῖχος πεφυκόσι κρημνοῖς καὶ σκοπέλοις προσήρασσον, ἅμα φθόρῳ πολλῷ τῶν ἐπιβατῶν συντριβομένων. πολλάκις δὲ μετέωρος ἐξαρθεῖσα ναῦς ἀπὸ τῆς θαλάσσης δεῦρο κἀκεῖσε περιδινουμένη καὶ κρεμαμένη θέαμα φρικῶδες ἦν, μέχρι οὗ τῶν ἀνδρῶν ἀπορριφέντων καὶ διασφενδονηθέντων κενὴ προσπέσοι τοῖς τείχεσιν ἢ περιολίσθοι τῆς λαβῆς ἀνείσης. ἣν δὲ ὁ Μάρκελλος ἀπὸ τοῦ ζεύγματος ἐπῆγε μηχανήν, σαμβύκη μὲν ἐκαλεῖτο δι' ὁμοιότητά τινα σχήματος πρὸς τὸ μουσικὸν ὄργανον, ἔτι δὲ ἄπωθεν αὐτῆς προσφερομένης πρὸς τὸ τεῖχος ἐξήλατο λίθος δεκατάλαντος ὁλκήν, εἶτα ἕτερος ἐπὶ τούτῳ καὶ τρίτος, ὧν οἱ μὲν αὐτῇ[1] ἐμπεσόντες μεγάλῳ κτύπῳ καὶ κλύδωνι τῆς μηχανῆς τήν τε βάσιν συνηλόησαν καὶ τὸ γόμφωμα διέσεισαν καὶ διέσπασαν τοῦ ζεύγματος, ὥστε τὸν Μάρκελλον ἀπορούμενον αὐτόν τε ταῖς ναυσὶν ἀποπλεῖν κατὰ τάχος καὶ τοῖς πεζοῖς ἀναχώρησιν παρεγγυῆσαι.

Βουλευομένοις δὲ ἔδοξεν αὐτοῖς ἔτι νυκτός, ἂν δύνωνται, προσμῖξαι τοῖς τείχεσι· τοὺς γὰρ τόνους, οἷς χρῆσθαι τὸν Ἀρχιμήδην, ῥύμην ἔχοντας ὑπερπετεῖς ποιήσεσθαι τὰς τῶν βελῶν ἀφέσεις, ἐγγύθεν δὲ καὶ τελέως ἀπράκτους εἶναι διάστημα τῆς πληγῆς οὐκ ἐχούσης. ὁ δ' ἦν, ὡς ἔοικεν, ἐπὶ ταῦτα πάλαι παρεσκευασμένος ὀργάνων τε συμμέτρους πρὸς πᾶν διάστημα κινήσεις καὶ βέλη βραχέα, καὶ διὰ ⟨τὸ τεῖχος[2]⟩ οὐ μεγάλων, πολλῶν

[1] αὐτῇ Coraës, αὐτῆς codd.
[2] τὸ τεῖχος add. Sintenis ex Polyb.

plunged down on their sterns, or were twisted round and turned about by means of ropes within the city, and dashed against the cliffs set by Nature under the wall and against the rocks, with great destruction of the crews, who were crushed to pieces. Often there was the fearful sight of a ship lifted out of the sea into mid-air and whirled about as it hung there, until the men had been thrown out and shot in all directions, when it would fall empty upon the walls or slip from the grip that had held it. As for the engine which Marcellus was bringing up from the platform of ships, and which was called *sambuca* from some resemblance in its shape to the musical instrument,[a] while it was still some distance away as it was being carried to the wall a stone ten talents in weight was discharged at it, and after this a second and a third; some of these, falling upon it with a great crash and sending up a wave, crushed the base of the engine, shook the framework and dislodged it from the barrier, so that Marcellus in perplexity sailed away in his ships and passed the word to his land forces to retire.

In a council of war it was decided to approach the walls, if they could, while it was still night; for they thought that the ropes used by Archimedes, since they gave a powerful impetus, would send the missiles over their heads and would fail in their object at close quarters since there was no space for the cast. But Archimedes, it seems, had long ago prepared for such a contingency engines adapted to all distances and missiles of short range, and through openings in the

[a] The *σαμβύκη* was a triangular musical instrument with four strings. Polybius (viii. 6) states that Marcellus had eight quinqueremes in pairs locked together, and on each pair a "sambuca" had been erected; it served as a pent-house for raising soldiers on to the battlements.

δὲ καὶ συνεχῶν τρημάτων ⟨ὄντων[1]⟩, οἱ σκορπίοι βραχύτονοι μέν, ἐγγύθεν δὲ πλῆξαι παρεστήκεσαν ἀόρατοι τοῖς πολεμίοις.

Ὡς οὖν προσέμιξαν οἰόμενοι λανθάνειν, αὖθις αὖ βέλεσι πολλοῖς ἐντυγχάνοντες καὶ πληγαῖς, πετρῶν μὲν ἐκ κεφαλῆς ἐπ' αὐτοὺς φερομένων ὥσπερ πρὸς κάθετον, τοῦ δὲ τείχους τοξεύματα πανταχόθεν ἀναπέμποντος, ἀνεχώρουν ὀπίσω. κἀνταῦθα πάλιν αὐτῶν εἰς μῆκος ἐκτεταγμένων, βελῶν ἐκθεόντων καὶ καταλαμβανόντων ἀπιόντας ἐγίνετο πολὺς μὲν αὐτῶν φθόρος, πολὺς δὲ τῶν νεῶν συγκρουσμός, οὐδὲν ἀντιδρᾶσαι τοὺς πολεμίους δυναμένων. τὰ γὰρ πλεῖστα τῶν ὀργάνων ὑπὸ τὸ τεῖχος ἐσκευοποίητο τῷ Ἀρχιμήδει, καὶ θεομαχοῦσιν ἐῴκεσαν οἱ Ῥωμαῖοι, μυρίων αὐτοῖς κακῶν ἐξ ἀφανοῦς ἐπιχεομένων.

Οὐ μὴν ἀλλ' ὁ Μάρκελλος ἀπέφυγέ τε καὶ τοὺς σὺν ἑαυτῷ σκώπτων τεχνίτας καὶ μηχανοποιοὺς ἔλεγεν· "οὐ παυσόμεθα πρὸς τὸν γεωμετρικὸν τοῦτον Βριάρεων πολεμοῦντες, ὃς ταῖς μὲν ναυσίν[2] ἡμῶν κυαθίζει ἐκ τῆς θαλάσσης, τὴν δὲ σαμβύκην ῥαπίζων[2] μετ' αἰσχύνης ἐκβέβληκε, τοὺς δὲ μυθικοὺς ἑκατόγχειρας ὑπεραίρει τοσαῦτα βάλλων ἅμα βέλη καθ' ἡμῶν;" τῷ γὰρ ὄντι πάντες οἱ λοιποὶ Συρακούσιοι σῶμα τῆς Ἀρχιμήδους παρασκευῆς ἦσαν, ἡ δὲ κινοῦσα πάντα καὶ στρέφουσα ψυχὴ μία, τῶν μὲν ἄλλων ὅπλων ἀτρέμα κειμένων, μόνοις δὲ τοῖς ἐκείνου τότε τῆς πόλεως χρωμένης καὶ πρὸς ἄμυναν καὶ πρὸς ἀσφάλειαν. τέλος δὲ τοὺς Ῥωμαίους οὕτω περιφόβους γεγονότας ὁρῶν ὁ Μάρκελλος ὥστ', εἰ καλῴδιον ἢ ξύλον ὑπὲρ τοῦ

[1] ὄντων add. Sintenis ex Polyb.

wall, small in size but many and continuous, short-ranged engines called scorpions could be trained on objects close at hand without being seen by the enemy.

When, therefore, the Romans approached the walls, thinking to escape notice, once again they were met by the impact of many missiles; stones fell down on them almost perpendicularly, the wall shot out arrows at them from all points, and they withdrew to the rear. Here again, when they were drawn up some distance away, missiles flew forth and caught them as they were retiring, and caused much destruction among them; many of the ships, also, were dashed together and they could not retaliate upon the enemy. For Archimedes had made the greater part of his engines under the wall, and the Romans seemed to be fighting against the gods, inasmuch as countless evils were poured upon them from an unseen source.

Nevertheless Marcellus escaped, and, twitting his artificers and craftsmen, he said: "Shall we not cease fighting against this geometrical Briareus, who uses our ships like cups to ladle water from the sea, who has whipped our *sambuca* and driven it off in disgrace, and who outdoes all the hundred-handed monsters of fable in hurling so many missiles against us all at once?" For in reality all the other Syracusans were only a body for Archimedes' apparatus, and his the one soul moving and turning everything: all other weapons lay idle, and the city then used his alone, both for offence and for defence. In the end the Romans became so filled with fear that, if they saw a little piece of rope or of wood projecting over

[a] *ταῖς μὲν ναυσὶν . . . ῥαπίζων* an anonymous correction from Polybius, *τὰς μὲν ναῦς ἡμῶν καθίζων πρὸς τὴν θάλασσαν παίζων* codd.

τείχους μικρὸν ὀφθείη προτεινόμενον, τοῦτο ἐκεῖνο, μηχανήν τινα κινεῖν ἐπ' αὐτοὺς Ἀρχιμήδη βοῶντας ἀποτρέπεσθαι καὶ φεύγειν, ἀπέσχετο μάχης ἁπάσης καὶ προσβολῆς, τὸ λοιπὸν ἐπὶ τῷ χρόνῳ τὴν πολιορκίαν θέμενος.

Τηλικοῦτον μέντοι φρόνημα καὶ βάθος ψυχῆς καὶ τοσοῦτον ἐκέκτητο θεωρημάτων πλοῦτον Ἀρχιμήδης ὥστε, ἐφ' οἷς ὄνομα καὶ δόξαν οὐκ ἀνθρωπίνης, ἀλλὰ δαιμονίου τινὸς ἔσχε συνέσεως, μηθὲν ἐθελῆσαι σύγγραμμα περὶ τούτων ἀπολιπεῖν, ἀλλὰ τὴν περὶ τὰ μηχανικὰ πραγματείαν καὶ πᾶσαν ὅλως τέχνην χρείας ἐφαπτομένην ἀγεννῆ καὶ βάναυσον ἡγησάμενος, εἰς ἐκεῖνα καταθέσθαι μόνα τὴν αὑτοῦ φιλοτιμίαν οἷς τὸ καλὸν καὶ περιττὸν ἀμιγὲς τοῦ ἀναγκαίου πρόσεστιν, ἀσύγκριτα μὲν ὄντα τοῖς ἄλλοις, ἔριν δὲ παρέχοντα πρὸς τὴν ὕλην τῇ ἀποδείξει, τῆς μὲν τὸ μέγεθος καὶ τὸ κάλλος, τῆς δὲ τὴν ἀκρίβειαν καὶ τὴν δύναμιν ὑπερφυῆ παρεχομένης· οὐ γὰρ ἔστιν ἐν γεωμετρίᾳ χαλεπωτέρας καὶ βαρυτέρας ὑποθέσεις ἐν ἁπλουστέροις λαβεῖν καὶ καθαρωτέροις στοιχείοις γραφομένας. καὶ τοῦθ' οἱ μὲν εὐφυΐᾳ τοῦ ἀνδρὸς προσάπτουσιν, οἱ δὲ ὑπερβολῇ τινι πόνου νομίζουσιν ἀπόνως πεποιημένῳ καὶ ῥᾳδίως ἕκαστον ἐοικὸς γεγονέναι. ζητῶν μὲν γὰρ οὐκ ἄν τις εὕροι δι' αὑτοῦ τὴν ἀπόδειξιν, ἅμα δὲ τῇ μαθήσει παρίσταται δόξα τοῦ κἂν αὐτὸν εὑρεῖν· οὕτω λείαν ὁδὸν ἄγει[1] καὶ ταχεῖαν ἐπὶ τὸ δεικνύμενον. οὔκουν οὐδὲ ἀπιστῆσαι τοῖς περὶ αὐτοῦ λεγομένοις ἐστίν, ὡς ὑπ' οἰκείας δή τινος καὶ συνοίκου θελγόμενος ἀεὶ σειρῆνος ἐλέληστο καὶ σίτου καὶ θεραπείας σώματος ἐξέλειπε, βίᾳ δὲ πολλάκις ἑλκόμενος ἐπ' ἄλειμμα καὶ

the wall, they cried, "There it is, Archimedes is training some engine upon us," and fled; seeing this Marcellus abandoned all fighting and assault, and for the future relied on a long siege.

Yet Archimedes possessed so lofty a spirit, so profound a soul, and such a wealth of scientific inquiry, that although he had acquired through his inventions a name and reputation for divine rather than human intelligence, he would not deign to leave behind a single writing on such subjects. Regarding the business of mechanics and every utilitarian art as ignoble and vulgar, he gave his zealous devotion only to those subjects whose elegance and subtlety are untrammelled by the necessities of life; these subjects, he held, cannot be compared with any others; in them the subject-matter vies with the demonstration, the former possessing strength and beauty, the latter precision and surpassing power; for it is not possible to find in geometry more difficult and weighty questions treated in simpler and purer terms. Some attribute this to the natural endowments of the man, others think it was the result of exceeding labour that everything done by him appeared to have been done without labour and with ease. For although by his own efforts no one could discover the proof, yet as soon as he learns it, he takes credit that he could have discovered it: so smooth and rapid is the path by which he leads to the conclusion. For these reasons there is no need to disbelieve the stories told about him—how, continually bewitched by some familiar siren dwelling with him, he forgot his food and neglected the care of his body; and how, when he was dragged by main force, as often happened, to the

[1] ἄγει Bryan, ἄγειν codd.

λουτρόν, ἐν ταῖς ἐσχάραις ἔγραφε σχήματα τῶν γεωμετρικῶν, καὶ τοῦ σώματος ἀληλιμμένου διῆγε τῷ δακτύλῳ γραμμάς, ὑπὸ ἡδονῆς μεγάλης κάτοχος ὢν καὶ μουσόληπτος ἀληθῶς. πολλῶν δὲ καὶ καλῶν εὑρετὴς γεγονὼς λέγεται τῶν φίλων δεηθῆναι καὶ τῶν συγγενῶν ὅπως αὐτοῦ μετὰ τὴν τελευτὴν ἐπιστήσωσι τῷ τάφῳ τὸν περιλαμβάνοντα τὴν σφαῖραν ἐντὸς κυλίνδρου, ἐπιγράψαντες τὸν λόγον τῆς ὑπεροχῆς τοῦ περιέχοντος στερεοῦ πρὸς τὸ περιεχόμενον.

Ibid. xix. 4-6

Μάλιστα δὲ τὸ Ἀρχιμήδους πάθος ἠνίασε Μάρκελλον. ἔτυχε μὲν γὰρ αὐτός τι καθ' ἑαυτὸν ἀνασκοπῶν ἐπὶ διαγράμματος· καὶ τῇ θεωρίᾳ δεδωκὼς ἅμα τήν τε διάνοιαν καὶ τὴν πρόσοψιν οὐ προῄσθετο τὴν καταδρομὴν τῶν Ῥωμαίων οὐδὲ τὴν ἅλωσιν τῆς πόλεως, ἄφνω δὲ ἐπιστάντος αὐτῷ στρατιώτου καὶ κελεύοντος ἀκολουθεῖν πρὸς Μάρκελλον οὐκ ἐβούλετο πρὶν ἢ τελέσαι τὸ πρόβλημα καὶ καταστῆσαι πρὸς τὴν ἀπόδειξιν. ὁ δὲ ὀργισθεὶς καὶ σπασάμενος τὸ ξίφος ἀνεῖλεν αὐτόν. ἕτεροι μὲν οὖν λέγουσιν ἐπιστῆναι μὲν εὐθὺς ὡς ἀποκτενοῦντα ξιφήρη τὸν Ῥωμαῖον, ἐκεῖνον δ' ἰδόντα δεῖσθαι καὶ ἀντιβολεῖν ἀναμεῖναι βραχὺν χρόνον, ὡς μὴ καταλίπῃ τὸ ζητούμενον ἀτελὲς καὶ ἀθεώρητον, τὸν δὲ οὐ φροντίσαντα διαχρήσασθαι. καὶ τρίτος ἐστὶ λόγος, ὡς κομίζοντι πρὸς Μάρκελλον αὐτῷ τῶν μαθηματικῶν ὀργάνων σκιόθηρα καὶ σφαίρας καὶ γωνίας, αἷς ἐναρμόττει

[a] Cicero, when quaestor in Sicily, found this tomb over-

place for bathing and anointing, he would draw geometrical figures in the hearths, and draw lines with his finger in the oil with which his body was anointed, being overcome by great pleasure and in truth inspired of the Muses. And though he made many elegant discoveries, he is said to have besought his friends and kinsmen to place on his grave after his death a cylinder enclosing a sphere, with an inscription giving the proportion by which the including solid exceeds the included.[a]

Ibid. xix. 4-6

But what specially grieved Marcellus was the death of Archimedes. For it chanced that he was alone, examining a diagram closely; and having fixed both his mind and his eyes on the object of his inquiry, he perceived neither the inroad of the Romans nor the taking of the city. Suddenly a soldier came up to him and bade him follow to Marcellus, but he would not go until he had finished the problem and worked it out to the demonstration. Thereupon the soldier became enraged, drew his sword and dispatched him. Others, however, say that the Roman came upon him with drawn sword intending to kill him at once, and that Archimedes, on seeing him, besought and entreated him to wait a little while so that he might not leave the question unfinished and only partly investigated; but the soldier did not understand and slew him. There is also a third story, that as he was carrying to Marcellus some of his mathematical instruments, such as sundials, spheres and

grown with vegetation, but still bearing the cylinder with the sphere, and he restored it (*Tusc. Disp.* v. 64-66). The theorem proving the proportion is given *infra*, pp. 124-127.

τὸ τοῦ ἡλίου μέγεθος πρὸς τὴν ὄψιν, στρατιῶται περιτυχόντες καὶ χρυσίον ἐν τῷ τεύχει δόξαντες φέρειν ἀπέκτειναν. ὅτι μέντοι Μάρκελλος ἤλγησε καὶ τὸν αὐτόχειρα τοῦ ἀνδρὸς ἀπεστράφη καθάπερ ἐναγῆ, τοὺς δὲ οἰκείους ἀνευρὼν ἐτίμησεν, ὁμολογεῖται.

Papp. *Coll.* viii. 11. 19, ed. Hultsch 1060. 1-4

Τῆς αὐτῆς δέ ἐστιν θεωρίας τὸ δοθὲν βάρος τῇ δοθείσῃ δυνάμει κινῆσαι· τοῦτο γὰρ Ἀρχιμήδους μὲν εὕρημα [λέγεται][1] μηχανικόν, ἐφ᾿ ᾧ λέγεται εἰρηκέναι· "δός μοί (φησι) ποῦ στῶ καὶ κινῶ τὴν γῆν."

Diod. Sic. i. 34. 2

Ποταμόχωστος γὰρ οὖσα καὶ κατάρρυτος πολλοὺς καὶ παντοδαποὺς ἐκφέρει καρπούς, τοῦ μὲν ποταμοῦ διὰ τὴν κατ᾿ ἔτος ἀνάβασιν νεαρὰν ἰλὺν ἀεὶ καταχέοντος, τῶν δ᾿ ἀνθρώπων ῥᾳδίως ἅπασαν ἀρδευόντων διά τινος μηχανῆς, ἣν ἐπενόησε μὲν Ἀρχιμήδης ὁ Συρακόσιος, ὀνομάζεται δὲ ἀπὸ τοῦ σχήματος κοχλίας.

Ibid. v. 37. 3

Τὸ πάντων παραδοξότατον, ἀπαρύτουσι τὰς ῥύσεις τῶν ὑδάτων τοῖς Αἰγυπτιακοῖς λεγομένοις κοχλίαις, οὓς Ἀρχιμήδης ὁ Συρακόσιος εὗρεν, ὅτε παρέβαλεν εἰς Αἴγυπτον.

[1] λέγεται om. Hultsch.

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[a] Diodorus is writing of the island in the delta of the Nile.

[b] It may be inferred that he studied with the successors of Euclid at Alexandria, and it was there perhaps that he made the acquaintance of Conon of Samos, to whom, as

angles adjusted to the apparent size of the sun, some soldiers fell in with him and, under the impression that he carried treasure in the box, killed him. What is, however, agreed is that Marcellus was distressed, and turned away from the slayer as from a polluted person, and sought out the relatives of Archimedes to do them honour.

Pappus, *Collection* viii. 11. 19, ed. Hultsch 1060. 1-4

To the same type of inquiry belongs the problem: *To move a given weight by a given force.* This is one of Archimedes' discoveries in mechanics, whereupon he is said to have exclaimed: "Give me somewhere to stand and I will move the earth."

Diodorus Siculus i. 34. 2

As it is made of silt watered by the river after being deposited, it[a] bears an abundance of fruits of all kinds; for in the annual rising the river continually pours over it fresh alluvium, and men easily irrigate the whole of it by means of a certain instrument conceived by Archimedes of Syracuse, and which gets its name because it has the form of a spiral or *screw*.

Ibid. v. 37. 3

Most remarkable of all, they draw off streams of water by the so-called Egyptian screws, which Archimedes of Syracuse invented when he went by ship to Egypt.[b]

the preface to his books *On the Sphere and Cylinder* shows, he used to communicate his discoveries before publication, and Eratosthenes of Cyrene, to whom he sent the *Method* and probably the *Cattle Problem*.

Vitr. *De Arch.* ix., Praef. 9-12

Archimedis vero cum multa miranda inventa et varia fuerint, ex omnibus etiam infinita sollertia id quod exponam videtur esse expressum. Nimium Hiero Syracusis auctus regia potestate, rebus bene gestis cum auream coronam votivam diis immortalibus in quodam fano constituisset ponendam, manupretio locavit faciendam et aurum ad sacoma adpendit redemptori. Is ad tempus opus manu factum subtiliter regi adprobavit et ad sacoma pondus coronae visus est praestitisse. Posteaquam indicium est factum dempto auro tantundem argenti in id coronarium opus admixtum esse, indignatus Hiero se contemptum esse neque inveniens qua ratione id furtum deprehenderet, rogavit Archimeden uti insumeret sibi de eo cogitationem. Tunc is cum haberet eius rei curam, casu venit in balineum ibique cum in solium descenderet, animadvertit quantum corporis sui in eo insideret tantum aquae extra solium effluere. Idque cum eius rei rationem explicationis ostendisset, non est moratus sed exsiluit gaudio motus de solio et nudus vadens domum versus significabat clara voce invenisse quod quaereret. Nam currens identidem graece clamabat εὕρηκα εὕρηκα.[a]

Tum vero ex eo inventionis ingressu duas fecisse dicitur massas aequo pondere quo etiam fuerat corona, unam ex auro et alteram ex argento. Cum ita fecisset, vas amplum ad summa labra implevit

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[a] "I have found, I have found."

ARCHIMEDES

Vitruvius, *On Architecture* ix., Preface 9-1

Archimedes made many wonderful discoveries of different kinds, but of all these that which I shall now explain seems to exhibit a boundless ingenuity. When Hiero was greatly exalted in the royal power at Syracuse, in return for the success of his policy he determined to set up in a certain shrine a golden crown as a votive offering to the immortal gods. He let out the work for a stipulated payment, and weighed out the exact amount of gold for the contractor. At the appointed time the contractor brought his work skilfully executed for the king's approval, and he seemed to have fulfilled exactly the requirement about the weight of the crown. Later information was given that gold had been removed and an equal weight of silver added in the making of the crown. Hiero was indignant at this disrespect for himself, and, being unable to discover any means by which he might unmask the fraud, he asked Archimedes to give it his attention. While Archimedes was turning the problem over, he chanced to come to the place of bathing, and there, as he was sitting down in the tub, he noticed that the amount of water which flowed over the tub was equal to the amount by which his body was immersed. This indicated to him a means of solving the problem, and he did not delay, but in his joy leapt out of the tub and, rushing naked towards his home, he cried out with a loud voice that he had found what he sought. For as he ran he repeatedly shouted in Greek, *heureka*, *heureka*.[a]

Then, following up his discovery, he is said to have made two masses of the same weight as the crown, the one of gold and the other of silver. When he had so done, he filled a large vessel right up to the brim

aqua, in quo demisit argenteam massam. Cuius quanta magnitudo in vase depressa est, tantum aquae effluxit. Ita exempta massa quanto minus factum fuerat refudit sextario mensus, ut eodem modo quo prius fuerat ad labra aequaretur. Ita ex eo invenit quantum pondus argenti ad certam aquae mensuram responderet.

Cum id expertus esset, tum auream massam similiter pleno vase demisit et ea exempta eadem ratione mensura addita invenit deesse aquae non tantum sed minus, quanto minus magno corpore eodem pondere auri massa esset quam argenti. Postea vero repleto vase in eadem aqua ipsa corona demissa invenit plus aquae defluxisse in coronam quam in auream eodem pondere massam, et ita ex eo quod defuerit plus aquae in corona quam in massa, ratiocinatus deprehendit argenti in auro mixtionem et manifestum furtum redemptoris.

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[a] The method may be thus expressed analytically.

Let w be the weight of the crown, and let it be made up of a weight w_1 of gold and a weight w_2 of silver, so that $w = w_1 + w_2$.

Let the crown displace a volume v of water.

Let the weight w of gold displace a volume v_1 of water; then a weight w_1 of gold displaces a volume $\frac{w_1}{w} \cdot v_1$ of water.

Let the weight w of silver displace a volume v_2 of water;

with water, into which he dropped the silver mass. The amount by which it was immersed in the vessel was the amount of water which overflowed. Taking out the mass, he poured back the amount by which the water had been depleted, measuring it with a pint pot, so that as before the water was made level with the brim. In this way he found what weight of silver answered to a certain measure of water.

When he had made this test, in like manner he dropped the golden mass into the full vessel. Taking it out again, for the same reason he added a measured quantity of water, and found that the deficiency of water was not the same, but less; and the amount by which it was less corresponded with the excess of a mass of silver, having the same weight, over a mass of gold. After filling the vessel again, he then dropped the crown itself into the water, and found that more water overflowed in the case of the crown than in the case of the golden mass of identical weight; and so, from the fact that more water was needed to make up the deficiency in the case of the crown than in the case of the mass, he calculated and detected the mixture of silver with the gold and the contractor's fraud stood revealed.[a]

then a weight w_2 of silver displaces a volume $\frac{w_2}{w} \cdot v_2$ of water.

It follows that
$$v = \frac{w_1}{w} \cdot v_1 + \frac{w_2}{w} \cdot v_2$$
$$= \frac{w_1 v_1 + w_2 v_2}{w_1 + w_2},$$
so that
$$\frac{w_1}{w_2} = \frac{v_2 - v}{v - v_1}.$$

For an alternative method of solving the problem, v. *infra*, pp. 248-251.

(b) Surface and Volume of the Cylinder and Sphere

Archim. *De Sphaera et Cyl.* i., Archim. ed. Heiberg i. 2–132. 3

Ἀρχιμήδης Δοσιθέῳ χαίρειν

Πρότερον μὲν ἀπέσταλκά σοι τῶν ὑφ' ἡμῶν τεθεωρημένων γράψας μετὰ ἀποδείξεως, ὅτι πᾶν τμῆμα τὸ περιεχόμενον ὑπό τε εὐθείας καὶ ὀρθογωνίου κώνου τομῆς ἐπίτριτόν ἐστι τριγώνου τοῦ βάσιν τὴν αὐτὴν ἔχοντος τῷ τμήματι καὶ ὕψος ἴσον· ὕστερον δὲ ἡμῖν ὑποπεσόντων θεωρημάτων ἀξίων λόγου[1] πεπραγματεύμεθα περὶ τὰς ἀποδείξεις αὐτῶν. ἔστιν δὲ τάδε· πρῶτον μέν, ὅτι πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶν τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ· ἔπειτα δέ, ὅτι παντὸς τμήματος σφαίρας τῇ ἐπιφανείᾳ ἴσος ἐστὶ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἐστὶ τῇ εὐθείᾳ τῇ ἀπὸ τῆς κορυφῆς τοῦ τμήματος ἀγομένῃ ἐπὶ τὴν περιφέρειαν τοῦ κύκλου, ὅς ἐστι βάσις τοῦ τμήματος·

[1] ἀξίων λόγου cod., ἀνελέγκτων coni. Heath.

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[a] The chief results of this book are described in the prefatory letter to Dositheus. In this selection as much as possible is given of what is essential to finding the proportions between the surface and volume of the sphere and the surface and volume of the enclosing cylinder, which Archimedes regarded as his crowning achievement (*supra*, p. 32). In the case of the surface, the whole series of propositions is reproduced so that the reader may follow in detail the majestic chain of reasoning by which Archimedes, starting from seemingly remote premises, reaches the desired conclusion; in the case of the volume only the final proposition (34) can be given, for reasons of space, but the reader will be able to prove the omitted theorems for himself. *Pari passu* with

(b) Surface and Volume of the Cylinder and Sphere

Archimedes, *On the Sphere and Cylinder* i., Archim. ed. Heiberg i. 2–132. 3[a]

Archimedes to Dositheus greeting

On a previous occasion I sent you, together with the proof, so much of my investigations as I had set down in writing, namely, that *any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base as the segment and equal height.*[b] Subsequently certain theorems deserving notice occurred to me, and I have worked out the proofs. They are these: first, that *the surface of any sphere is four times the greatest of the circles in it*[c]; then, that *the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment*[d]; and,

this demonstration, Archimedes finds the surface and volume of any segment of a sphere. The method in each case is to inscribe in the sphere or segment of a sphere, and to circumscribe about it, figures composed of cones and frusta of cones. The sphere or segment of a sphere is intermediate in surface and volume between the inscribed and circumscribed figures, and in the limit, when the number of sides in the inscribed and circumscribed figures is indefinitely increased, it would become identical with them. It will readily be appreciated that Archimedes' method is fundamentally the same as integration, and on p. 116 n. *b* this will be shown trigonometrically.

[b] This is proved in Props. 17 and 24 of the *Quadrature of the Parabola*, sent to Dositheus of Pelusium with a prefatory letter, v. pp. 228–243, *infra*.

[c] *De Sphaera et Cyl.* i. 30. "The greatest of the circles," here and elsewhere, is equivalent to "a great circle."

[d] *Ibid.* i. 42, 43.

πρὸς δὲ τούτοις, ὅτι πάσης σφαίρας ὁ κύλινδρος ὁ βάσιν μὲν ἔχων ἴσην τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ ἴσον τῇ διαμέτρῳ τῆς σφαίρας αὐτός τε ἡμιόλιός ἐστιν τῆς σφαίρας, καὶ ἡ ἐπιφάνεια αὐτοῦ τῆς ἐπιφανείας τῆς σφαίρας. ταῦτα δὲ τὰ συμπτώματα τῇ φύσει προυπῆρχεν περὶ τὰ εἰρημένα σχήματα, ἠγνοεῖτο δὲ ὑπὸ τῶν πρὸ ἡμῶν περὶ γεωμετρίαν ἀνεστραμμένων οὐδενὸς αὐτῶν ἐπινενοηκότος, ὅτι τούτων τῶν σχημάτων ἐστὶν συμμετρία. . . . ἐξέσται δὲ περὶ τούτων ἐπισκέψασθαι τοῖς δυνησομένοις. ὤφειλε μὲν οὖν Κόνωνος ἔτι ζῶντος ἐκδίδοσθαι ταῦτα· τῆνον γὰρ ὑπολαμβάνομέν που μάλιστα ἂν δύνασθαι κατανοῆσαι ταῦτα καὶ τὴν ἁρμόζουσαν ὑπὲρ αὐτῶν ἀπόφασιν ποιήσασθαι· δοκιμάζοντες δὲ καλῶς ἔχειν μεταδιδόναι τοῖς οἰκείοις τῶν μαθημάτων ἀποστέλλομέν σοι τὰς ἀποδείξεις ἀναγράψαντες, ὑπὲρ ὧν ἐξέσται τοῖς περὶ τὰ μαθήματα ἀναστρεφομένοις ἐπισκέψασθαι. ἐρρωμένως.

Γράφονται πρῶτον τά τε ἀξιώματα καὶ τὰ λαμβανόμενα εἰς τὰς ἀποδείξεις αὐτῶν.

Ἀξιώματα

α΄. Εἰσί τινες ἐν ἐπιπέδῳ καμπύλαι γραμμαὶ πεπερασμέναι, αἳ τῶν τὰ πέρατα ἐπιζευγνυουσῶν αὐτῶν εὐθειῶν ἤτοι ὅλαι ἐπὶ τὰ αὐτά εἰσιν ἢ οὐδὲν ἔχουσιν ἐπὶ τὰ ἕτερα.

β΄. Ἐπὶ τὰ αὐτὰ δὴ κοίλην καλῶ τὴν τοιαύτην γραμμήν, ἐν ᾗ ἐὰν δύο σημείων λαμβανομένων

[a] *De Sphaera et Cyl.* i. 34 coroll. The surface of the cylinder here includes the bases.

further, that, *in the case of any sphere, the cylinder having its base equal to the greatest of the circles in the sphere, and height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface is one-and-a-half times the surface of the sphere.*[a] Now these properties were inherent in the nature of the figures mentioned, but they were unknown to all who studied geometry before me, nor did any of them suspect such a relationship in these figures.[b] . . . But now it will be possible for those who have the capacity to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I opine that he more than others would have been able to grasp them and pronounce a fitting verdict upon them; but, holding it well to communicate them to students of mathematics, I send you the proofs that I have written out, which proofs will now be open to those who are conversant with mathematics. Farewell.

In the first place, the axioms and the assumptions used for the proofs of these theorems are here set out.

AXIOMS[c]

1. There are in a plane certain finite bent lines which either lie wholly on the same side of the straight lines joining their extremities or have no part on the other side.

2. I call *concave in the same direction* a line such that, if any two points whatsoever are taken on it, either

[b] In the omitted passage which follows, Archimedes compares his discoveries with those of Eudoxus; it has already been given, vol. i. pp. 408-411.

[c] These so-called axioms are more in the nature of definitions.

ὁποιωνοῦν αἱ μεταξὺ τῶν σημείων εὐθεῖαι ἤτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς γραμμῆς, ἢ τινὲς μὲν ἐπὶ τὰ αὐτά, τινὲς δὲ κατ᾿ αὐτῆς, ἐπὶ τὰ ἕτερα δὲ μηδεμία.

γ΄. Ὁμοίως δὴ καὶ ἐπιφάνειαί τινές εἰσιν πεπερασμέναι, αὐταὶ μὲν οὐκ ἐν ἐπιπέδῳ, τὰ δὲ πέρατα ἔχουσαι ἐν ἐπιπέδῳ, αἳ τοῦ ἐπιπέδου, ἐν ᾧ τὰ πέρατα ἔχουσιν, ἤτοι ὅλαι ἐπὶ τὰ αὐτὰ ἔσονται ἢ οὐδὲν ἔχουσιν ἐπὶ τὰ ἕτερα.

δ΄. Ἐπὶ τὰ αὐτὰ δὴ κοίλας καλῶ τὰς τοιαύτας ἐπιφανείας, ἐν αἷς ἂν δύο σημείων λαμβανομένων αἱ μεταξὺ τῶν σημείων εὐθεῖαι ἤτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς ἐπιφανείας, ἢ τινὲς μὲν ἐπὶ τὰ αὐτά, τινὲς δὲ κατ᾿ αὐτῆς, ἐπὶ τὰ ἕτερα δὲ μηδεμία.

ε΄. Τομέα δὲ στερεὸν καλῶ, ἐπειδὰν σφαῖραν κῶνος τέμνῃ κορυφὴν ἔχων πρὸς τῷ κέντρῳ τῆς σφαίρας, τὸ ἐμπεριεχόμενον σχῆμα ὑπό τε τῆς ἐπιφανείας τοῦ κώνου καὶ τῆς ἐπιφανείας τῆς σφαίρας ἐντὸς τοῦ κώνου.

ϛ΄. Ῥόμβον δὲ καλῶ στερεόν, ἐπειδὰν δύο κῶνοι τὴν αὐτὴν βάσιν ἔχοντες τὰς κορυφὰς ἔχωσιν ἐφ᾿ ἑκάτερα τοῦ ἐπιπέδου τῆς βάσεως, ὅπως οἱ ἄξονες αὐτῶν ἐπ᾿ εὐθείας ὦσι κείμενοι, τὸ ἐξ ἀμφοῖν τοῖν κώνοιν συγκείμενον στερεὸν σχῆμα.

Λαμβανόμενα

Λαμβάνω δὲ ταῦτα·

α΄. Τῶν τὰ αὐτὰ πέρατα ἐχουσῶν γραμμῶν ἐλαχίστην εἶναι τὴν εὐθεῖαν.

all the straight lines joining the points fall on the same side of the line, or some fall on one and the same side while others fall along the line itself, but none fall on the other side.

3. Similarly also there are certain finite surfaces, not in a plane themselves but having their extremities in a plane, and such that they will either lie wholly on the same side of the plane containing their extremities or will have no part on the other side.

4. I call *concave in the same direction* surfaces such that, if any two points on them are taken, either the straight lines between the points all fall upon the same side of the surface, or some fall on one and the same side while others fall along the surface itself, but none falls on the other side.

5. When a cone cuts a sphere, and has its vertex at the centre of the sphere, I call the figure comprehended by the surface of the cone and the surface of the sphere within the cone a *solid sector*.

6. When two cones having the same base have their vertices on opposite sides of the plane of the base in such a way that their axes lie in a straight line, I call the solid figure formed by the two cones a *solid rhombus*.

POSTULATES

I make these postulates :

1. Of all lines which have the same extremities the straight line is the least.[a]

[a] Proclus (*in Eucl.*, ed. Friedlein 110. 10-14) saw in this statement a connexion with Euclid's definition of a straight line as lying evenly with the points on itself : ὁ δ' αὖ Ἀρχιμήδης τὴν εὐθεῖαν ὡρίσατο γραμμὴν ἐλαχίστην τῶν τὰ αὐτὰ πέρατα ἐχουσῶν. διότι γάρ, ὡς ὁ Εὐκλείδειος λόγος φησίν, ἐξ ἴσου κεῖται τοῖς ἐφ' ἑαυτῆς σημείοις, διὰ τοῦτο ἐλαχίστη ἐστὶν τῶν τὰ αὐτὰ πέρατα ἐχουσῶν.

β'. Τῶν δὲ ἄλλων γραμμῶν, ἐὰν ἐν ἐπιπέδῳ οὖσαι τὰ αὐτὰ πέρατα ἔχωσιν, ἀνίσους εἶναι τὰς τοιαύτας, ἐπειδὰν ὦσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ κοῖλαι, καὶ ἤτοι ὅλη περιλαμβάνηται ἡ ἑτέρα αὐτῶν ὑπὸ τῆς ἑτέρας καὶ τῆς εὐθείας τῆς τὰ αὐτὰ πέρατα ἐχούσης αὐτῇ, ἢ τινὰ μὲν περιλαμβάνηται, τινὰ δὲ κοινὰ ἔχῃ, καὶ ἐλάσσονα εἶναι τὴν περιλαμβανομένην.

γ'. Ὁμοίως δὲ καὶ τῶν ἐπιφανειῶν τῶν τὰ αὐτὰ πέρατα ἐχουσῶν, ἐὰν ἐν ἐπιπέδῳ τὰ πέρατα ἔχωσιν, ἐλάσσονα εἶναι τὴν ἐπίπεδον.

δ'. Τῶν δὲ ἄλλων ἐπιφανειῶν καὶ τὰ αὐτὰ πέρατα ἐχουσῶν, ἐὰν ἐν ἐπιπέδῳ τὰ πέρατα ᾖ, ἀνίσους εἶναι τὰς τοιαύτας, ἐπειδὰν ὦσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ κοῖλαι, καὶ ἤτοι ὅλη περιλαμβάνηται ὑπὸ τῆς ἑτέρας ἡ ἑτέρα ἐπιφάνεια καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἐχούσης αὐτῇ, ἢ τινὰ μὲν περιλαμβάνηται, τινὰ δὲ κοινὰ ἔχῃ, καὶ ἐλάσσονα εἶναι τὴν περιλαμβανομένην.

ε'. Ἔτι δὲ τῶν ἀνίσων γραμμῶν καὶ τῶν ἀνίσων ἐπιφανειῶν καὶ τῶν ἀνίσων στερεῶν τὸ μεῖζον τοῦ ἐλάσσονος ὑπερέχειν τοιούτῳ, ὃ συντιθέμενον αὐτὸ ἑαυτῷ δυνατόν ἐστιν ὑπερέχειν παντὸς τοῦ προτεθέντος τῶν πρὸς ἄλληλα λεγομένων.

Τούτων δὲ ὑποκειμένων, ἐὰν εἰς κύκλον πολύγωνον ἐγγραφῇ, φανερόν, ὅτι ἡ περίμετρος τοῦ ἐγγραφέντος πολυγώνου ἐλάσσων ἐστὶν τῆς τοῦ κύκλου περιφερείας· ἑκάστη γὰρ τῶν τοῦ πολυγώνου πλευρῶν ἐλάσσων ἐστὶ τῆς τοῦ κύκλου περιφερείας τῆς ὑπὸ τῆς αὐτῆς ἀποτεμνομένης.

* This famous " Axiom of Archimedes " is, in fact, generally used by him in the alternative form in which it is proved

2. Of other lines lying in a plane and having the same extremities, [any two] such are unequal when both are concave in the same direction and one is either wholly included between the other and the straight line having the same extremities with it, or is partly included by and partly common with the other ; and the included line is the lesser.

3. Similarly, of surfaces which have the same extremities, if those extremities be in a plane, the plane is the least.

4. Of other surfaces having the same extremities, if the extremities be in a plane, [any two] such are unequal when both are concave in the same direction, and one surface is either wholly included between the other and the plane having the same extremities with it, or is partly included by and partly common with the other ; and the included surface is the lesser.

5. Further, of unequal lines and unequal surfaces and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those comparable with one another.[a]

With these premises, *if a polygon be inscribed in a circle, it is clear that the perimeter of the inscribed polygon is less than the circumference of the circle* ; for each of the sides of the polygon is less than the arc of the circle cut off by it.

in Euclid x. 1, for which *v.* vol. i. pp. 452-455. The axiom can be shown to be equivalent to Dedekind's principle, that a section of the rational points in which they are divided into two classes is made by a single point. Applied to straight lines, it is equivalent to saying that there is a complete correspondence between the aggregate of real numbers and the aggregate of points in a straight line; *v.* E. W. Hobson, *The Theory of Functions of a Real Variable*, 2nd ed., vol. i. p. 55.

α΄

Ἐὰν περὶ κύκλον πολύγωνον περιγραφῇ, ἡ τοῦ περιγραφέντος πολυγώνου περίμετρος μείζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Περὶ γὰρ κύκλον πολύγωνον περιγεγράφθω τὸ ὑποκείμενον. λέγω, ὅτι ἡ περίμετρος τοῦ πολυγώνου μείζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Ἐπεὶ γὰρ συναμφότερος ἡ ΒΑΛ μείζων ἐστὶ τῆς ΒΛ περιφερείας διὰ τὸ τὰ αὐτὰ πέρατα ἔχουσαν περιλαμβάνειν τὴν περιφέρειαν, ὁμοίως δὲ καὶ συναμφότερος μὲν ἡ ΔΓ, ΓΒ τῆς ΔΒ, συναμφότερος δὲ ἡ ΛΚ, ΚΘ τῆς ΛΘ, συναμφότερος δὲ ἡ ΖΗΘ τῆς ΖΘ, ἔτι δὲ συναμφότερος ἡ ΔΕ, ΕΖ τῆς ΔΖ, ὅλη ἄρα ἡ περίμετρος τοῦ πολυγώνου μείζων ἐστὶ τῆς περιφερείας τοῦ κύκλου.

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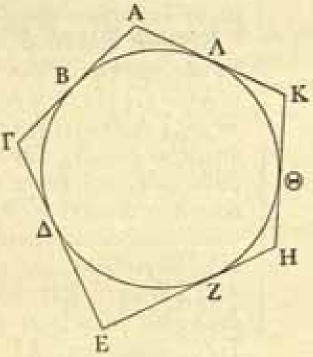
[a] It is here indicated, as in Prop. 3, that Archimedes added a figure to his own demonstration.

Prop. 1

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the circumference of the circle.

For let the polygon be circumscribed about the circle as below.[a] I say that the perimeter of the polygon is greater than the circumference of the circle.



For since $\quad \mathrm{BA} + \mathrm{A\Lambda} > \text{arc } \mathrm{B\Lambda},$

owing to the fact that they have the same extremities as the arc and include it, and similarly

$$\Delta\Gamma + \Gamma\mathrm{B} > [\text{arc}]\ \Delta\mathrm{B},$$
$$\Lambda\mathrm{K} + \mathrm{K}\Theta > [\text{arc}]\ \Lambda\Theta,$$
$$\mathrm{ZH} + \mathrm{H}\Theta > [\text{arc}]\ \mathrm{Z}\Theta,$$

and further $\quad \Delta\mathrm{E} + \mathrm{EZ} > [\text{arc}]\ \Delta\mathrm{Z},$

therefore the whole perimeter of the polygon is greater than the circumference of the circle.

β'

Δύο μεγεθῶν ἀνίσων δοθέντων δυνατόν ἐστιν εὑρεῖν δύο εὐθείας ἀνίσους, ὥστε τὴν μείζονα εὐθεῖαν πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἔλασσον.

Ἔστω δύο μεγέθη ἄνισα τὰ ΑΒ, Δ, καὶ ἔστω μεῖζον τὸ ΑΒ. λέγω, ὅτι δυνατόν ἐστι δύο εὐθείας ἀνίσους εὑρεῖν τὸ εἰρημένον ἐπίταγμα ποιούσας.

Κείσθω διὰ τὸ β' τοῦ α' τῶν Εὐκλείδου τῷ Δ ἴσον τὸ ΒΓ, καὶ κείσθω τις εὐθεῖα γραμμὴ ἡ ΖΗ· τὸ δὴ ΓΑ ἑαυτῷ ἐπισυντιθέμενον ὑπερέξει τοῦ Δ. πεπολλαπλασιάσθω οὖν, καὶ ἔστω τὸ ΑΘ, καὶ ὁσαπλάσιόν ἐστι τὸ ΑΘ τοῦ ΑΓ, τοσαυταπλάσιος ἔστω ἡ ΖΗ τῆς ΗΕ· ἔστιν ἄρα, ὡς τὸ ΘΑ πρὸς ΑΓ, οὕτως ἡ ΖΗ πρὸς ΗΕ· καὶ ἀνάπαλίν ἐστιν, ὡς ἡ ΕΗ πρὸς ΗΖ, οὕτως τὸ ΑΓ πρὸς ΑΘ. καὶ ἐπεὶ μεῖζόν ἐστιν τὸ ΑΘ τοῦ Δ, τουτέστι τοῦ ΓΒ, τὸ ἄρα ΓΑ πρὸς τὸ ΑΘ λόγον ἐλάσσονα ἔχει ἤπερ τὸ ΓΑ πρὸς ΓΒ. ἀλλ' ὡς τὸ ΓΑ πρὸς ΑΘ, οὕτως ἡ ΕΗ πρὸς ΗΖ· ἡ ΕΗ ἄρα πρὸς ΗΖ ἐλάσσονα λόγον ἔχει ἤπερ τὸ ΓΑ πρὸς ΓΒ· καὶ συνθέντι ἡ ΕΖ [ἄρα][1] πρὸς ΖΗ ἐλάσσονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς ΒΓ [διὰ λῆμμα].[2] ἴσον δὲ τὸ ΒΓ τῷ Δ· ἡ ΕΖ ἄρα πρὸς ΖΗ ἐλάσσονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς τὸ Δ.

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Prop. 2

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let AB, Δ be two unequal magnitudes, and let AB be the greater. I say that it is possible to find two unequal straight lines satisfying the aforesaid requirement.

By the second proposition in the first book of Euclid let BΓ be placed equal to Δ, and let ZH be any straight line; then ΓA, if added to itself, will exceed Δ. [Post. 5.] Let it be multiplied, therefore, and let the result be AΘ, and as AΘ is to AΓ, so let ZH be to HE; therefore

$$\Theta A : A\Gamma = ZH : HE \qquad [\textit{cf.}\ \text{Eucl. v. 15}$$

and conversely, $EH : HZ = A\Gamma : A\Theta$.

[Eucl. v. 7, coroll.

And since	$A\Theta > \Delta$	
	$> \Gamma B$,	
therefore	$\Gamma A : A\Theta < \Gamma A : \Gamma B$.	[Eucl. v. 8
But	$\Gamma A : A\Theta = EH : HZ$;	
therefore	$EH : HZ < \Gamma A : \Gamma B$;	
componendo,	$EZ : ZH < AB : B\Gamma$.[a]	
Now	$B\Gamma = \Delta$;	
therefore	$EZ : ZH < AB : \Delta$.	

[a] This and related propositions are proved by Eutocius [Archim. ed. Heiberg iii. 16. 11–18. 22] and by Pappus, *Coll.* ed. Hultsch 684. 20 ff. It may be simply proved thus. If $a : b < c : d$, it is required to prove that $a + b : b < c + d : d$. Let e be taken so that $a : b = e : d$. Then $e : d < c : d$. Therefore $e < c$, and $e + d : d < c + d : d$. But $e + d : d = a + b : b$ (*ex hypothesi, componendo*). Therefore $a + b : b < c + d : d$.

[1] ἄρα om. Heiberg. [2] διὰ λῆμμα om. Heiberg.

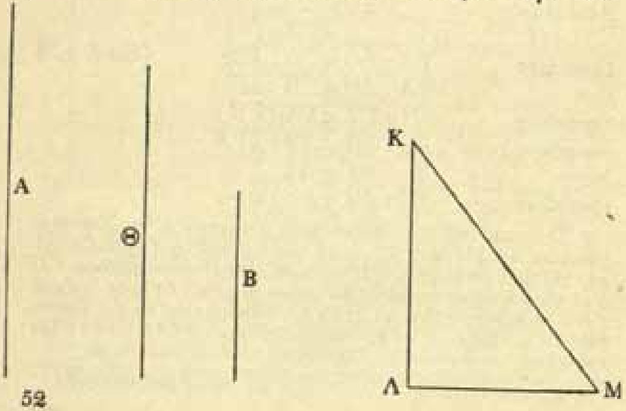
Εὑρημέναι εἰσὶν ἄρα δύο εὐθεῖαι ἄνισοι ποιοῦσαι τὸ εἰρημένον ἐπίταγμα [τουτέστιν τὴν μείζονα πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἔλασσον].[1]

γʹ

Δύο μεγεθῶν ἀνίσων δοθέντων καὶ κύκλου δυνατόν ἐστιν εἰς τὸν κύκλον πολύγωνον ἐγγράψαι καὶ ἄλλο περιγράψαι, ὅπως ἡ τοῦ περιγραφομένου πολυγώνου πλευρὰ πρὸς τὴν τοῦ ἐγγραφομένου πολυγώνου πλευρὰν ἐλάσσονα λόγον ἔχῃ ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἔλαττον.

Ἔστω τὰ δοθέντα δύο μεγέθη τὰ Α, Β, ὁ δὲ δοθεὶς κύκλος ὁ ὑποκείμενος. λέγω οὖν, ὅτι δυνατόν ἐστι ποιεῖν τὸ ἐπίταγμα.

Εὑρήσθωσαν γὰρ δύο εὐθεῖαι αἱ Θ, ΚΛ, ὧν μείζων ἔστω ἡ Θ, ὥστε τὴν Θ πρὸς τὴν ΚΛ

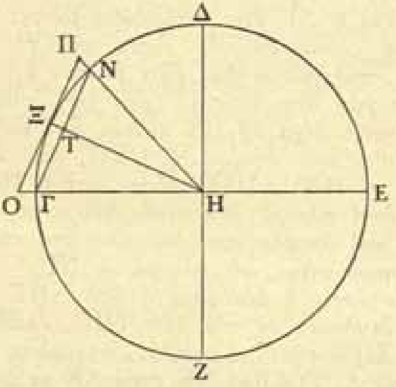


Accordingly there have been discovered two unequal straight lines fulfilling the aforesaid requirement.

Prop. 3

Given two unequal magnitudes and a circle, it is possible to inscribe a [regular] polygon in the circle and to circumscribe another, in such a manner that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which the greater magnitude has to the less.

Let Α, Β be the two given magnitudes, and let the



given circle be that set out below. I say then that it is possible to do what is required.

For let there be found two straight lines Θ, ΚΛ, of which Θ is the greater, such that Θ has to ΚΛ a ratio

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[1] τουτέστιν . . . ἔλασσον verba subditiva esse suspicatur Heiberg.

ἐλάσσονα λόγον ἔχειν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἔλαττον, καὶ ἤχθω ἀπὸ τοῦ Λ *τῇ* ΛΚ *πρὸς ὀρθὰς ἡ* ΛΜ, *καὶ ἀπὸ τοῦ* Κ *τῇ* Θ *ἴση κατήχθω ἡ* ΚΜ [*δυνατὸν γὰρ τοῦτο*],[1] *καὶ ἤχθωσαν τοῦ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ* ΓΕ, ΔΖ. *τέμνοντες οὖν τὴν ὑπὸ τῶν* ΔΗΓ *γωνίαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ αἰεὶ τοῦτο ποιοῦντες λείψομέν τινα γωνίαν ἐλάσσονα ἢ διπλασίαν τῆς ὑπὸ* ΛΚΜ. *λελείφθω καὶ ἔστω ἡ ὑπὸ* ΝΗΓ, *καὶ ἐπεζεύχθω ἡ* ΝΓ· *ἡ ἄρα* ΝΓ *πολυγώνου ἐστὶ πλευρὰ ἰσοπλεύρου* [*ἐπείπερ ἡ ὑπὸ* ΝΗΓ *γωνία μετρεῖ τὴν ὑπὸ* ΔΗΓ *ὀρθὴν οὖσαν, καὶ ἡ* ΝΓ *ἄρα περιφέρεια μετρεῖ τὴν* ΓΔ *τέταρτον οὖσαν κύκλου· ὥστε καὶ τὸν κύκλον μετρεῖ. πολυγώνου ἄρα ἐστὶ πλευρὰ ἰσοπλεύρου· φανερὸν γάρ ἐστι τοῦτο*].[2] *καὶ τετμήσθω ἡ ὑπὸ* ΓΗΝ *γωνία δίχα τῇ* ΗΞ *εὐθείᾳ, καὶ ἀπὸ τοῦ* Ξ *ἐφαπτέσθω τοῦ κύκλου ἡ* ΟΞΠ, *καὶ ἐκβεβλήσθωσαν αἱ* ΗΝΠ, ΗΓΟ· *ὥστε καὶ ἡ* ΠΟ *πολυγώνου ἐστὶ πλευρὰ τοῦ περιγραφομένου περὶ τὸν κύκλον καὶ ἰσοπλεύρου* [*φανερόν, ὅτι καὶ ὁμοίου τῷ ἐγγραφομένῳ, οὗ πλευρὰ ἡ* ΝΓ].[3] *ἐπεὶ δὲ ἐλάσσων ἐστὶν ἢ διπλασία ἡ ὑπὸ* ΝΗΓ *τῆς ὑπὸ* ΛΚΜ, *διπλασία δὲ τῆς ὑπὸ* ΤΗΓ, *ἐλάσσων ἄρα ἡ ὑπὸ* ΤΗΓ *τῆς ὑπὸ* ΛΚΜ. *καί εἰσιν ὀρθαὶ αἱ πρὸς τοῖς* Λ, Τ· *ἡ ἄρα* ΜΚ *πρὸς* ΛΚ *μείζονα λόγον ἔχει ἤπερ ἡ* ΓΗ *πρὸς* ΗΤ. *ἴση δὲ ἡ* ΓΗ *τῇ* ΗΞ· *ὥστε ἡ* ΗΞ *πρὸς* ΗΤ *ἐλάσσονα λόγον ἔχει, τουτέστιν ἡ* ΠΟ *πρὸς* ΝΓ, *ἤπερ ἡ* ΜΚ *πρὸς* ΚΛ· *ἔτι δὲ ἡ* ΜΚ *πρὸς* ΚΛ *ἐλάσσονα λόγον ἔχει ἤπερ τὸ* Α *πρὸς τὸ* Β. *καί ἐστιν ἡ μὲν* ΠΟ *πλευρὰ*

less than that which the greater magnitude has to the less [Prop. 2], and from Λ let ΛΜ be drawn at right angles to ΛΚ, and from Κ let ΚΜ be drawn equal to Θ, and let there be drawn two diameters of the circle, ΓΕ, ΔΖ, at right angles one to another. If we bisect the angle ΔΗΓ and then bisect the half and so on continually we shall leave a certain angle less than double the angle ΛΚΜ. Let it be left and let it be the angle ΝΗΓ, and let ΝΓ be joined; then ΝΓ is the side of an equilateral polygon. Let the angle ΓΗΝ be bisected by the straight line ΗΞ, and through Ξ let the tangent ΟΞΠ be drawn, and let ΗΝΠ, ΗΓΟ be produced; then ΠΟ is a side of an equilateral polygon circumscribed about the circle. Since the angle ΝΗΓ is less than double the angle ΛΚΜ and is double the angle ΤΗΓ, therefore the angle ΤΗΓ is less than the angle ΛΚΜ. And the angles at Λ, Τ are right; therefore

$$\mathrm{MK} : \mathrm{\Lambda K} > \Gamma\mathrm{H} : \mathrm{HT}.^{a}$$

But $\Gamma\mathrm{H} = \mathrm{H}\Xi.$

Therefore $\mathrm{H}\Xi : \mathrm{HT} < \mathrm{MK} : \mathrm{K}\Lambda,$

that is, $\Pi\mathrm{O} : \mathrm{N}\Gamma < \mathrm{MK} : \mathrm{K}\Lambda.^{b}$

Further, $\mathrm{MK} : \mathrm{K}\Lambda < \mathrm{A} : \mathrm{B}.^{c}$

[Therefore $\Pi\mathrm{O} : \mathrm{N}\Gamma < \mathrm{A} : \mathrm{B}.$]

[a] This is proved by Eutocius and is equivalent to the assertion that if $\alpha < \beta \leq \frac{\pi}{2}$, cosec $\beta >$ cosec α.

[b] For $\mathrm{H}\Xi : \mathrm{HT} = \Pi\mathrm{O} : \mathrm{N}\Gamma$, since $\mathrm{H}\Xi : \mathrm{HT} = \mathrm{O}\Xi : \Gamma\mathrm{T} = 2\mathrm{O}\Xi : 2\Gamma\mathrm{T} = \Pi\mathrm{O} : \Gamma\mathrm{N}$.

[c] For by hypothesis $\Theta : \mathrm{K}\Lambda < \mathrm{A} : \mathrm{B}$, and $\Theta = \mathrm{MK}$.

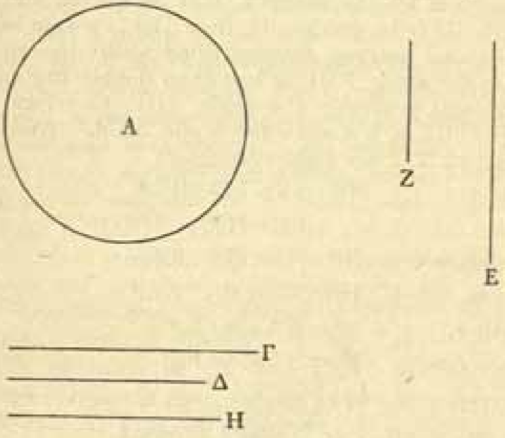
[1] δυνατὸν . . . τοῦτο om. Heiberg.
[2] ἐπείπερ . . . τοῦτο om. Heiberg.
[3] φανερὸν . . . ἡ ΝΓ om. Heiberg.

τοῦ περιγραφομένου πολυγώνου, ἡ δὲ ΓΝ τοῦ ἐγγραφομένου· ὅπερ προέκειτο εὑρεῖν.

ε´

Κύκλου δοθέντος καὶ δύο μεγεθῶν ἀνίσων περιγράψαι περὶ τὸν κύκλον πολύγωνον καὶ ἄλλο ἐγγράψαι, ὥστε τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχειν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἔλασσον.

Ἐκκείσθω κύκλος ὁ Α καὶ δύο μεγέθη ἄνισα



τὰ Ε, Ζ καὶ μεῖζον τὸ Ε· δεῖ οὖν πολύγωνον ἐγγράψαι εἰς τὸν κύκλον καὶ ἄλλο περιγράψαι, ἵνα γένηται τὸ ἐπιταχθέν.

Λαμβάνω γὰρ δύο εὐθείας ἀνίσους τὰς Γ, Δ, ὧν μείζων ἔστω ἡ Γ, ὥστε τὴν Γ πρὸς τὴν Δ

And ΠΟ is a side of the circumscribed polygon, ΓΝ of the inscribed; which was to be found.

Prop. 5

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another, so that the circumscribed polygon has to the inscribed polygon a ratio less than the greater magnitude has to the less.

Let there be set out the circle A and the two unequal magnitudes E, Z, and let E be the greater; it is therefore required to inscribe a polygon in the circle and to circumscribe another, so that what is required may be done.

For I take two unequal straight lines Γ, Δ, of which let Γ be the greater, so that Γ has to Δ a ratio

ἐλάσσονα λόγον ἔχειν ἢ τὴν Ε πρὸς τὴν Ζ· καὶ τῶν Γ, Δ μέσης ἀνάλογον ληφθείσης τῆς Η μείζων ἄρα καὶ ἡ Γ τῆς Η. περιγεγράφθω δὴ περὶ κύκλον πολύγωνον καὶ ἄλλο ἐγγεγράφθω, ὥστε τὴν τοῦ περιγραφέντος πολυγώνου πλευρὰν πρὸς τὴν τοῦ ἐγγραφέντος ἐλάσσονα λόγον ἔχειν ἢ τὴν Γ πρὸς τὴν Η [καθὼς ἐμάθομεν][1]· διὰ τοῦτο δὴ καὶ ὁ διπλάσιος λόγος τοῦ διπλασίου ἐλάσσων ἐστί. καὶ τοῦ μὲν τῆς πλευρᾶς πρὸς τὴν πλευρὰν διπλάσιός ἐστι ὁ τοῦ πολυγώνου πρὸς τὸν πολύγωνον [ὅμοια γάρ],[2] τῆς δὲ Γ πρὸς τὴν Η ὁ τῆς Γ πρὸς τὴν Δ· καὶ τὸ περιγραφὲν ἄρα πολύγωνον πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχει ἤπερ ἡ Γ πρὸς τὴν Δ· πολλῷ ἄρα τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Ζ.

η′

Ἐὰν περὶ κῶνον ἰσοσκελῆ πυραμὶς περιγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως ἴση ἐστὶν τριγώνῳ βάσιν μὲν ἔχοντι τὴν ἴσην τῇ περιμέτρῳ τῆς βάσεως, ὕψος δὲ τὴν πλευρὰν τοῦ κώνου. . . .

θ′

Ἐὰν κώνου τινὸς ἰσοσκελοῦς εἰς τὸν κύκλον, ὅς ἐστι βάσις τοῦ κώνου, εὐθεῖα γραμμὴ ἐμπέσῃ, ἀπὸ δὲ τῶν περάτων αὐτῆς εὐθεῖαι γραμμαὶ ἀχθῶσιν ἐπὶ τὴν κορυφὴν τοῦ κώνου, τὸ περιληφθὲν τρίγωνον ὑπό τε τῆς ἐμπεσούσης καὶ τῶν ἐπιζευχθεισῶν ἐπὶ τὴν κορυφὴν ἔλασσον ἔσται τῆς

less than that which E has to Z [Prop. 2]; if a mean proportional H be taken between Γ, Δ, then Γ will be greater than H [Eucl. vi. 13]. Let a polygon be circumscribed about the circle and another inscribed, so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which Γ has to H [Prop. 3]; it follows that the duplicate ratio is less than the duplicate ratio. Now the duplicate ratio of the sides is the ratio of the polygons [Eucl. vi. 20], and the duplicate ratio of Γ to H is the ratio of Γ to Δ [Eucl. v. Def. 9]; therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which Γ has to Δ; by much more therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which E has to Z.

Prop. 8

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid without the base is equal to a triangle having its base equal to the perimeter of the base [of the pyramid] and its height equal to the side of the cone. . . .[a]

Prop. 9

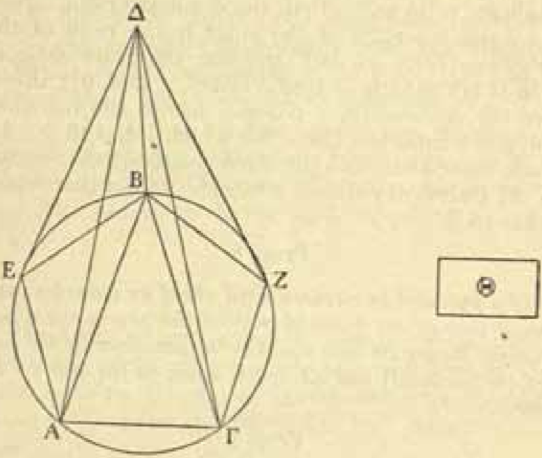
If in an isosceles cone a straight line [chord] fall in the circle which is the base of the cone, and from its extremities straight lines be drawn to the vertex of the cone, the triangle formed by the chord and the lines joining it to

[a] The "side of the cone" is a generator. The proof is obvious.

[1] καθὼς ἐμάθομεν om. Heiberg.
[2] ὅμοια γάρ om. Heiberg.

ἐπιφανείας τοῦ κώνου τῆς μεταξὺ τῶν ἐπὶ τὴν κορυφὴν ἐπιζευχθεισῶν.

Ἔστω κώνου ἰσοσκελοῦς βάσις ὁ ΑΒΓ κύκλος, κορυφὴ δὲ τὸ Δ, καὶ διήχθω τις εἰς αὐτὸν εὐθεῖα ἡ ΑΓ, καὶ ἀπὸ τῆς κορυφῆς ἐπὶ τὰ Α, Γ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ· λέγω, ὅτι τὸ ΑΔΓ τρίγωνον



ἔλασσόν ἐστιν τῆς ἐπιφανείας τῆς κωνικῆς τῆς μεταξὺ τῶν ΑΔΓ.

Τετμήσθω ἡ ΑΒΓ περιφέρεια δίχα κατὰ τὸ Β, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΓΒ, ΔΒ· ἔσται δὴ τὰ ΑΒΔ, ΒΓΔ τρίγωνα μείζονα τοῦ ΑΔΓ τριγώνου.[1] ᾧ δὴ ὑπερέχει τὰ εἰρημένα τρίγωνα τοῦ ΑΔΓ τριγώνου, ἔστω τὸ Θ. τὸ δὴ Θ ἤτοι τῶν ΑΒ, ΒΓ τμημάτων ἔλασσόν ἐστιν ἢ οὔ.

the vertex will be less than the surface of the cone between the lines drawn to the vertex.

Let the circle ΑΒΓ be the base of an isosceles cone, let Δ be its vertex, let the straight line ΑΓ be drawn in it, and let ΑΔ, ΔΓ be drawn from the vertex to Α, Γ; I say that the triangle ΑΔΓ is less than the surface of the cone between ΑΔ, ΔΓ.

Let the arc ΑΒΓ be bisected at Β, and let ΑΒ, ΓΒ, ΔΒ be joined; then the triangles ΑΒΔ, ΒΓΔ will be greater than the triangle ΑΔΓ.[a] Let Θ be the excess by which the aforesaid triangles exceed the triangle ΑΔΓ. Now Θ is either less than the sum of the segments ΑΒ, ΒΓ or not less.

[a] For if h be the length of a generator of the isosceles cone, triangle $\text{ΑΒΔ} = \frac{1}{2}h \,.\, \text{ΑΒ}$, triangle $\text{ΒΓΔ} = \frac{1}{2}h \,.\, \text{ΒΓ}$, triangle $\text{ΑΔΓ} = \frac{1}{2}h \,.\, \text{ΑΓ}$, and $\text{ΑΒ} + \text{ΒΓ} > \text{ΑΓ}$.

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[1] ἔσται . . . τριγώνου: ex Eutocio videtur Archimedem scripsisse: μείζονα ἄρα ἐστὶ τὰ ΑΒΔ, ΒΔΓ τρίγωνα τοῦ ΑΔΓ τριγώνου.

Ἔστω μὴ ἔλασσον πρότερον. ἐπεὶ οὖν δύο εἰσὶν ἐπιφάνειαι ἥ τε κωνικὴ ἡ μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμήματος καὶ ἡ τοῦ ΑΔΒ τριγώνου τὸ αὐτὸ πέρας ἔχουσαι τὴν περίμετρον τοῦ τριγώνου τοῦ ΑΔΒ, μείζων ἔσται ἡ περιλαμβάνουσα τῆς περιλαμβανομένης· μείζων ἄρα ἐστὶν ἡ κωνικὴ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμήματος τοῦ ΑΒΔ τριγώνου. ὁμοίως δὲ καὶ ἡ μεταξὺ τῶν ΒΔΓ μετὰ τοῦ ΓΖΒ τμήματος μείζων ἐστὶν τοῦ ΒΔΓ τριγώνου· ὅλη ἄρα ἡ κωνικὴ ἐπιφάνεια μετὰ τοῦ Θ χωρίου μείζων ἐστὶ τῶν εἰρημένων τριγώνων. τὰ δὲ εἰρημένα τρίγωνα ἴσα ἐστὶν τῷ τε ΑΔΓ τριγώνῳ καὶ τῷ Θ χωρίῳ. κοινὸν ἀφῃρήσθω τὸ Θ χωρίον· λοιπὴ ἄρα ἡ κωνικὴ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μείζων ἐστὶν τοῦ ΑΔΓ τριγώνου.

Ἔστω δὴ τὸ Θ ἔλασσον τῶν ΑΒ, ΒΓ τμημάτων. τέμνοντες δὴ τὰς ΑΒ, ΒΓ περιφερείας δίχα καὶ τὰς ἡμισείας αὐτῶν δίχα λείψομεν τμήματα ἐλάσσονα ὄντα τοῦ Θ χωρίου. λελείφθω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ εὐθειῶν, καὶ ἐπεζεύχθωσαν αἱ ΔΕ, ΔΖ. πάλιν τοίνυν κατὰ τὰ αὐτὰ ἡ μὲν ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΔΕ μετὰ τοῦ ἐπὶ τῆς ΑΕ τμήματος μείζων ἐστὶν τοῦ ΑΔΕ τριγώνου, ἡ δὲ μεταξὺ τῶν ΕΔΒ μετὰ τοῦ ἐπὶ τῆς ΕΒ τμήματος μείζων ἐστὶν τοῦ ΕΔΒ τριγώνου· ἡ ἄρα ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἐπὶ τῶν ΑΕ, ΕΒ τμημάτων μείζων ἐστὶν τῶν ΑΔΕ, ΕΒΔ τριγώνων. ἐπεὶ δὲ τὰ ΑΕΔ, ΔΕΒ τρίγωνα μείζονά ἐστιν τοῦ ΑΒΔ τριγώνου, καθὼς δέδεικται, πολλῷ ἄρα ἡ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἐπὶ τῶν ΑΕ,

Firstly, let it be not less. Then since there are two surfaces, the surface of the cone between ΑΔ, ΔΒ together with the segment ΑΕΒ and the triangle ΑΔΒ, having the same extremity, that is, the perimeter of the triangle ΑΔΒ, the surface which includes the other is greater than the included surface [Post. 3]; therefore the surface of the cone between the straight lines ΑΔ, ΔΒ together with the segment ΑΕΒ is greater than the triangle ΑΒΔ. Similarly the [surface of the cone] between ΒΔ, ΔΓ together with the segment ΓΖΒ is greater than the triangle ΒΔΓ; therefore the whole surface of the cone together with the area Θ is greater than the aforesaid triangles. Now the aforesaid triangles are equal to the triangle ΑΔΓ and the area Θ. Let the common area Θ be taken away; therefore the remainder, the surface of the cone between ΑΔ, ΔΓ is greater than the triangle ΑΔΓ.

Now let Θ be less than the segments ΑΒ, ΒΓ. Bisecting the arcs ΑΒ, ΒΓ and then bisecting their halves, we shall leave segments less than the area Θ [Eucl. xii. 2]. Let the segments so left be those on the straight lines ΑΕ, ΕΒ, ΒΖ, ΖΓ, and let ΔΕ, ΔΖ be joined. Then once more by the same reasoning the surface of the cone between ΑΔ, ΔΕ together with the segment ΑΕ is greater than the triangle ΑΔΕ, while that between ΕΔ, ΔΒ together with the segment ΕΒ is greater than the triangle ΕΔΒ; therefore the surface between ΑΔ, ΔΒ together with the segments ΑΕ, ΕΒ is greater than the triangles ΑΔΕ, ΕΒΔ. Now since the triangles ΑΕΔ, ΔΕΒ are greater than the triangle ΑΒΔ, as was proved, by much more therefore the surface of the cone between ΑΔ, ΔΒ together with the segments ΑΕ, ΕΒ is

ΕΒ τμημάτων μείζων ἐστὶ τοῦ ΑΔΒ τριγώνου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΒΔΓ μετὰ τῶν ἐπὶ τῶν ΒΖ, ΖΓ τμημάτων μείζων ἐστὶν τοῦ ΒΔΓ τριγώνου· ὅλη ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μετὰ τῶν εἰρημένων τμημάτων μείζων ἐστὶ τῶν ΑΒΔ, ΔΒΓ τριγώνων. ταῦτα δέ ἐστιν ἴσα τῷ ΑΔΓ τριγώνῳ καὶ τῷ Θ χωρίῳ· ὧν τὰ εἰρημένα τμήματα ἐλάσσονα τοῦ Θ χωρίου· λοιπὴ ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μείζων ἐστὶν τοῦ ΑΔΓ τριγώνου.

ι′

Ἐὰν ἐπιψαύουσαι ἀχθῶσιν τοῦ κύκλου, ὅς ἐστι βάσις τοῦ κώνου, ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι τῷ κύκλῳ καὶ συμπίπτουσαι ἀλλήλαις, ἀπὸ δὲ τῶν ἁφῶν καὶ τῆς συμπτώσεως ἐπὶ τὴν κορυφὴν τοῦ κώνου εὐθεῖαι ἀχθῶσιν, τὰ περιεχόμενα τρίγωνα ὑπὸ τῶν ἐπιψαυουσῶν καὶ τῶν ἐπὶ τὴν κορυφὴν τοῦ κώνου ἐπιζευχθεισῶν εὐθειῶν μείζονά ἐστιν τῆς τοῦ κώνου ἐπιφανείας τῆς ἀπολαμβανομένης ὑπ' αὐτῶν. . . .

ιβ′

. . . Τούτων δὴ δεδειγμένων φανερὸν [ἐπὶ μὲν τῶν προειρημένων],[1] ὅτι, ἐὰν εἰς κῶνον ἰσοσκελῆ πυραμὶς ἐγγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως ἐλάσσων ἐστὶ τῆς κωνικῆς ἐπιφανείας [ἕκαστον γὰρ τῶν περιεχόντων τὴν πυραμίδα τριγώνων ἔλασσόν ἐστιν τῆς κωνικῆς ἐπιφανείας τῆς μεταξὺ τῶν τοῦ τριγώνου πλευρῶν· ὥστε καὶ ὅλη ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς

greater than the triangle ΑΔΒ. By the same reasoning the surface between ΒΔ, ΔΓ together with the segments ΒΖ, ΖΓ is greater than the triangle ΒΔΓ; therefore the whole surface between ΑΔ, ΔΓ together with the aforesaid segments is greater than the triangles ΑΒΔ, ΔΒΓ. Now these are equal to the triangle ΑΔΓ and the area Θ; and the aforesaid segments are less than the area Θ; therefore the remainder, the surface between ΑΔ, ΔΓ is greater than the triangle ΑΔΓ.

Prop. 10

If tangents be drawn to the circle which is the base of an [isosceles] cone, being in the same plane as the circle and meeting one another, and from the points of contact and the point of meeting straight lines be drawn to the vertex of the cone, the triangles formed by the tangents and the lines drawn to the vertex of the cone are together greater than the portion of the surface of the cone included by them. . . .[a]

Prop. 12

. . . From what has been proved it is clear that, if a pyramid is inscribed in an isosceles cone, the surface of the pyramid without the base is less than the surface of the cone [Prop. 9], and that, if a pyramid

[a] The proof is on lines similar to the preceding proposition.

[1] ἐπὶ . . . προειρημένων om. Heiberg.

βάσεως ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου χωρὶς τῆς βάσεως],[1] καὶ ὅτι, ἐὰν περὶ κῶνον ἰσοσκελῆ πυραμὶς περιγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως μείζων ἐστὶν τῆς ἐπιφανείας τοῦ κώνου χωρὶς τῆς βάσεως [κατὰ τὸ συνεχὲς ἐκείνῳ].[2]

Φανερὸν δὲ ἐκ τῶν ἀποδεδειγμένων, ὅτι τε, ἐὰν εἰς κύλινδρον ὀρθὸν πρίσμα ἐγγραφῇ, ἡ ἐπιφάνεια τοῦ πρίσματος ἡ ἐκ τῶν παραλληλογράμμων συγκειμένη ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρὶς τῆς βάσεως [ἔλασσον γὰρ ἕκαστον παραλληλόγραμμον τοῦ πρίσματός ἐστι τῆς καθ' αὑτὸ τοῦ κυλίνδρου ἐπιφανείας],[3] καὶ ὅτι, ἐὰν περὶ κύλινδρον ὀρθὸν πρίσμα περιγραφῇ, ἡ ἐπιφάνεια τοῦ πρίσματος ἡ ἐκ τῶν παραλληλογράμμων συγκειμένη μείζων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρὶς τῆς βάσεως.

ιγ'

Παντὸς κυλίνδρου ὀρθοῦ ἡ ἐπιφάνεια χωρὶς τῆς βάσεως ἴση ἐστὶ κύκλῳ, οὗ ἡ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς διαμέτρου τῆς βάσεως τοῦ κυλίνδρου.

Ἔστω κυλίνδρου τινὸς ὀρθοῦ βάσις ὁ Α κύκλος, καὶ ἔστω τῇ μὲν διαμέτρῳ τοῦ Α κύκλου ἴση ἡ ΓΔ, τῇ δὲ πλευρᾷ τοῦ κυλίνδρου ἡ ΕΖ, ἐχέτω δὲ μέσον λόγον τῶν ΔΓ, ΕΖ ἡ Η, καὶ κείσθω κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἐστὶ τῇ Η, ὁ Β· δεικτέον, ὅτι ὁ Β κύκλος ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κυλίνδρου χωρὶς τῆς βάσεως.

Εἰ γὰρ μή ἐστιν ἴσος, ἤτοι μείζων ἐστὶ ἢ

is circumscribed about an isosceles cone, the surface of the pyramid without the base is greater than the surface of the cone without the base [Prop. 10].

From what has been demonstrated it is also clear that, if a right prism be inscribed in a cylinder, the surface of the prism composed of the parallelograms is less than the surface of the cylinder excluding the bases [a] [Prop. 11], and if a right prism be circumscribed about a cylinder, the surface of the prism composed of the parallelograms is greater than the surface of the cylinder excluding the bases.

Prop. 13

The surface of any right cylinder excluding the bases [b] *is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base of the cylinder.*

Let the circle A be the base of a right cylinder, let ΓΔ be equal to the diameter of the circle A, let EZ be equal to the side of the cylinder, let H be a mean proportional between ΔΓ, EZ, and let there be set out a circle, B, whose radius is equal to H; it is required to prove that the circle B is equal to the surface of the cylinder excluding the bases.[b]

For if it is not equal, it is either greater or less.

[a] Here, and in other places in this and the next proposition, Archimedes must have written *χωρὶς τῶν βάσεων*, not *χωρὶς τῆς βάσεως*.

[b] See preceding note.

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[1] *ἕκαστον . . . βάσεως*. Heiberg suspects that this demonstration is interpolated. Why give a proof of what is *φανερόν*?

[2] *κατὰ . . . ἐκείνῳ* om. Heiberg.

[3] *ἔλασσον . . . ἐπιφανείας*. Heiberg suspects that this proof is interpolated.

ἐλάσσων. ἔστω πρότερον, εἰ δυνατόν, ἐλάσσων. δύο δὴ μεγεθῶν ὄντων ἀνίσων τῆς τε ἐπιφανείας τοῦ κυλίνδρου καὶ τοῦ Β κύκλου δυνατόν ἐστιν εἰς τὸν Β κύκλον ἰσόπλευρον πολύγωνον ἐγγράψαι καὶ ἄλλο περιγράψαι, ὥστε τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν Β κύκλον. νοείσθω δὴ περιγεγραμμένον καὶ ἐγγεγραμμένον, καὶ περὶ τὸν Α κύκλον περιγεγράφθω εὐθύγραμμον ὅμοιον τῷ περὶ τὸν Β περιγεγραμμένῳ, καὶ ἀναγεγράφθω ἀπὸ τοῦ εὐθυγράμμου πρίσμα· ἔσται δὴ περὶ τὸν κύλινδρον περιγεγραμμένον. ἔστω δὲ καὶ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ περὶ

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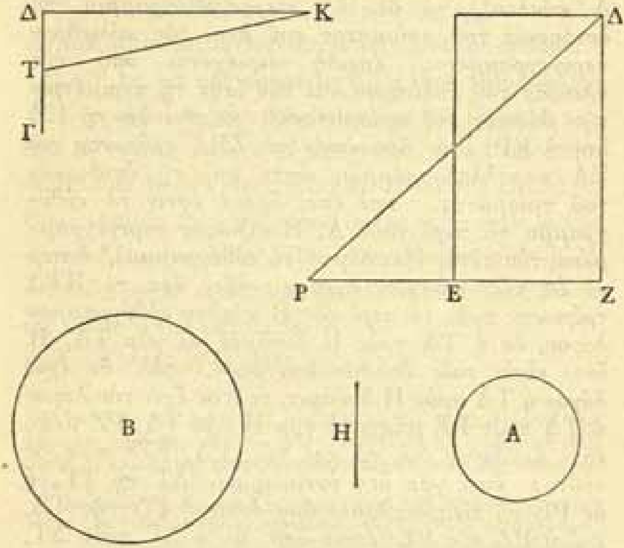
[a] One MS. has the marginal note, "equalis altitudinis chylindro," on which Heiberg comments: "nec hoc omiserat Archimedes." Heiberg notes several places in which the text is clearly not that written by Archimedes.

Let it first be, if possible, less. Now there are two unequal magnitudes, the surface of the cylinder and the circle B, and it is possible to inscribe in the circle B an equilateral polygon, and to circumscribe another, so that the circumscribed has to the inscribed a ratio



less than that which the surface of the cylinder has to the circle B [Prop. 5]. Let the circumscribed and inscribed polygons be imagined, and about the circle A let there be circumscribed a rectilineal figure similar to that circumscribed about B, and on the rectilineal figure let a prism be erected [a]; it will be circumscribed about the cylinder. Let KΔ be equal

τὸν Α κύκλον ἴση ἡ ΚΔ καὶ τῇ ΚΔ ἴση ἡ ΛΖ, τῆς δὲ ΓΔ ἡμίσεια ἔστω ἡ ΓΤ· ἔσται δὴ τὸ ΚΔΤ τρίγωνον ἴσον τῷ περιγεγραμμένῳ εὐθυγράμμῳ περὶ τὸν Α κύκλον [ἐπειδὴ βάσιν μὲν ἔχει τῇ περιμέτρῳ ἴσην, ὕψος δὲ ἴσον τῇ ἐκ τοῦ κέντρου τοῦ Α κύκλου],[1] τὸ δὲ ΕΛ παραλληλόγραμμον τῇ ἐπιφανείᾳ τοῦ πρίσματος τοῦ περὶ τὸν κύλινδρον περιγεγραμμένου [ἐπειδὴ περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἴσης τῇ περιμέτρῳ τῆς βάσεως τοῦ πρίσματος].[2] κείσθω δὴ τῇ ΕΖ ἴση ἡ ΕΡ· ἴσον ἄρα ἐστὶν τὸ ΖΡΛ τρίγωνον τῷ ΕΛ παραλληλογράμμῳ, ὥστε καὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὅμοιά ἐστιν τὰ εὐθύγραμμα τὰ περὶ τοὺς Α, Β κύκλους περιγεγραμμένα, τὸν αὐτὸν ἕξει λόγον [τὰ εὐθύγραμμα],[3] ὅνπερ αἱ ἐκ τῶν κέντρων δυνάμει· ἕξει ἄρα τὸ ΚΤΔ τρίγωνον πρὸς τὸ περὶ τὸν Β κύκλον εὐθύγραμμον λόγον, ὃν ἡ ΤΔ πρὸς Η δυνάμει [αἱ γὰρ ΤΔ, Η ἴσαι εἰσὶν ταῖς ἐκ τῶν κέντρων]. ἀλλ' ὃν ἔχει λόγον ἡ ΤΔ πρὸς Η δυνάμει, τοῦτον ἔχει τὸν λόγον ἡ ΤΔ πρὸς ΡΖ μήκει [ἡ γὰρ Η τῶν ΤΔ, ΡΖ μέση ἐστὶ ἀνάλογον διὰ τὸ καὶ τῶν ΓΔ, ΕΖ· πῶς δὲ τοῦτο; ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΔΤ τῇ ΤΓ, ἡ δὲ ΡΕ τῇ ΕΖ, διπλασία ἄρα ἐστὶν ἡ ΓΔ τῆς ΤΔ, καὶ ἡ ΡΖ τῆς ΡΕ· ἔστιν ἄρα, ὡς ἡ ΔΓ πρὸς ΔΤ, οὕτως ἡ ΡΖ πρὸς ΖΕ. τὸ ἄρα ὑπὸ τῶν ΓΔ, ΕΖ ἴσον ἐστὶν τῷ ὑπὸ τῶν ΤΔ, ΡΖ. τῷ δὲ ὑπὸ τῶν ΓΔ, ΕΖ ἴσον ἐστὶν τὸ ἀπὸ Η· καὶ τῷ ὑπὸ τῶν ΤΔ, ΡΖ ἄρα ἴσον ἐστὶ τὸ ἀπὸ τῆς Η. ἔστιν ἄρα,

[1] ἐπειδὴ . . . κύκλου om. Heiberg.
[2] ἐπειδὴ . . . πρίσματος om. Heiberg.
[3] τὰ εὐθύγραμμα om. Torellius.

to the perimeter of the rectilineal figure about the circle A, let ΛZ be equal to KΔ, and let ΓT be half of ΓΔ; then the triangle KΔT will be equal to the rectilineal figure circumscribed about the circle A,[a] while the parallelogram EΛ will be equal to the surface of the prism circumscribed about the cylinder.[b] Let EP be set out equal to EZ; then the triangle ZPΛ is equal to the parallelogram EΛ [Eucl. i. 41], and so to the surface of the prism. And since the rectilineal figures circumscribed about the circles A, B are similar, they will stand in the same ratio as the squares on the radii[c]; therefore the triangle KTΔ will have to the rectilineal figure circumscribed about the circle B the ratio $T\Delta^2 : H^2$.

But $$T\Delta^2 : H^2 = T\Delta : PZ.$$[d]

[a] Because the base KΔ is equal to the perimeter of the polygon, and the altitude ΔT is equal to the radius of the circle A, *i.e.*, to the perpendiculars drawn from the centre of A to the sides of the polygon.

[b] Because the base ΛZ is made equal to ΔK and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude EZ is equal to the side of the cylinder and therefore to the height of the prism.

[c] Eutocius supplies a proof based on Eucl. xii. 1, which proves a similar theorem for inscribed figures.

[d] For, by hypothesis,
$$\begin{aligned} H^2 &= \Delta\Gamma \cdot EZ \\ &= 2T\Delta \cdot \tfrac{1}{2}PZ \\ &= T\Delta \cdot PZ \end{aligned}$$

Heiberg would delete the demonstration in the text on the ground of excessive verbosity, as Nizze had already perceived to be necessary.

ὡς ἡ ΤΔ πρὸς Η, οὕτως ἡ Η πρὸς ΡΖ· ἔστιν ἄρα, ὡς ἡ ΤΔ πρὸς ΡΖ, τὸ ἀπὸ τῆς ΤΔ πρὸς τὸ ἀπὸ τῆς Η· [ἐὰν γὰρ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν, ὡς ἡ πρώτη πρὸς τὴν τρίτην, τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας εἶδος τὸ ὅμοιον καὶ ὁμοίως ἀναγεγραμμένον][1]. ὃν δὲ λόγον ἔχει ἡ ΤΔ πρὸς ΡΖ μήκει, τοῦτον ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ ΡΛΖ [ἐπειδήπερ ἴσαι εἰσὶν αἱ ΚΔ, ΛΖ][2]· τὸν αὐτὸν ἄρα λόγον ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον, ὅνπερ τὸ ΤΚΔ τρίγωνον πρὸς τὸ ΡΖΛ τρίγωνον. ἴσον ἄρα ἐστὶν τὸ ΖΛΡ τρίγωνον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ εὐθυγράμμῳ· ὥστε καὶ ἡ ἐπιφάνεια τοῦ πρίσματος τοῦ περὶ τὸν Α κύλινδρον περιγεγραμμένου τῷ εὐθυγράμμῳ τῷ περὶ τὸν Β κύκλον ἴση ἐστίν. καὶ ἐπεὶ ἐλάσσονα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον πρὸς τὸ ἐγγεγραμμένον ἐν τῷ κύκλῳ τοῦ, ὃν ἔχει ἡ ἐπιφάνεια τοῦ Α κυλίνδρου πρὸς τὸν Β κύκλον, ἐλάσσονα λόγον ἕξει καὶ ἡ ἐπιφάνεια τοῦ πρίσματος τοῦ περὶ τὸν κύλινδρον περιγεγραμμένου πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ κύκλῳ τῷ Β ἐγγεγραμμένον ἤπερ ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν Β κύκλον· καὶ ἐναλλάξ· ὅπερ ἀδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τοῦ πρίσματος τοῦ περιγεγραμμένου περὶ τὸν κύλινδρον μείζων οὖσα δέδεικται τῆς ἐπιφανείας τοῦ κυλίνδρου, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ Β κύκλῳ ἔλασσόν ἐστιν τοῦ Β κύκλου].[3] οὐκ ἄρα ἐστὶν ὁ Β κύκλος ἐλάσσων τῆς ἐπιφανείας τοῦ κυλίνδρου.

[1] ἐὰν γὰρ . . . ὁμοίως ἀναγεγραμμένον om. Heiberg.
[2] ἐπειδήπερ . . . ΚΔ, ΛΖ om. Heiberg.

And $\quad T\Delta : PZ = \text{triangle } KT\Delta : \text{triangle } P\Lambda Z.$[a]

Therefore the ratio which the triangle KTΔ has to the rectilineal figure circumscribed about the circle B is the same as the ratio of the triangle TKΔ to the triangle PZΛ. Therefore the triangle TKΔ is equal to the rectilineal figure circumscribed about the circle B [Eucl. v. 9]; and so the surface of the prism circumscribed about the cylinder A is equal to the rectilineal figure about B. And since the rectilineal figure about the circle B has to the inscribed figure in the circle a ratio less than that which the surface of the cylinder A has to the circle B [*ex hypothesi*], the surface of the prism circumscribed about the cylinder will have to the rectilineal figure inscribed in the circle B a ratio less than that which the surface of the cylinder has to the circle B; and, *permutando*, [the prism will have to the cylinder a ratio less than that which the rectilineal figure inscribed in the circle B has to the circle B][b]; which is absurd.[c] Therefore the circle B is not less than the surface of the cylinder.

[a] By Eucl. vi. 1, since ΛZ = KΔ.

[b] From Eutocius's comment it appears that Archimedes wrote, in place of *καὶ ἐναλλάξ· ὅπερ ἀδύνατον* in our text: *ἐναλλὰξ ἄρα ἐλάσσονα λόγον ἔχει τὸ πρίσμα πρὸς τὸν κύλινδρον ἤπερ τὸ ἐγγεγραμμένον εἰς τὸν* B *κύκλον πολύγωνον πρὸς τὸν* B *κύκλον ὅπερ ἄτοπον*. This is what I translate.

[c] For the surface of the prism is greater than the surface of the cylinder [Prop. 12], but the inscribed figure is less than the circle B: the explanation in our text to this effect is shown to be an interpolation by the fact that Eutocius supplies a proof in his own words.

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[a] *ἡ μὲν . . . τοῦ* B *κύκλου* om. Heiberg ex Eutocio.

Ἔστω δή, εἰ δυνατόν, μείζων. πάλιν δὴ νοείσθω εἰς τὸν Β κύκλον εὐθύγραμμον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν ἢ τὸν Β κύκλον πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἐγγεγράφθω εἰς τὸν Α κύκλον πολύγωνον ὅμοιον τῷ εἰς τὸν Β κύκλον ἐγγεγραμμένῳ, καὶ πρίσμα ἀναγεγράφθω ἀπὸ τοῦ ἐν τῷ κύκλῳ ἐγγεγραμμένου πολυγώνου· καὶ πάλιν ἡ ΚΔ ἴση ἔστω τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου, καὶ ἡ ΖΛ ἴση αὐτῇ ἔστω. ἔσται δὴ τὸ μὲν ΚΤΔ τρίγωνον μεῖζον τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου [διότι βάσιν μὲν ἔχει τὴν περίμετρον αὐτοῦ, ὕψος δὲ μεῖζον τῆς ἀπὸ τοῦ κέντρου ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου ἀγομένης καθέτου],[1] τὸ δὲ ΕΛ παραλληλόγραμμον ἴσον τῇ ἐπιφανείᾳ τοῦ πρίσματος τῇ ἐκ τῶν παραλληλογράμμων συγκειμένῃ [διότι περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἴσης τῇ περιμέτρῳ τοῦ εὐθυγράμμου, ὅ ἐστιν βάσις τοῦ πρίσματος]· ὥστε καὶ τὸ ΡΛΖ τρίγωνον ἴσον ἐστὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὅμοιά ἐστι τὰ εὐθύγραμμα τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἔχει λόγον πρὸς ἄλληλα, ὃν αἱ ἐκ τῶν κέντρων αὐτῶν δυνάμει. ἔχει δὲ καὶ τὰ ΚΤΔ, ΖΡΛ τρίγωνα πρὸς ἄλληλα λόγον, ὃν αἱ ἐκ τῶν κέντρων τῶν κύκλων δυνάμει· τὸν αὐτὸν ἄρα λόγον ἔχει

[1] διότι . . . καθέτου om. Heiberg.

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[a] For the base ΚΔ is equal to the perimeter of the polygon and the altitude ΔΤ, which is equal to the radius of the

Now let it be, if possible, greater. Again, let there be imagined a rectilineal figure inscribed in the circle B, and another circumscribed, so that the circumscribed figure has to the inscribed a ratio less than that which the circle B has to the surface of the cylinder [Prop. 5], and let there be inscribed in the circle A a polygon similar to the figure inscribed in the circle B, and let a prism be erected on the polygon inscribed in the circle [A]; and again let KΔ be equal to the perimeter of the rectilineal figure inscribed in the circle A, and let ZΛ be equal to it. Then the triangle KTΔ will be greater than the rectilineal figure inscribed in the circle A,[a] and the parallelogram EΛ will be equal to the surface of the prism composed of the parallelograms[b]; and so the triangle PΛZ is equal to the surface of the prism. And since the rectilineal figures inscribed in the circles A, B are similar, they have the same ratio one to the other as the squares of their radii [Eucl. xii. 1]. But the triangles KTΔ, ZPΛ have one to the other the same ratio as the squares of the radii[c]; therefore the rectilineal figure inscribed in

circle A, is greater than the perpendiculars drawn from the centre of the circle to the sides of the polygon: but Heiberg regards the explanation to this effect in the text as an interpolation.

[b] Because the base ZΛ is made equal to KΔ, and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude EZ is equal to the side of the cylinder and therefore to the height of the prism.

[c] For triangle KTΔ : triangle ZPΛ = TΔ : ZP

$$= T\Delta^2 : H^2$$

[*cf.* p. 71 n. *d*.

But TΔ is equal to the radius of the circle A, and H to the radius of the circle B.

τὸ εὐθύγραμμον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ Β ἐγγεγραμμένον καὶ τὸ ΚΤΔ τρίγωνον πρὸς τὸ ΛΖΡ τρίγωνον. ἔλασσον δέ ἐστι τὸ εὐθύγραμμον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον τοῦ ΚΤΔ τριγώνου· ἔλασσον ἄρα καὶ τὸ εὐθύγραμμον τὸ ἐν τῷ Β κύκλῳ ἐγγεγραμμένον τοῦ ΖΡΛ τριγώνου· ὥστε καὶ τῆς ἐπιφανείας τοῦ πρίσματος τοῦ ἐν τῷ κυλίνδρῳ ἐγγεγραμμένου· ὅπερ ἀδύνατον [ἐπεὶ γὰρ ἐλάσσονα λόγον ἔχει τὸ περιγεγραμμένον εὐθύγραμμον περὶ τὸν Β κύκλον πρὸς τὸ ἐγγεγραμμένον ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἐναλλάξ, μεῖζον δέ ἐστι τὸ περιγεγραμμένον περὶ τὸν Β κύκλον τοῦ Β κύκλου, μεῖζον ἄρα ἐστὶν τὸ ἐγγεγραμμένον ἐν τῷ Β κύκλῳ τῆς ἐπιφανείας τοῦ κυλίνδρου· ὥστε καὶ τῆς ἐπιφανείας τοῦ πρίσματος].[1] οὐκ ἄρα μείζων ἐστὶν ὁ Β κύκλος τῆς ἐπιφανείας τοῦ κυλίνδρου. ἐδείχθη δέ, ὅτι οὐδὲ ἐλάσσων· ἴσος ἄρα ἐστίν.

ιδʹ

Παντὸς κώνου ἰσοσκελοῦς χωρὶς τῆς βάσεως ἡ ἐπιφάνεια ἴση ἐστὶ κύκλῳ, οὗ ἡ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς πλευρᾶς τοῦ κώνου καὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ὅς ἐστιν βάσις τοῦ κώνου.

Ἔστω κῶνος ἰσοσκελής, οὗ βάσις ὁ Α κύκλος, ἡ δὲ ἐκ τοῦ κέντρου ἔστω ἡ Γ, τῇ δὲ πλευρᾷ τοῦ

[1] ἐπεὶ . . . πρίσματος om. Heiberg.

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[a] For since the figure circumscribed about the circle B has to the inscribed figure a ratio less than that which the circle B has to the surface of the cylinder [*ex hypothesi*], and the circle B is less than the circumscribed figure, therefore the

the circle A has to the rectilineal figure inscribed in the circle B the same ratio as the triangle KTΔ has to the triangle ΛZP. But the rectilineal figure inscribed in the circle A is less than the triangle KTΔ; therefore the rectilineal figure inscribed in the circle B is less than the triangle ZPΛ; and so it is less than the surface of the prism inscribed in the cylinder; which is impossible.[a] Therefore the circle B is not greater than the surface of the cylinder. But it was proved not to be less. Therefore it is equal.

Prop. 14

The surface of any cone without the base is equal to a circle, whose radius is a mean proportional between the side of the cone and the radius of the circle which is the base of the cone.

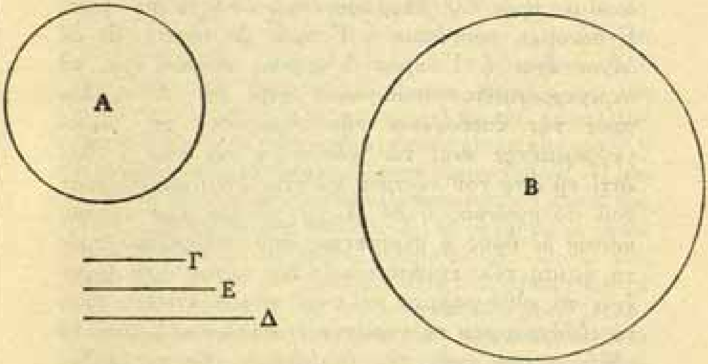
Let there be an isosceles cone, whose base is the circle A, and let its radius be Γ, and let Δ be equal

inscribed figure is greater than the surface of the cylinder, and *a fortiori* is greater than the surface of the prism [Prop. 12]. An explanation on these lines is found in our text, but as the corresponding proof in the first half of the proposition was unknown to Eutocius, this also must be presumed an interpolation.

κώνου ἔστω ἴση ἡ Δ, τῶν δὲ Γ, Δ μέση ἀνάλογον ἡ Ε, ὁ δὲ Β κύκλος ἐχέτω τὴν ἐκ τοῦ κέντρου τῇ Ε ἴσην· λέγω, ὅτι ὁ Β κύκλος ἐστὶν ἴσος τῇ ἐπιφανείᾳ τοῦ κώνου χωρὶς τῆς βάσεως.

Εἰ γὰρ μή ἐστιν ἴσος, ἤτοι μείζων ἐστὶν ἢ ἐλάσσων. ἔστω πρότερον ἐλάσσων. ἔστι δὴ δύο μεγέθη ἄνισα ἥ τε ἐπιφάνεια τοῦ κώνου καὶ ὁ Β κύκλος, καὶ μείζων ἡ ἐπιφάνεια τοῦ κώνου· δυνατὸν ἄρα εἰς τὸν Β κύκλον πολύγωνον ἰσόπλευρον ἐγγράψαι καὶ ἄλλο περιγράψαι ὅμοιον τῷ ἐγγεγραμμένῳ, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλον. νοείσθω δὴ καὶ περὶ τὸν Α κύκλον πολύγωνον περιγεγραμμένον ὅμοιον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ, καὶ ἀπὸ τοῦ περὶ τὸν Α κύκλον περιγεγραμμένου πολυγώνου πυραμὶς ἀνεστάτω ἀναγεγραμμένη τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ. ἐπεὶ οὖν ὅμοιά ἐστιν τὰ πολύγωνα τὰ περὶ

to the side of the cone, and let E be a mean proportional between Γ, Δ, and let the circle B have its



radius equal to E; I say that the circle B is equal to the surface of the cone without the base.

For if it is not equal, it is either greater or less. First let it be less. Then there are two unequal magnitudes, the surface of the cone and the circle B, and the surface of the cone is the greater; it is therefore possible to inscribe an equilateral polygon in the circle B and to circumscribe another similar to the inscribed polygon, so that the circumscribed polygon has to the inscribed polygon a ratio less than that which the surface of the cone has to the circle B [Prop. 5]. Let this be imagined, and about the circle A let a polygon be circumscribed similar to the polygon circumscribed about the circle B, and on the polygon circumscribed about the circle A let a pyramid be raised having the same vertex as the cone. Now since the polygons circumscribed about

τοὺς Α, Β κύκλους περιγεγραμμένα, τὸν αὐτὸν ἔχει λόγον πρὸς ἄλληλα, ὃν αἱ ἐκ τοῦ κέντρου δυνάμει πρὸς ἀλλήλας, τουτέστιν ὃν ἔχει ἡ Γ πρὸς Ε δύναμει, τουτέστιν ἡ Γ πρὸς Δ μήκει. ὃν δὲ λόγον ἔχει ἡ Γ πρὸς Δ μήκει, τοῦτον ἔχει τὸ περιγεγραμμένον πολύγωνον περὶ τὸν Α κύκλον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κῶνον [ἡ μὲν γὰρ Γ ἴση ἐστὶ τῇ ἀπὸ τοῦ κέντρου καθέτῳ ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου, ἡ δὲ Δ τῇ πλευρᾷ τοῦ κώνου· κοινὸν δὲ ὕψος ἡ περίμετρος τοῦ πολυγώνου πρὸς τὰ ἡμίση τῶν ἐπιφανειῶν][1]· τὸν αὐτὸν ἄρα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Α κύκλον πρὸς τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον καὶ αὐτὸ τὸ εὐθύγραμμον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κῶνον· ὥστε ἴση ἐστὶν ἡ ἐπιφάνεια τῆς πυραμίδος τῷ εὐθυγράμμῳ τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ. ἐπεὶ οὖν ἐλάσσονα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἤπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλον, ἐλάσσονα λόγον ἕξει ἡ ἐπιφάνεια τῆς πυραμίδος τῆς περὶ τὸν κῶνον περιγεγραμμένης πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ Β κύκλῳ ἐγγεγραμμένον ἤπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλον· ὅπερ ἀδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τῆς πυραμίδος μείζων οὖσα δέδεικται τῆς ἐπιφανείας τοῦ κώνου, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ Β κύκλῳ ἔλασσον ἔσται τοῦ Β κύκλου].[2] οὐκ ἄρα ὁ Β κύκλος ἐλάσσων ἔσται τῆς ἐπιφανείας τοῦ κώνου.

the circles A, B are similar, they have the same ratio one toward the other as the square of the radii have one toward the other, that is $\Gamma^2 : E^2$, or $\Gamma : \Delta$ [Eucl. vi. 20, coroll. 2]. But $\Gamma : \Delta$ is the same ratio as that of the polygon circumscribed about the circle A to the surface of the pyramid circumscribed about the cone [a]; therefore the rectilineal figure about the circle A has to the rectilineal figure about the circle B the same ratio as this rectilineal figure [about A] has to the surface of the pyramid circumscribed about the cone; therefore the surface of the pyramid is equal to the rectilineal figure circumscribed about the circle B. Since the rectilineal figure circumscribed about the circle B has towards the inscribed [rectilineal figure] a ratio less than that which the surface of the cone has to the circle B, therefore the surface of the pyramid circumscribed about the cone will have to the rectilineal figure inscribed in the circle B a ratio less than that which the surface of the cone has to the circle B; which is impossible.[b] Therefore the circle B will not be less than the surface of the cone.

[a] For the circumscribed polygon is equal to a triangle, whose base is equal to the perimeter of the polygon and whose height is equal to Γ, while the surface of the pyramid is equal to a triangle having the same base and height Δ [Prop. 8]. There is an explanation to this effect in the Greek, but so obscurely worded that Heiberg attributes it to an interpolator.

[b] For the surface of the pyramid is greater than the surface of the cone [Prop. 12], while the inscribed polygon is less than the circle B.

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[1] ἡ μὲν . . . ἐπιφανειῶν om. Heiberg.
[2] ἡ μὲν . . . τοῦ Β κύκλου om. Heiberg.

Λέγω δή, ὅτι οὐδὲ μείζων. εἰ γὰρ δυνατόν ἐστιν, ἔστω μείζων. πάλιν δὴ νοείσθω εἰς τὸν Β κύκλον πολύγωνον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου, καὶ εἰς τὸν Α κύκλον νοείσθω ἐγγεγραμμένον πολύγωνον ὅμοιον τῷ εἰς τὸν Β κύκλον ἐγγεγραμμένῳ, καὶ ἀναγεγράφθω ἀπ' αὐτοῦ πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ. ἐπεὶ οὖν ὅμοιά ἐστι τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἕξει λόγον πρὸς ἄλληλα, ὃν αἱ ἐκ τῶν κέντρων δυνάμει πρὸς ἀλλήλας· τὸν αὐτὸν ἄρα λόγον ἔχει τὸ πολύγωνον πρὸς τὸ πολύγωνον καὶ ἡ Γ πρὸς τὴν Δ μήκει. ἡ δὲ Γ πρὸς τὴν Δ μείζονα λόγον ἔχει ἢ τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐγγεγραμμένης εἰς τὸν κῶνον [ἡ γὰρ ἐκ τοῦ κέντρου τοῦ Α κύκλου πρὸς τὴν πλευρὰν τοῦ κώνου μείζονα λόγον ἔχει ἤπερ ἡ ἀπὸ τοῦ κέντρου ἀγομένη κάθετος ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου πρὸς τὴν ἐπὶ τὴν πλευρὰν τοῦ πολυγώνου κάθετον ἀγομένην ἀπὸ τῆς κορυφῆς τοῦ κώνου][1]· μεί-

[1] ἡ γὰρ . . . τοῦ κώνου om. Heiberg.

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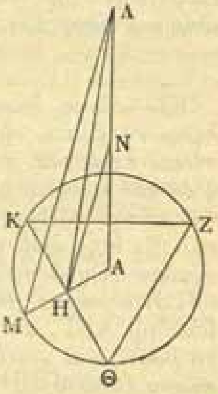
[a] Eutocius supplies a proof. ZΘK is the polygon inscribed in the circle A (of centre A), AH is drawn perpendicular to

I say now that neither will it be greater. For if it is possible, let it be greater. Then again let there be imagined a polygon inscribed in the circle B and another circumscribed, so that the circumscribed has to the inscribed a ratio less than that which the circle B has to the surface of the cone [Prop. 5], and in the circle A let there be imagined an inscribed polygon similar to that inscribed in the circle B, and on it let there be drawn a pyramid having the same vertex as the cone. Since the polygons inscribed in the circles A, B are similar, therefore they will have one toward the other the same ratio as the squares of the radii have one toward the other; therefore the one polygon has to the other polygon the same ratio as Γ to Δ [Eucl. vi. 20, coroll. 2]. But Γ has to Δ a ratio greater than that which the polygon inscribed in the circle A has to the surface of the pyramid inscribed in the cone [a]; therefore the polygon in-

KΘ and meets the circle in M, A is the vertex of the isosceles cone (so that ΛH is perpendicular to KΘ), and HN is drawn parallel to MΛ to meet AΛ in N. Then the area of the polygon inscribed in the circle $=\frac{1}{2}$ perimeter of polygon . ΛH, and the area of the pyramid inscribed in the cone $=\frac{1}{2}$ perimeter of polygon . AH, so that the area of the polygon has to the area of the pyramid the ratio ΛH : AH. Now, by similar triangles, AM : MΛ = AH : HN, and AH : HN > AH : HΛ, for HΛ > HN. Therefore AM : MΛ > AH : HΛ; that is, Γ : Δ exceeds the ratio of the polygon to the surface of the pyramid.



ζονα ἄρα λόγον ἔχει τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὸ πολύγωνον τὸ ἐν τῷ Β ἐγγεγραμμένον ἢ αὐτὸ τὸ πολύγωνον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος· μείζων ἄρα ἐστὶν ἡ ἐπιφάνεια τῆς πυραμίδος τοῦ ἐν τῷ Β πολυγώνου ἐγγεγραμμένου. ἐλάσσονα δὲ λόγον ἔχει τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· πολλῷ ἄρα τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἐγγεγραμμένης ἐλάσσονα λόγον ἔχει ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· ὅπερ ἀδύνατον [τὸ μὲν γὰρ περιγεγραμμένον πολύγωνον μεῖζόν ἐστιν τοῦ Β κύκλου, ἡ δὲ ἐπιφάνεια τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου].[1] οὐκ ἄρα οὐδὲ μείζων ἐστὶν ὁ κύκλος τῆς ἐπιφανείας τοῦ κώνου. ἐδείχθη δέ, ὅτι οὐδὲ ἐλάσσων· ἴσος ἄρα.

ις'

Ἐὰν κῶνος ἰσοσκελὴς ἐπιπέδῳ τμηθῇ παραλλήλῳ τῇ βάσει, τῇ μεταξὺ τῶν παραλλήλων ἐπιπέδων ἐπιφανείᾳ τοῦ κώνου ἴσος ἐστὶ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς τε πλευρᾶς τοῦ κώνου τῆς μεταξὺ τῶν παραλλήλων ἐπιπέδων καὶ τῆς ἴσης ἀμφοτέραις ταῖς ἐκ τῶν κέντρων τῶν κύκλων τῶν ἐν τοῖς παραλλήλοις ἐπιπέδοις.

Ἔστω κῶνος, οὗ τὸ διὰ τοῦ ἄξονος τρίγωνον ἴσον τῷ ΑΒΓ, καὶ τετμήσθω παραλλήλῳ ἐπιπέδῳ τῇ βάσει, καὶ ποιείτω τομὴν τὴν ΔΕ, ἄξων δὲ τοῦ κώνου ἔστω ὁ ΒΗ κύκλος δέ τις ἐκκείσθω, οὗ ἡ

scribed in the circle A has to the polygon inscribed in the circle B a ratio greater than that which the same polygon [inscribed in the circle A] has to the surface of the pyramid; therefore the surface of the pyramid is greater than the polygon inscribed in B. Now the polygon circumscribed about the circle B has to the inscribed polygon a ratio less than that which the circle B has to the surface of the cone; by much more therefore the polygon circumscribed about the circle B has to the surface of the pyramid inscribed in the cone a ratio less than that which the circle B has to the surface of the cone; which is impossible.[a] Therefore the circle is not greater than the surface of the cone. And it was proved not to be less; therefore it is equal.

Prop. 16

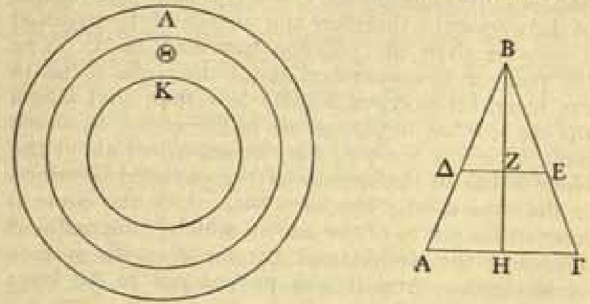
If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between the portion of the side of the cone between the parallel planes and a straight line equal to the sum of the radii of the circles in the parallel planes.

Let there be a cone, in which the triangle through the axis is equal to ΑΒΓ, and let it be cut by a plane parallel to the base, and let [the cutting plane] make the section ΔΕ, and let ΒΗ be the axis of the cone,

[a] For the circumscribed polygon is greater than the circle B, but the surface of the inscribed pyramid is less than the surface of the cone [Prop. 12]: the explanation to this effect in the text is attributed by Heiberg to an interpolator.

[1] τὸ μὲν . . . τοῦ κώνου om. Heiberg.

ἐκ τοῦ κέντρου μέση ἀνάλογόν ἐστι τῆς τε ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΗΑ, ἔστω δὲ κύκλος ὁ Θ·



λέγω, ὅτι ὁ Θ κύκλος ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΔΕ, ΑΓ.

Ἐκκείσθωσαν γὰρ κύκλοι οἱ Λ, Κ, καὶ τοῦ μὲν Κ κύκλου ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΔΖ, τοῦ δὲ Λ ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΑΗ· ὁ μὲν ἄρα Λ κύκλος ἴσος ἐστὶν τῇ ἐπιφανείᾳ τοῦ ΑΒΓ κώνου, ὁ δὲ Κ κύκλος ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΔΕΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΒΑ, ΑΗ ἴσον ἐστὶ τῷ τε ὑπὸ τῶν ΒΔ, ΔΖ καὶ τῷ ὑπὸ τῆς ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ διὰ τὸ παράλληλον εἶναι τὴν ΔΖ τῇ ΑΗ, ἀλλὰ τὸ μὲν ὑπὸ ΑΒ, ΑΗ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Λ κύκλου, τὸ δὲ ὑπὸ ΒΔ, ΔΖ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Κ κύκλου, τὸ δὲ ὑπὸ τῆς ΔΑ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Θ, τὸ ἄρα ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Λ κύκλου ἴσον ἐστὶ τοῖς ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Κ, Θ κύκλων· ὥστε καὶ ὁ Λ κύκλος ἴσος ἐστὶ τοῖς Κ, Θ κύκλοις.

and let there be set out a circle whose radius is a mean proportional between $A\Delta$ and the sum of ΔZ, HA, and let Θ be the circle ; I say that the circle Θ is equal to the portion of the surface of the cone between ΔE, $A\Gamma$.

For let the circles Λ, K be set out, and let the square of the radius of K be equal to the rectangle contained by $B\Delta$, ΔZ, and let the square of the radius of Λ be equal to the rectangle contained by BA, AH ; therefore the circle Λ is equal to the surface of the cone $AB\Gamma$, while the circle K is equal to the surface of the cone ΔEB [Prop. 14]. And since

$$BA \,.\, AH = B\Delta \,.\, \Delta Z + A\Delta \,.\, (\Delta Z + AH)$$

because ΔZ is parallel to AH,[a] while the square of the radius of Λ is equal to $AB \,.\, AH$, the square of the radius of K is equal to $B\Delta \,.\, \Delta Z$, and the square of the radius of Θ is equal to $\Delta A \,.\, (\Delta Z + AH)$, therefore the square on the radius of the circle Λ is equal to the sum of the squares on the radii of the circles K, Θ; so that the circle Λ is equal to the sum of the circles

[a] The proof is given by Eutocius as follows:

$BA : AH = B\Delta : \Delta Z$

$\therefore$ $BA \,.\, \Delta Z = B\Delta \,.\, AH.$ [Eucl. vi. 16

But $BA \,.\, \Delta Z = B\Delta \,.\, \Delta Z + A\Delta \,.\, \Delta Z.$ [Eucl. ii. 1

$\therefore$ $B\Delta \,.\, AH = B\Delta \,.\, \Delta Z + A\Delta \,.\, \Delta Z.$

Let $\Delta A \,.\, AH$ be added to both sides.

Then $B\Delta \,.\, AH + \Delta A + AH$,

i.e. $BA \,.\, AH = B\Delta \,.\, \Delta Z + A\Delta \,.\, \Delta Z + A\Delta \,.\, AH.$

ἀλλ' ὁ μὲν Λ ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΒΑΓ κώνου, ὁ δὲ Κ τῇ ἐπιφανείᾳ τοῦ ΔΒΕ κώνου· λοιπὴ ἄρα ἡ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν παραλλήλων ἐπιπέδων τῶν ΔΕ, ΑΓ ἴση ἐστὶ τῷ Θ κύκλῳ.

κα'

Ἐὰν εἰς κύκλον πολύγωνον ἐγγραφῇ ἀρτιοπλευρόν τε καὶ ἰσόπλευρον, καὶ διαχθῶσιν εὐθεῖαι ἐπιζευγνύουσαι τὰς πλευρὰς τοῦ πολυγώνου, ὥστε αὐτὰς παραλλήλους εἶναι μιᾷ ὁποιᾳοῦν τῶν ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτεινουσῶν, αἱ ἐπιζευγνύουσαι πᾶσαι πρὸς τὴν τοῦ κύκλου διάμετρον τοῦτον ἔχουσι τὸν λόγον, ὃν ἔχει ἡ ὑποτείνουσα τὰς μιᾷ ἐλάσσονας τῶν ἡμίσεων πρὸς τὴν πλευρὰν τοῦ πολυγώνου.

Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθω τὸ ΑΕΖΒΗΘΓΜΝΔΛΚ, καὶ ἐπεζεύχθωσαν αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ· δῆλον δή, ὅτι παράλληλοί εἰσιν τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτεινούσῃ· λέγω οὖν, ὅτι αἱ εἰρημέναι πᾶσαι πρὸς τὴν τοῦ κύκλου διάμετρον τὴν ΑΓ τὸν αὐτὸν λόγον ἔχουσι τῷ τῆς ΓΕ πρὸς ΕΑ.

Ἐπεζεύχθωσαν γὰρ αἱ ΖΚ, ΛΒ, ΗΔ, ΘΝ· παράλληλος ἄρα ἡ μὲν ΖΚ τῇ ΕΑ, ἡ δὲ ΒΛ τῇ ΖΚ, καὶ ἔτι ἡ μὲν ΔΗ τῇ ΒΛ, ἡ δὲ ΘΝ τῇ ΔΗ, καὶ ἡ ΓΜ τῇ ΘΝ [καὶ ἐπεὶ δύο παράλληλοί εἰσιν αἱ ΕΑ, ΚΖ, καὶ δύο διηγμέναι εἰσὶν αἱ ΕΚ, ΑΟ][1]· ἔστιν ἄρα, ὡς ἡ ΕΞ πρὸς ΞΑ, ὁ ΚΞ πρὸς ΞΟ. ὡς δ' ἡ ΚΞ πρὸς ΞΟ, ἡ ΖΠ πρὸς ΠΟ, ὡς δὲ

[1] καὶ ἐπεὶ . . . ΕΚ, ΑΟ om. Heiberg.

K, Θ. But Λ is equal to the surface of the cone ΒΑΓ, while Κ is equal to the surface of the cone ΔΒΕ; therefore the remainder, the portion of the surface of the cone between the parallel planes ΔΕ, ΑΓ, is equal to the circle Θ.

Prop. 21

If a regular polygon with an even number of sides be inscribed in a circle, and straight lines be drawn joining the angles [a] *of the polygon, in such a manner as to be parallel to any one whatsoever of the lines subtended by two sides of the polygon, the sum of these connecting lines bears to the diameter of the circle the same ratio as the straight line subtended by half the sides less one bears to the side of the polygon.*

Let ΑΒΓΔ be a circle, and in it let the polygon ΑΕΖΒΗΘΓΜΝΔΛΚ be inscribed, and let ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ be joined; then it is clear that they are parallel to a straight line subtended by two sides of the polygon [b]; I say therefore that the sum of the aforementioned straight lines bears to ΑΓ, the diameter of the circle, the same ratio as ΓΕ bears to ΕΑ.

For let ΖΚ, ΛΒ, ΗΔ, ΘΝ be joined; then ΖΚ is parallel to ΕΑ, ΒΛ to ΖΚ, also ΔΗ to ΒΛ, ΘΝ to ΔΗ and ΓΜ to ΘΝ [c]; therefore

$$\mathrm{E\Xi} : \mathrm{\Xi A} = \mathrm{K\Xi} : \mathrm{\Xi O}.$$

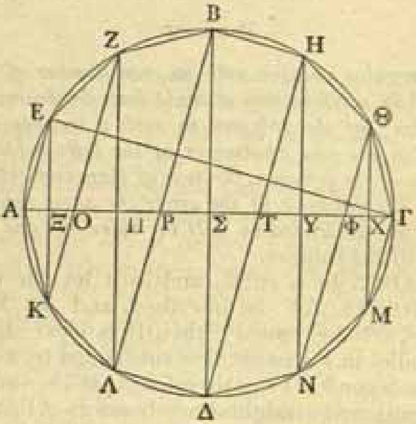
But $\mathrm{K\Xi} : \mathrm{\Xi O} = \mathrm{Z\Pi} : \mathrm{\Pi O}$, [Eucl. vi. 4

[a] "Sides" according to the text, but Heiberg thinks Archimedes probably wrote γωνίας where we have πλευράς.

[b] For, because the arcs ΚΑ, ΕΖ are equal, ∠ΕΚΖ = ∠ΚΖΑ [Eucl. iii. 27]; therefore ΕΚ is parallel to ΑΖ; and so on.

[c] For, as the arcs ΑΚ, ΕΖ are equal, ∠ΑΕΚ = ∠ΕΚΖ, and therefore ΑΕ is parallel to ΖΚ; and so on.

ἡ ΖΠ πρὸς ΠΟ, ἡ ΛΠ πρὸς ΠΡ, ὡς δὲ ἡ ΛΠ πρὸς ΠΡ, οὕτως ἡ ΒΣ πρὸς ΣΡ, καὶ ἔτι, ὡς ἡ μὲν ΒΣ πρὸς ΣΡ, ἡ ΔΣ πρὸς ΣΤ, ὡς δὲ ἡ ΔΣ πρὸς ΣΤ, ἡ ΗΥ πρὸς ΥΤ, καὶ ἔτι, ὡς ἡ μὲν ΗΥ



πρὸς ΥΤ, ἡ ΝΥ πρὸς ΥΦ, ὡς δὲ ἡ ΝΥ πρὸς ΥΦ, ἡ ΘΧ πρὸς ΧΦ, καὶ ἔτι, ὡς μὲν ἡ ΘΧ πρὸς ΧΦ, ἡ ΜΧ πρὸς ΧΓ [καὶ πάντα ἄρα πρὸς πάντα ἐστίν, ὡς εἷς τῶν λόγων πρὸς ἕνα][1]· ὡς ἄρα ἡ ΕΞ πρὸς ΞΑ, οὕτως αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον. ὡς δὲ ἡ ΕΞ πρὸς ΞΑ, οὕτως ἡ ΓΕ πρὸς ΕΑ· ἔσται ἄρα καί, ὡς ἡ ΓΕ πρὸς ΕΑ, οὕτω πᾶσαι αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον.

[1] καὶ . . . ἕνα om. Heiberg.

while	$Z\Pi : \Pi O = A\Pi : \Pi P$,	[*ibid.*
and	$A\Pi : \Pi P = B\Sigma : \Sigma P$.	[*ibid.*
Again,	$B\Sigma : \Sigma P = \Delta\Sigma : \Sigma T$,	[*ibid.*
while	$\Delta\Sigma : \Sigma T = H\Upsilon : \Upsilon T$.	[*ibid.*
Again,	$H\Upsilon : \Upsilon T = N\Upsilon : \Upsilon\Phi$,	[*ibid.*
while	$N\Upsilon : \Upsilon\Phi = \Theta X : X\Phi$.	[*ibid.*
Again,	$\Theta X : X\Phi = MX : X\Gamma$,	[*ibid.*
therefore	$E\Xi : \Xi A = EK + Z\Lambda + B\Delta + HN + \Theta M : A\Gamma$.[a]	[Eucl. v. 12
But	$E\Xi : \Xi A = \Gamma E : EA$;	[Eucl. vi. 4
therefore	$\Gamma E : EA = EK + Z\Lambda + B\Delta + HN + \Theta M : A\Gamma$.[b]	

[a] By adding all the antecedents and consequents, for

$$E\Xi : \Xi A = E\Xi + K\Xi + Z\Pi + A\Pi + B\Sigma + \Delta\Sigma + H\Upsilon + N\Upsilon + \Theta X + MX : \Xi A + \Xi O + \Pi O + \Pi P + \Sigma P + \Sigma T + \Upsilon T + \Upsilon\Phi + X\Phi + X\Gamma$$
$$= EK + Z\Lambda + B\Delta + HN + \Theta M : A\Gamma.$$

[b] If the polygon has $4n$ sides, then

$$\angle E\Gamma K = \frac{\pi}{2n} \quad \text{and} \quad EK : A\Gamma = \sin\frac{\pi}{2n},$$
$$\angle Z\Gamma\Lambda = \frac{2\pi}{2n} \quad \text{and} \quad Z\Lambda : A\Gamma = \sin\frac{2\pi}{2n},$$
$$\angle \Theta\Gamma M = (2n-1)\frac{\pi}{2n} \quad \text{and} \quad \Theta M : A\Gamma = \sin(2n-1)\frac{\pi}{2n}.$$

. . .

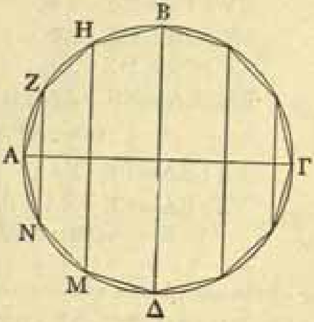
Further, $\angle A\Gamma E = \frac{\pi}{4n}$ and $\Gamma E : EA = \cot\frac{\pi}{4n}$.

Therefore the proposition shows that

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin(2n-1)\frac{\pi}{2n} = \cot\frac{\pi}{4n}.$$

κγ´

Ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐγγεγράφθω εἰς αὐτὸν πολύγωνον ἰσόπλευρον, τὸ



δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρείσθω ὑπὸ τετράδος, αἱ δὲ ΑΓ, ΔΒ διάμετροι ἔστωσαν. ἐὰν δὴ μενούσης τῆς ΑΓ διαμέτρου περιενεχθῇ ὁ ΑΒΓΔ κύκλος ἔχων τὸ πολύγωνον, δῆλον, ὅτι ἡ μὲν περιφέρεια αὐτοῦ κατὰ τῆς ἐπιφανείας τῆς σφαίρας ἐνεχθήσεται, αἱ δὲ τοῦ πολυγώνου γωνίαι χωρὶς τῶν πρὸς τοῖς Α, Γ σημείοις κατὰ κύκλων περιφερειῶν ἐνεχθήσονται ἐν τῇ ἐπιφανείᾳ τῆς σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΑΒΓΔ κύκλον· διάμετροι δὲ αὐτῶν ἔσονται αἱ ἐπιζευγνύουσαι τὰς γωνίας τοῦ πολυγώνου παρὰ τὴν ΒΔ οὖσαι. αἱ δὲ τοῦ πολυγώνου πλευραὶ κατά τινων κώνων ἐνεχθήσονται, αἱ μὲν ΑΖ, ΑΝ κατ' ἐπιφανείας κώνου, οὗ βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΖΝ, κορυφὴ δὲ τὸ Α σημεῖον, αἱ δὲ

Prop. 23

Let ΑΒΓΔ be the greatest circle in a sphere, and let there be inscribed in it an equilateral polygon, the number of whose sides is divisible by four, and let ΑΓ, ΔΒ be diameters. If the diameter ΑΓ remain stationary and the circle ΑΒΓΔ containing the polygon be rotated, it is clear that the circumference of the circle will traverse the surface of the sphere, while the angles of the polygon, except those at the points Α, Γ, will traverse the circumferences of circles described on the surface of the sphere at right angles to the circle ΑΒΓΔ; their diameters will be the [straight lines] joining the angles of the polygon, being parallel to ΒΔ. Now the sides of the polygon will traverse certain cones; ΑΖ, ΑΝ will traverse the surface of a cone whose base is the circle about the diameter ΖΝ and whose vertex is the point Α; ΖΗ,

ΖΗ, ΜΝ κατά τινος κωνικῆς ἐπιφανείας οἰσθήσονται, ἧς βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΜΗ, κορυφὴ δὲ τὸ σημεῖον, καθ' ὃ συμβάλλουσιν ἐκβαλλόμεναι αἱ ΖΗ, ΜΝ ἀλλήλαις τε καὶ τῇ ΑΓ, αἱ δὲ ΒΗ, ΜΔ πλευραὶ κατὰ κωνικῆς ἐπιφανείας οἰσθήσονται, ἧς βάσις μέν ἐστιν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΒΔ ὀρθὸς πρὸς τὸν ΑΒΓΔ κύκλον, κορυφὴ δὲ τὸ σημεῖον, καθ' ὃ συμβάλλουσιν ἐκβαλλόμεναι αἱ ΒΗ, ΔΜ ἀλλήλαις τε καὶ τῇ ΓΑ· ὁμοίως δὲ καὶ αἱ ἐν τῷ ἑτέρῳ ἡμικυκλίῳ πλευραὶ κατὰ κωνικῶν ἐπιφανειῶν οἰσθήσονται πάλιν ὁμοίων ταύταις. ἔσται δή τι σχῆμα ἐγγεγραμμένον ἐν τῇ σφαίρᾳ ὑπὸ κωνικῶν ἐπιφανειῶν περιεχόμενον τῶν προειρημένων, οὗ ἡ ἐπιφάνεια ἐλάσσων ἔσται τῆς ἐπιφανείας τῆς σφαίρας.

Διαιρεθείσης γὰρ τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν ΒΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον ἡ ἐπιφάνεια τοῦ ἑτέρου ἡμισφαιρίου καὶ ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν αὐτῷ ἐγγεγραμμένου τὰ αὐτὰ πέρατα ἔχουσιν ἐν ἑνὶ ἐπιπέδῳ· ἀμφοτέρων γὰρ τῶν ἐπιφανειῶν πέρας ἐστὶν τοῦ κύκλου ἡ περιφέρεια τοῦ περὶ διάμετρον τὴν ΒΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον· καί εἰσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ κοῖλαι, καὶ περιλαμβάνεται αὐτῶν ἡ ἑτέρα ὑπὸ τῆς ἑτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἐχούσης αὐτῇ. ὁμοίως δὲ καὶ τοῦ ἐν τῷ ἑτέρῳ ἡμισφαιρίῳ σχήματος ἡ ἐπιφάνεια ἐλάσσων ἐστὶν τῆς τοῦ ἡμισφαιρίου ἐπιφανείας· καὶ ὅλη οὖν ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν τῇ σφαίρᾳ ἐλάσσων ἐστὶν τῆς ἐπιφανείας τῆς σφαίρας.

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[a] Archimedes would not have omitted to make the deduc-

MN will traverse the surface of a certain cone whose base is the circle about the diameter MH and whose vertex is the point in which ZH, MN produced meet one another and with ΑΓ; the sides BH, MΔ will traverse the surface of a cone whose base is the circle about the diameter BΔ at right angles to the circle ΑΒΓΔ and whose vertex is the point in which BH, ΔM produced meet one another and with ΓΑ; in the same way the sides in the other semicircle will traverse surfaces of cones similar to these. As a result there will be inscribed in the sphere and bounded by the aforesaid surfaces of cones a figure whose surface will be less than the surface of the sphere.

For, if the sphere be cut by the plane through BΔ at right angles to the circle ΑΒΓΔ, the surface of one of the hemispheres and the surface of the figure inscribed in it have the same extremities in one plane; for the extremity of both surfaces is the circumference of the circle about the diameter BΔ at right angles to the circle ΑΒΓΔ; and both are concave in the same direction, and one of them is included by the other surface and the plane having the same extremities with it.[a] Similarly the surface of the figure inscribed in the other hemisphere is less than the surface of the hemisphere; and therefore the whole surface of the figure in the sphere is less than the surface of the sphere.

tion, from Postulate 4, that the surface of the figure inscribed in the hemisphere is less than the surface of the hemisphere.

κδʹ

Ἡ τοῦ ἐγγραφομένου σχήματος εἰς τὴν σφαῖραν ἐπιφάνεια ἴση ἐστὶ κύκλῳ, οὗ ἡ ἐκ τοῦ κέντρου δύναται τὸ περιεχόμενον ὑπό τε τῆς πλευρᾶς τοῦ σχήματος καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνυούσαις τὰς πλευρὰς τοῦ πολυγώνου παραλλήλοις οὔσαις τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτεινούσῃ εὐθείᾳ.

Ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθω ἰσόπλευρον, οὗ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπὸ τοῦ πολυγώνου τοῦ ἐγγεγραμμένου νοείσθω τι εἰς τὴν σφαῖραν ἐγγραφὲν σχῆμα, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ παράλληλοι οὖσαι τῇ ὑπὸ δύο πλευρὰς ὑποτεινούσῃ εὐθείᾳ, κύκλος δέ τις ἐκκείσθω ὁ Ξ, οὗ ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΑΕ καὶ τῆς ἴσης ταῖς ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· λέγω, ὅτι ὁ κύκλος οὗτος ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ εἰς τὴν σφαῖραν ἐγγραφομένου σχήματος.

Ἐκκείσθωσαν γὰρ κύκλοι οἱ Ο, Π, Ρ, Σ, Τ, Υ, καὶ τοῦ μὲν Ο ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΕΑ καὶ τῆς ἡμισείας τῆς ΕΖ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Π δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΕΑ καὶ τῆς ἡμισείας τῶν ΕΖ, ΗΘ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Ρ δυνάσθω τὸ περιεχόμενον ὑπὸ τῆς ΕΑ καὶ τῆς ἡμισείας τῶν ΗΘ, ΓΔ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Σ δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΕΑ καὶ τῆς ἡμισείας τῶν ΓΔ, ΚΛ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Τ δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΑΕ καὶ τῆς ἡμισείας

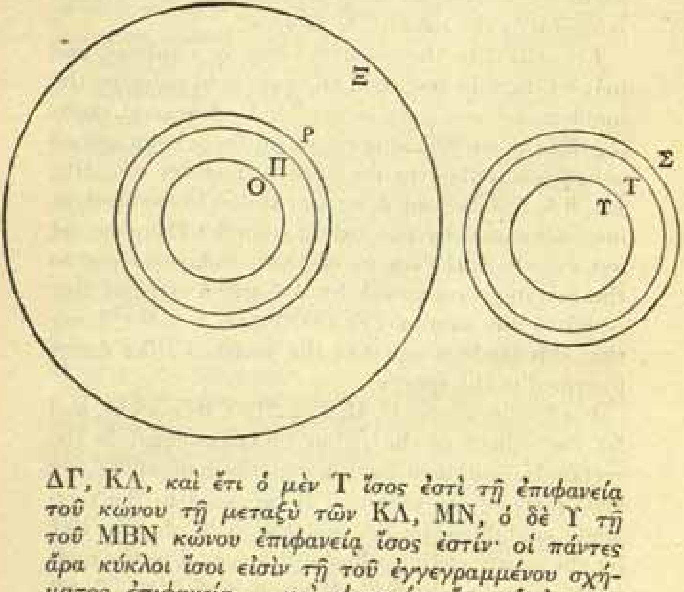
Prop. 24

The surface of the figure inscribed in the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by the side of the figure and a straight line equal to the sum of the straight lines joining the angles of the polygon, being parallel to the straight line subtended by two sides of the polygon.

Let ΑΒΓΔ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from the inscribed polygon, let there be imagined a figure inscribed in the sphere, and let ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ be joined, being parallel to the straight line subtended by two sides; now let there be set out a circle Ξ, the square of whose radius is equal to the rectangle contained by ΑΕ and a straight line equal to the sum of ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ; I say that this circle is equal to the surface of the figure inscribed in the sphere.

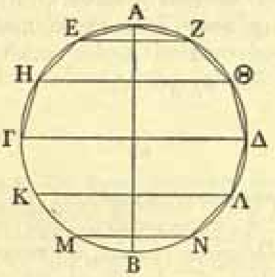
For let the circles Ο, Π, Ρ, Σ, Τ, Υ be set out, and let the square of the radius of Ο be equal to the rectangle contained by ΕΑ and the half of ΕΖ, let the square of the radius of Π be equal to the rectangle contained by ΕΑ and the half of ΕΖ + ΗΘ, let the square of the radius of Ρ be equal to the rectangle contained by ΕΑ and the half of ΗΘ + ΓΔ, let the square of the radius of Σ be equal to the rectangle contained by ΕΑ and the half of ΓΔ + ΚΛ, let the square of the radius of Τ be equal to the rectangle

τῶν ΚΛ, ΜΝ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Υ δυνάσθω τὸ περιεχόμενον ὑπό τε τῆς ΑΕ καὶ τῆς ἡμισείας τῆς ΜΝ. διὰ δὴ ταῦτα ὁ μὲν Ο κύκλος ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΑΕΖ κώνου, ὁ δὲ Π τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΕΖ, ΗΘ, ὁ δὲ Ρ τῇ μεταξὺ τῶν ΗΘ, ΓΔ, ὁ δὲ Σ τῇ μεταξὺ τῶν



ΔΓ, ΚΛ, καὶ ἔτι ὁ μὲν Τ ἴσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΚΛ, ΜΝ, ὁ δὲ Υ τῇ τοῦ ΜΒΝ κώνου ἐπιφανείᾳ ἴσος ἐστίν· οἱ πάντες ἄρα κύκλοι ἴσοι εἰσὶν τῇ τοῦ ἐγγεγραμμένου σχήματος ἐπιφανείᾳ. καὶ φανερόν, ὅτι αἱ ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ὑπό τε τῆς ΑΕ καὶ δὶς τῶν ἡμίσεων τῆς ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ, αἳ ὅλαι εἰσὶν

contained by AE and the half of ΚΛ + MN, and let the square of the radius of Υ be equal to the rectangle contained by AE and the half of MN. Now by these constructions the circle O is equal to the surface of the cone AEZ [Prop. 14], the circle Π is equal to the surface of the conical frustum between EZ and HΘ, the circle P is equal to the surface of the conical frustum between HΘ and ΓΔ, the circle Σ is



equal to the surface of the conical frustum between ΔΓ and ΚΛ, the circle T is equal to the surface of the conical frustum between ΚΛ, MN [Prop. 16], and the circle Υ is equal to the surface of the cone MBN [Prop. 14]; the sum of the circles is therefore equal to the surface of the inscribed figure. And it is manifest that the sum of the squares of the radii of the circles O, Π, P, Σ, T, Υ is equal to the rectangle contained by AE and twice the sum of the halves of EZ, HΘ, ΓΔ, ΚΛ, MN, that is to say, the sum of EZ,

αἱ ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· αἱ ἄρα ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ὑπό τε τῆς ΑΕ καὶ πασῶν τῶν ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. ἀλλὰ καὶ ἡ ἐκ τοῦ κέντρου τοῦ Ξ κύκλου δύναται τὸ ὑπὸ τῆς ΑΕ καὶ τῆς συγκειμένης ἐκ πασῶν τῶν ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· ἡ ἄρα ἐκ τοῦ κέντρου τοῦ Ξ κύκλου δύναται τὰ ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων· καὶ ὁ κύκλος ἄρα ὁ Ξ ἴσος ἐστὶ τοῖς Ο, Π, Ρ, Σ, Τ, Υ κύκλοις. οἱ δὲ Ο, Π, Ρ, Σ, Τ, Υ κύκλοι ἀπεδείχθησαν ἴσοι τῇ εἰρημένῃ τοῦ σχήματος ἐπιφανείᾳ· καὶ ὁ Ξ ἄρα κύκλος ἴσος ἔσται τῇ ἐπιφανείᾳ τοῦ σχήματος.[a]

κϵ΄

Τοῦ ἐγγεγραμμένου σχήματος εἰς τὴν σφαῖραν ἡ ἐπιφάνεια ἡ περιεχομένη ὑπὸ τῶν κωνικῶν ἐπιφανειῶν ἐλάσσων ἐστὶν ἢ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

Ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ ἐγγεγράφθω πολύγωνον [ἀρτιόγωνον] ἰσόπλευρον, οὗ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπ᾽ αὐτοῦ νοείσθω ἐπιφάνεια ἡ ὑπὸ τῶν

[a] If the radius of the sphere is a this proposition shows that

Surface of inscribed figure = circle Ξ

$$= \pi \,.\, \text{ΑΕ} \,.\, (\text{ΕΖ} + \text{ΗΘ} + \text{ΓΔ} + \text{ΚΛ} + \text{ΜΝ}).$$

Now $\text{ΑΕ} = 2a \sin \frac{\pi}{4n}$, and by p. 91 n. b

HΘ, ΓΔ, KΛ, MN; therefore the sum of the squares of the radii of the circles O, Π, P, Σ, T, Υ is equal to the rectangle contained by AE and the sum of EZ, HΘ, ΓΔ, KΛ, MN. But the square of the radius of the circle Ξ is equal to the rectangle contained by AE and a straight line made up of EZ, HΘ, ΓΔ, KΛ, MN [*ex hypothesi*]; therefore the square of the radius of the circle Ξ is equal to the sum of the squares of the radii of the circles O, Π, P, Σ, T, Υ; and therefore the circle Ξ is equal to the sum of the circles O, Π, P, Σ, T, Υ. Now the sum of the circles O, Π, P, Σ, T, Υ was shown to be equal to the surface of the aforesaid figure; and therefore the circle Ξ will be equal to the surface of the figure.[a]

Prop. 25

The surface of the figure inscribed in the sphere and bounded by the surfaces of cones is less than four times the greatest of the circles in the sphere.

Let ABΓΔ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from it, let a surface bounded by surfaces of

$$EZ + H\Theta + \Gamma\Delta + K\Lambda + MN = 2a\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin(2n-1)\frac{\pi}{2n}\right].$$

$$\therefore \text{ Surface of inscribed figure} = 4\pi a^2 \sin\frac{\pi}{4n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin(2n-1)\frac{\pi}{2n}\right]$$

$$= 4\pi a^2 \cos\frac{\pi}{4n}$$

[by p. 91 n. *b*.

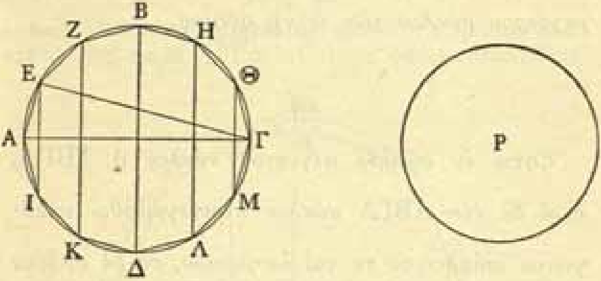
κωνικῶν ἐπιφανειῶν περιεχομένη· λέγω, ὅτι ἡ ἐπιφάνεια τοῦ ἐγγραφέντος ἐλάσσων ἐστὶν ἢ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

Ἐπεζεύχθωσαν γὰρ αἱ ὑπὸ δύο πλευρὰς ὑποτείνουσαι τοῦ πολυγώνου αἱ ΕΙ, ΘΜ καὶ ταύταις παράλληλοι αἱ ΖΚ, ΔΒ, ΗΛ, ἐκκείσθω δέ τις κύκλος ὁ Ρ, οὗ ἡ ἐκ τοῦ κέντρου δύναται τὸ ὑπὸ τῆς ΕΑ καὶ τῆς ἴσης πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ· διὰ δὴ τὸ προδειχθὲν ἴσος ἐστὶν ὁ κύκλος τῇ τοῦ εἰρημένου σχήματος ἐπιφανείᾳ. καὶ ἐπεὶ ἐδείχθη, ὅτι ἐστίν, ὡς ἡ ἴση πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ πρὸς τὴν διάμετρον τοῦ κύκλου τὴν ΑΓ, οὕτως ἡ ΓΕ πρὸς ΕΑ, τὸ ἄρα ὑπὸ τῆς ἴσης πάσαις ταῖς εἰρημέναις καὶ τῆς ΕΑ, τουτέστιν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ κύκλου, ἴσον ἐστὶν τῷ ὑπὸ τῶν ΑΓ, ΓΕ. ἀλλὰ καὶ τὸ ὑπὸ ΑΓ, ΓΕ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΑΓ· ἔλασσον ἄρα ἐστὶν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ τοῦ ἀπὸ τῆς ΑΓ [ἐλάσσων ἄρα ἐστὶν ἡ ἐκ τοῦ κέντρου τοῦ Ρ τῆς ΑΓ· ὥστε ἡ διάμετρος τοῦ Ρ κύκλου ἐλάσσων ἐστὶν ἢ διπλασία τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, καὶ δύο ἄρα τοῦ ΑΒΓΔ κύκλου διάμετροι μείζους εἰσὶ τῆς διαμέτρου τοῦ Ρ κύκλου, καὶ τὸ τετράκις ἀπὸ τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, τουτέστι τῆς ΑΓ, μεῖζόν ἐστι τοῦ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου. ὡς δὲ τὸ τετράκις ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου, οὕτως τέσσαρες κύκλοι οἱ ΑΒΓΔ πρὸς τὸν Ρ κύκλον· τέσσαρες ἄρα κύκλοι οἱ ΑΒΓΔ μείζους εἰσὶν τοῦ Ρ κύκλου][1]· ὁ ἄρα κύκλος ὁ Ρ ἐλάσσων ἐστὶν ἢ τετραπλάσιος τοῦ

[1] ἐλάσσων . . . κύκλου om. Heiberg.

cones be imagined; I say that the surface of the inscribed figure is less than four times the greatest of the circles inscribed in the sphere.

For let EI, ΘM, subtended by two sides of the polygon, be joined, and let ZK, ΔB, HΛ be parallel



to them, and let there be set out a circle P, the square of whose radius is equal to the rectangle contained by EA and a straight line equal to the sum of EI, ZK, BΔ, HΛ, ΘM; by what has been proved above, the circle is equal to the surface of the aforesaid figure. And since it was proved that the ratio of the sum of EI, ZK, BΔ, HΛ, ΘM to AΓ, the diameter of the circle, is equal to the ratio of ΓE to EA [Prop. 21], therefore

$$EA \,.\, (EI + ZK + B\Delta + H\Lambda + \Theta M)$$

that is, the square on the radius of the circle P

[*ex hyp.*

$$= A\Gamma \,.\, \Gamma E. \qquad \text{[Eucl. vi. 16}$$

But $\quad A\Gamma \,.\, \Gamma E < A\Gamma^2.$ [Eucl. iii. 15

Therefore the square on the radius of P is less than the square on AΓ; therefore the circle P is less

μεγιστου κύκλου. ὁ δὲ Ρ κύκλος ἴσος ἐδείχθη τῇ εἰρημένῃ ἐπιφανείᾳ τοῦ σχήματος· ἡ ἄρα ἐπιφάνεια τοῦ σχήματος ἐλάσσων ἐστὶν ἢ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

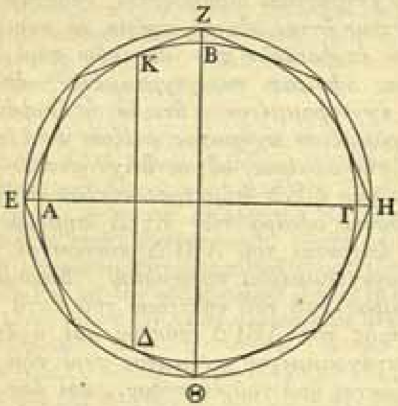
κη΄

Ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, περὶ δὲ τὸν ΑΒΓΔ κύκλον περιγεγράφθω πολύγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον, τὸ δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρείσθω ὑπὸ τετράδος, τὸ δὲ περὶ τὸν κύκλον περιγεγραμμένον πολύγωνον κύκλος περιγεγραμμένος περιλαμβανέτω περὶ τὸ αὐτὸ κέντρον γινόμενος τῷ ΑΒΓΔ. μενούσης δὴ τῆς ΕΗ περιενεχθήτω τὸ ΕΖΗΘ ἐπίπεδον, ἐν ᾧ τό τε πολύγωνον καὶ ὁ κύκλος· δῆλον οὖν, ὅτι ἡ μὲν περιφέρεια τοῦ ΑΒΓΔ κύκλου κατὰ τῆς ἐπιφανείας τῆς σφαίρας οἰσθήσεται, ἡ δὲ περιφέρεια τοῦ ΕΖΗΘ κατ' ἄλλης ἐπιφανείας σφαίρας τὸ

than four times the greatest circle. But the circle P was proved equal to the aforesaid surface of the figure; therefore the surface of the figure is less than four times the greatest of the circles in the sphere.

Prop. 28

Let ΑΒΓΔ be the greatest circle in a sphere, and about the circle ΑΒΓΔ let there be circumscribed



an equilateral and equiangular polygon, the number of whose sides is divisible by four, and let a circle be described about the polygon circumscribing the circle, having the same centre as ΑΒΓΔ. While EH remains stationary, let the plane EZHΘ, in which lie both the polygon and the circle, be rotated; it is clear that the circumference of the circle ΑΒΓΔ will traverse the surface of the sphere, while the circumference of EZHΘ will traverse the surface of another

αὐτὸ κέντρον ἐχούσης τῇ ἐλάσσονι οἰσθήσεται, αἱ δὲ ἁφαί, καθ' ἃς ἐπιψαύουσιν αἱ πλευραί, γράφουσιν κύκλους ὀρθοὺς πρὸς τὸν ΑΒΓΔ κύκλον ἐν τῇ ἐλάσσονι σφαίρᾳ, αἱ δὲ γωνίαι τοῦ πολυγώνου χωρὶς τῶν πρὸς τοῖς Ε, Η σημείοις κατὰ κύκλων περιφερειῶν οἰσθήσονται ἐν τῇ ἐπιφανείᾳ τῆς μείζονος σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΕΖΗΘ κύκλον, αἱ δὲ πλευραὶ τοῦ πολυγώνου κατὰ κωνικῶν ἐπιφανειῶν οἰσθήσονται, καθάπερ ἐπὶ τῶν πρὸ τούτου· ἔσται οὖν τὸ σχῆμα τὸ περιεχόμενον ὑπὸ τῶν ἐπιφανειῶν τῶν κωνικῶν περὶ μὲν τὴν ἐλάσσονα σφαῖραν περιγεγραμμένον, ἐν δὲ τῇ μείζονι ἐγγεγραμμένον. ὅτι δὲ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος μείζων ἐστὶ τῆς ἐπιφανείας τῆς σφαίρας, οὕτως δειχθήσεται.

Ἔστω γὰρ ἡ ΚΔ διάμετρος κύκλου τινὸς τῶν ἐν τῇ ἐλάσσονι σφαίρᾳ τῶν Κ, Δ σημείων ὄντων, καθ' ἃ ἅπτονται τοῦ ΑΒΓΔ κύκλου αἱ πλευραὶ[1] τοῦ περιγεγραμμένου πολυγώνου. διῃρημένης δὴ τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν ΚΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον καὶ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν διαιρεθήσεται ὑπὸ τοῦ ἐπιπέδου. καὶ φανερόν, ὅτι τὰ αὐτὰ πέρατα ἔχουσιν ἐν ἐπιπέδῳ· ἀμφοτέρων γὰρ τῶν ἐπιπέδων πέρας ἐστὶν ἡ τοῦ κύκλου περιφέρεια τοῦ περὶ διάμετρον τὴν ΚΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον· καί εἰσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ κοῖλαι, καὶ περιλαμβάνεται ἡ ἑτέρα αὐτῶν ὑπὸ τῆς ἑτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἐχούσης· ἐλάσσων οὖν ἐστιν ἡ περιλαμβανομένη τοῦ τμήματος τῆς σφαίρας ἐπιφάνεια τῆς ἐπιφανείας τοῦ σχήματος τοῦ περι-

sphere, having the same centre as the lesser sphere; the points of contact in which the sides touch [the smaller circle] will describe circles on the lesser sphere at right angles to the circle ΑΒΓΔ, and the angles of the polygon, except those at the points E, H will traverse the circumferences of circles on the surface of the greater sphere at right angles to the circle ΕΖΗΘ, while the sides of the polygon will traverse surfaces of cones, as in the former case; there will therefore be a figure, bounded by surfaces of cones, described about the lesser sphere and inscribed in the greater. That the surface of the circumscribed figure is greater than the surface of the sphere will be proved thus.

Let ΚΔ be a diameter of one of the circles in the lesser sphere, K, Δ being points at which the sides of the circumscribed polygon touch the circle ΑΒΓΔ. Now, since the sphere is divided by the plane containing ΚΔ at right angles to the circle ΑΒΓΔ, the surface of the figure circumscribed about the sphere will be divided by the same plane. And it is manifest that they [a] have the same extremities in a plane; for the extremity of both surfaces [b] is the circumference of the circle about the diameter ΚΔ at right angles to the circle ΑΒΓΔ; and they are both concave in the same direction, and one of them is included by the other and the plane having the same extremities; therefore the included surface of the segment of the sphere is less than the surface of

[a] *i.e.*, the surface formed by the revolution of the circular segment ΚΑΔ and the surface formed by the revolution of the portion K . . . E . . . Δ of the polygon.

[b] In the text *ἐπιπέδων* should obviously be *ἐπιφανειῶν*.

[1] *αἱ πλευραὶ* Heiberg; om. codd.

γεγραμμένου περὶ αὐτήν. ὁμοίως δὲ καὶ ἡ τοῦ λοιποῦ τμήματος τῆς σφαίρας ἐπιφάνεια ἐλάσσων ἐστὶν τῆς ἐπιφανείας τοῦ σχήματος τοῦ περιγεγραμμένου περὶ αὐτήν· δῆλον οὖν, ὅτι καὶ ὅλη ἡ ἐπιφάνεια τῆς σφαίρας ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ σχήματος τοῦ περιγεγραμμένου περὶ αὐτήν.

κθʹ

Τῇ ἐπιφανείᾳ τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν ἴσος ἐστὶ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴσον δύναται τῷ περιεχομένῳ ὑπό τε μιᾶς πλευρᾶς τοῦ πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνυούσαις τὰς γωνίας τοῦ πολυγώνου οὔσαις παρά τινα τῶν ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτεινουσῶν.

Τὸ γὰρ περιγεγραμμένον περὶ τὴν ἐλάσσονα σφαῖραν ἐγγέγραπται εἰς τὴν μείζονα σφαῖραν· τοῦ δὲ ἐγγεγραμμένου ἐν τῇ σφαίρᾳ περιεχομένου ὑπὸ τῶν ἐπιφανειῶν τῶν κωνικῶν δέδεικται ὅτι τῇ ἐπιφανείᾳ ἴσος ἐστὶν ὁ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου δύναται τὸ περιεχόμενον ὑπό τε μιᾶς πλευρᾶς τοῦ πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνυούσαις τὰς γωνίας τοῦ πολυγώνου οὔσαις παρά τινα τῶν ὑπὸ δύο πλευρὰς ὑποτεινουσῶν· δῆλον οὖν ἐστι τὸ προειρημένον.

[a] If the radius of the inner sphere is a and that of the outer sphere a', and the regular polygon has $4n$ sides, then

$$a' = a \sec \frac{\pi}{4n}.$$

This proposition shows that

$$\frac{\text{Area of figure circumscribed}}{\text{to circle of radius } a} = \frac{\text{Area of figure inscribed in}}{\text{circle of radius } a'}$$

the figure circumscribed about it [Post. 4]. Similarly the surface of the remaining segment of the sphere is less than the surface of the figure circumscribed about it; it is clear therefore that the whole surface of the sphere is less than the surface of the figure circumscribed about it.

Prop. 29

The surface of the figure circumscribed about the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides of the polygon.

For the figure circumscribed about the lesser sphere is inscribed in the greater sphere [Prop. 28]; and it has been proved that the surface of the figure inscribed in the sphere and formed by surfaces of cones is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides [Prop. 24]; what was aforesaid is therefore obvious.[a]

$$=4\pi a^2 \sin\frac{\pi}{4n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin(2n-1)\frac{\pi}{2n}\right],$$

$$\text{or } 4\pi a^2 \cos\frac{\pi}{4n} \text{ [by p. 91 n. } b$$

$$=4\pi a^2 \sec^2\frac{\pi}{4n}\sin\frac{\pi}{4n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin(2n-1)\frac{\pi}{2n}\right],$$

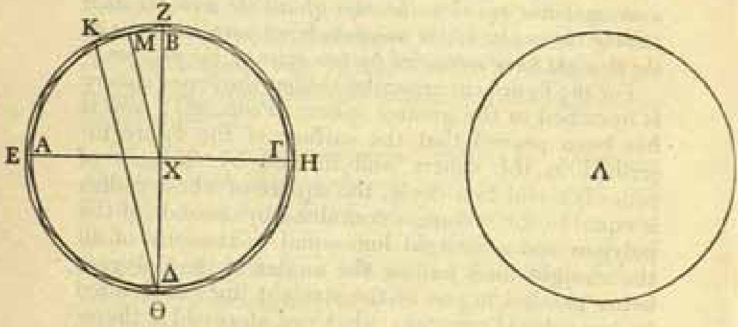
$$\text{or } 4\pi a^2 \sec\frac{\pi}{4n}.$$

χʹ

Τοῦ σχήματος τοῦ περιγεγραμμένου περὶ τὴν σφαῖραν ἡ ἐπιφάνεια μείζων ἐστὶν ἢ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

Ἔστω γὰρ ἥ τε σφαῖρα καὶ ὁ κύκλος καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρότερον προκειμένοις, καὶ ὁ Λ κύκλος ἴσος τῇ ἐπιφανείᾳ ἔστω τοῦ προκειμένου περιγεγραμμένου περὶ τὴν ἐλάσσονα σφαῖραν.

Ἐπεὶ οὖν ἐν τῷ ΕΖΗΘ κύκλῳ πολύγωνον



ἰσόπλευρον ἐγγέγραπται καὶ ἀρτιογώνιον, αἱ ἐπιζευγνύουσαι τὰς τοῦ πολυγώνου πλευρὰς παράλληλοι οὖσαι τῇ ΖΘ πρὸς τὴν ΖΘ τὸν αὐτὸν λόγον ἔχουσιν, ὃν ἡ ΘΚ πρὸς ΚΖ· ἴσον ἄρα ἐστὶν τὸ περιεχόμενον σχῆμα ὑπό τε μιᾶς πλευρᾶς τοῦ πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνυούσαις τὰς γωνίας τοῦ πολυγώνου τῷ περιεχομένῳ ὑπὸ τῶν ΖΘΚ· ὥστε ἡ ἐκ τοῦ κέντρου τοῦ Λ κύκλου ἴσον δύναται τῷ ὑπὸ ΖΘΚ· μείζων ἄρα

Prop. 30

The surface of the figure circumscribed about the sphere is greater than four times the greatest of the circles in the sphere.

For let there be both the sphere and the circle and the other things the same as were posited before, and let the circle Λ be equal to the surface of the given figure circumscribed about the lesser sphere.

Therefore since in the circle ΕΖΗΘ there has been inscribed an equilateral polygon with an even number of angles, the [sum of the straight lines] joining the sides of the polygon, being parallel to ΖΘ, have the same ratio to ΖΘ as ΘΚ to ΚΖ [Prop. 21]; therefore the rectangle contained by one side of the polygon and the straight line equal to the sum of the straight lines joining the angles of the polygon is equal to the rectangle contained by ΖΘ, ΘΚ [Eucl. vi. 16]; so that the square of the radius of the circle Λ is equal to the rectangle contained by ΖΘ, ΘΚ

ἐστὶν ἡ ἐκ τοῦ κέντρου τοῦ Λ κύκλου τῆς ΘΚ. ἡ δὲ ΘΚ ἴση ἐστὶ τῇ διαμέτρῳ τοῦ ΑΒΓΔ κύκλου [διπλασία γάρ ἐστιν τῆς ΧΣ οὔσης ἐκ τοῦ κέντρου τοῦ ΑΒΓΔ κύκλου].[1] δῆλον οὖν, ὅτι μείζων ἐστὶν ἢ τετραπλάσιος ὁ Λ κύκλος, τουτέστιν ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν ἐλάσσονα σφαῖραν, τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

λγ΄

Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.

Ἔστω γὰρ σφαῖρά τις, καὶ ἔστω τετραπλάσιος τοῦ μεγίστου κύκλου ὁ Α· λέγω, ὅτι ὁ Α ἴσος ἐστὶν τῇ ἐπιφανείᾳ τῆς σφαίρας.

Εἰ γὰρ μή, ἤτοι μείζων ἐστὶν ἢ ἐλάσσων. ἔστω πρότερον μείζων ἡ ἐπιφάνεια τῆς σφαίρας τοῦ κύκλου. ἔστι δὴ δύο μεγέθη ἄνισα ἥ τε ἐπιφάνεια τῆς σφαίρας καὶ ὁ Α κύκλος· δυνατὸν ἄρα ἐστὶ λαβεῖν δύο εὐθείας ἀνίσους, ὥστε τὴν μείζονα πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα τοῦ, ὃν ἔχει ἡ

[1] διπλασία . . . κύκλου om. Heiberg.

[a] Because ΖΘ > ΘΚ [Eucl. iii. 15].

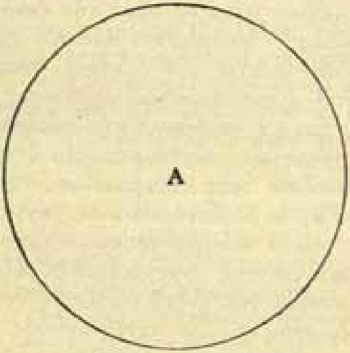
[Prop. 29]. Therefore the radius of the circle Λ is greater than ΘK.[a] Now ΘK is equal to the diameter of the circle ΑΒΓΔ; it is therefore clear that the circle Λ, that is, the surface of the figure circumscribed about the lesser sphere, is greater than four times the greatest of the circles in the sphere.

Prop. 33

The surface of any sphere is four times the greatest of the circles in it.

For let there be a sphere, and let A be four times the greatest circle; I say that A is equal to the surface of the sphere.

For if not, either it is greater or less. First, let the surface of the sphere be greater than the circle.

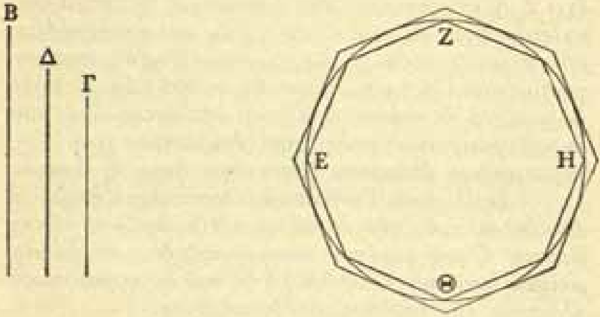


Then there are two unequal magnitudes, the surface of the sphere and the circle A; it is therefore possible to take two unequal straight lines so that the greater bears to the less a ratio less than that which the sur-

ἐπιφάνεια τῆς σφαίρας πρὸς τὸν κύκλον. εἰλήφθωσαν αἱ Β, Γ, καὶ τῶν Β, Γ μέση ἀνάλογον ἔστω ἡ Δ, νοείσθω δὲ καὶ ἡ σφαῖρα ἐπιπέδῳ τετμημένη διὰ τοῦ κέντρου κατὰ τὸν ΕΖΗΘ κύκλον, νοείσθω δὲ καὶ εἰς τὸν κύκλον ἐγγεγραμμένον καὶ περιγεγραμμένον πολύγωνον, ὥστε ὅμοιον εἶναι τὸ περιγεγραμμένον τῷ ἐγγεγραμμένῳ πολυγώνῳ καὶ τὴν τοῦ περιγεγραμμένου πλευρὰν ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ἡ Β πρὸς Δ [καὶ ὁ διπλάσιος ἄρα λόγος τοῦ διπλασίου λόγου ἐστὶν ἐλάσσων. καὶ τοῦ μὲν τῆς Β πρὸς Δ διπλάσιός ἐστιν ὁ τῆς Β πρὸς τὴν Γ, τῆς δὲ πλευρᾶς τοῦ περιγεγραμμένου πολυγώνου πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμμένου διπλάσιος ὁ τῆς ἐπιφανείας τοῦ περιγεγραμμένου στερεοῦ πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου][1]· ἡ ἐπιφάνεια ἄρα τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου σχήματος ἐλάσσονα λόγον ἔχει ἤπερ ἡ ἐπιφάνεια τῆς σφαίρας πρὸς τὸν Α κύκλον· ὅπερ ἄτοπον· ἡ μὲν γὰρ ἐπιφάνεια τοῦ περιγεγραμμένου τῆς ἐπιφανείας τῆς σφαίρας μείζων ἐστίν, ἡ δὲ ἐπιφάνεια τοῦ ἐγγεγραμμένου σχήματος τοῦ Α κύκλου ἐλάσσων ἐστί [δέδεικται γὰρ ἡ ἐπιφάνεια τοῦ ἐγγεγραμμένου ἐλάσσων τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ ἢ τετραπλασία, τοῦ δὲ μεγίστου κύκλου τετραπλάσιός ἐστιν ὁ Α κύκλος].[2] οὐκ ἄρα ἡ ἐπιφάνεια τῆς σφαίρας μείζων ἐστὶ τοῦ Α κύκλου.

[1] καὶ . . . ἐγγεγραμμένου om. Heiberg.
[2] δέδεικται . . . κύκλος "repetitionem inutilem Prop. 25." om. Heiberg.

face of the sphere bears to the circle [Prop. 2]. Let B, Γ be so taken, and let Δ be a mean proportional between B, Γ, and let the sphere be imagined as cut



through the centre along the [plane of the] circle EZHΘ, and let there be imagined a polygon inscribed in the circle and another circumscribed about it in such a manner that the circumscribed polygon is similar to the inscribed polygon and the side of the circumscribed polygon has [to the side of the inscribed polygon] [a] a ratio less than that which B has to Δ [Prop. 3]. Therefore the surface of the figure circumscribed about the sphere has to the surface of the inscribed figure a ratio less than that which the surface of the sphere has to the circle A; which is absurd; for the surface of the circumscribed figure is greater than the surface of the sphere [Prop. 28], while the surface of the inscribed figure is less than the circle A [Prop. 25]. Therefore the surface of the sphere is not greater than the circle A.

[a] Archimedes would not have omitted: πρὸς τὴν τοῦ ἐγγεγραμμένου.

Λέγω δή, ὅτι οὐδὲ ἐλάσσων. εἰ γὰρ δυνατόν, ἔστω· καὶ ὁμοίως εὑρήσθωσαν αἱ Β, Γ εὐθεῖαι, ὥστε τὴν Β πρὸς Γ ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας, καὶ τῶν Β, Γ μέση ἀνάλογον ἡ Δ, καὶ ἐγγεγράφθω καὶ περιγεγράφθω πάλιν, ὥστε τὴν τοῦ περιγεγραμμένου ἐλάσσονα λόγον ἔχειν τοῦ τῆς Β πρὸς Δ [καὶ τὰ διπλάσια ἄρα][1]· ἡ ἐπιφάνεια ἄρα τοῦ περιγεγραμμένου πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἔχει ἤπερ [ἡ Β πρὸς Γ. ἡ δὲ Β πρὸς Γ ἐλάσσονα λόγον ἔχει ἤπερ][2] ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας· ὅπερ ἄτοπον· ἡ μὲν γὰρ τοῦ περιγεγραμμένου ἐπιφάνεια μείζων ἐστὶ τοῦ Α κύκλου, ἡ δὲ τοῦ ἐγγεγραμμένου ἐλάσσων τῆς ἐπιφανείας τῆς σφαίρας.

Οὐκ ἄρα οὐδὲ ἐλάσσων ἡ ἐπιφάνεια τῆς σφαίρας τοῦ Α κύκλου. ἐδείχθη δέ, ὅτι οὐδὲ μείζων· ἡ ἄρα ἐπιφάνεια τῆς σφαίρας ἴση ἐστὶ τῷ Α κύκλῳ, τουτέστι τῷ τετραπλασίῳ τοῦ μεγίστου κύκλου.

[1] καὶ . . . ἄρα om. Heiberg.
[2] ἡ Β . . . ἤπερ om. Heiberg.

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[a] Archimedes would not have omitted these words.

[b] On p. 100 n. *a* it was proved that the area of the inscribed figure is

$$4\pi a^2 \sin\frac{\pi}{n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin(2n-1)\frac{\pi}{2n}\right],$$

$$\text{or } 4\pi a^2 \cos\frac{\pi}{4n}.$$

On p. 108 n. *a* it was proved that the area of the circumscribed figure is

$$4\pi a^2 \sec^2\frac{\pi}{4n}\sin\frac{\pi}{4n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin(2n-1)\frac{\pi}{2n}\right],$$

$$\text{or } 4\pi a^2 \sec\frac{\pi}{4n}.$$

I say now that neither is it less. For, if possible let it be; and let the straight lines Β, Γ be similarly found, so that Β has to Γ a less ratio than that which the circle Α has to the surface of the sphere, and let Δ be a mean proportional between Β, Γ, and let [polygons] be again inscribed and circumscribed, so that the [side] of the circumscribed polygon has [to the side of the inscribed polygon] [a] a less ratio than that of Β to Δ; then the surface of the circumscribed polygon has to the surface of the inscribed polygon a ratio less than that which the circle Α has to the surface of the sphere; which is absurd; for the surface of the circumscribed polygon is greater than the circle Α, while that of the inscribed polygon is less than the surface of the sphere.

Therefore the surface of the sphere is not less than the circle Α. And it was proved not to be greater; therefore the surface of the sphere is equal to the circle Α, that is to four times the greatest circle.[b]

When n is indefinitely increased, the inscribed and circumscribed figures become identical with one another and with the circle, and, since $\cos\frac{\pi}{4n}$ and $\sec\frac{\pi}{4n}$ both become unity, the above expressions both give the area of the circle as $4\pi a^2$.

But the first expressions are, when n is indefinitely increased, precisely what is meant by the integral

$$4\pi a^2 \cdot \tfrac{1}{2}\int_0^{\pi} \sin\phi \, d\phi,$$

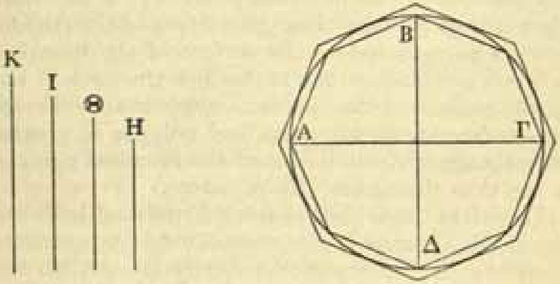
which is familiar to every student of the calculus as the formula for the area of a sphere and has the value $4\pi a^2$.

Thus Archimedes' procedure is equivalent to a genuine integration, but when it comes to the last stage, instead of saying, "Let the sides of the polygon be indefinitely

λδ'

Πᾶσα σφαῖρα τετραπλασία ἐστὶ κώνου τοῦ βάσιν μὲν ἔχοντος ἴσην τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ τὴν ἐκ τοῦ κέντρου τῆς σφαίρας.

Ἔστω γὰρ σφαῖρά τις καὶ ἐν αὐτῇ μέγιστος κύκλος ὁ ΑΒΓΔ. εἰ οὖν μή ἐστιν ἡ σφαῖρα τε-



τραπλασία τοῦ εἰρημένου κώνου, ἔστω, εἰ δυνατόν, μείζων ἢ τετραπλασία· ἔστω δὲ ὁ Ξ κῶνος βάσιν μὲν ἔχων τετραπλασίαν τοῦ ΑΒΓΔ κύκλου, ὕψος δὲ ἴσον τῇ ἐκ τοῦ κέντρου τῆς σφαίρας· μείζων οὖν ἐστιν ἡ σφαῖρα τοῦ Ξ κώνου. ἔσται δὴ δύο μεγέθη ἄνισα ἥ τε σφαῖρα καὶ ὁ κῶνος· δυνατὸν οὖν δύο εὐθείας λαβεῖν ἀνίσους, ὥστε ἔχειν τὴν

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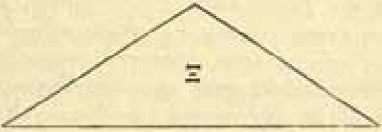
increased," he prefers to prove that the area of the sphere cannot be either greater or less than $4\pi a^2$. By this double *reductio ad absurdum* he avoids the logical difficulties of dealing with indefinitely small quantities, difficulties that were not fully overcome until recent times.

The procedure by which in this same book Archimedes

Prop. 34

Any sphere is four times as great as the cone having a base equal to the greatest of the circles in the sphere and height equal to the radius of the sphere.

For let there be a sphere in which ΑΒΓΔ is the greatest circle. If the sphere is not four times the



aforesaid cone, let it be, if possible, greater than four times; let Ξ be a cone having a base four times the circle ΑΒΓΔ and height equal to the radius of the sphere; then the sphere is greater than the cone Ξ. Accordingly there will be two unequal magnitudes, the sphere and the cone; it is therefore possible to take two unequal straight lines so that

finds the surface of the segment of a sphere is equivalent to the integration

$$\pi a^2 \int_0^\alpha 2 \sin \theta \, d\theta = 2\pi a^2 (1 - \cos \alpha).$$

Concurrently Archimedes finds the volumes of a sphere and segment of a sphere. He uses the same inscribed and circumscribed figures, and the procedure is equivalent to multiplying the above formulae by $\frac{1}{3}a$ throughout. Other "integrations" effected by Archimedes are the volume of a segment of a paraboloid of revolution, the volume of a segment of a hyperboloid of revolution, the volume of a segment of a spheroid, the area of a spiral and the area of a segment of a parabola. He also finds the area of an ellipse, but not by a method equivalent to integration. The subject is fully treated by Heath, *The Works of Archimedes*, pp. cxlii-cliv, to whom I am much indebted in writing this note.

μείζονα πρὸς τὴν ἐλάσσονα ἐλάσσονα λόγον τοῦ, ὃν ἔχει ἡ σφαῖρα πρὸς τὸν Ξ κῶνον. ἔστωσαν οὖν αἱ Κ, Η, αἱ δὲ Ι, Θ εἰλημμέναι, ὥστε τῷ ἴσῳ ἀλλήλων ὑπερέχειν τὴν Κ τῆς Ι καὶ τὴν Ι τῆς Θ καὶ τὴν Θ τῆς Η, νοείσθω δὲ καὶ εἰς τὸν ΑΒΓΔ κύκλον ἐγγεγραμμένον πολύγωνον, οὗ τὸ πλῆθος τῶν πλευρῶν μετρείσθω ὑπὸ τετράδος, καὶ ἄλλο περιγεγραμμένον ὅμοιον τῷ ἐγγεγραμμένῳ, καθάπερ ἐπὶ τῶν πρότερον, ἡ δὲ τοῦ περιγεγραμμένου πολυγώνου πλευρὰ πρὸς τὴν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἐχέτω τοῦ, ὃν ἔχει ἡ Κ πρὸς Ι, καὶ ἔστωσαν αἱ ΑΓ, ΒΔ διάμετροι πρὸς ὀρθὰς ἀλλήλαις. εἰ οὖν μενούσης τῆς ΑΓ διαμέτρου περιενεχθείη τὸ ἐπίπεδον, ἐν ᾧ τὰ πολύγωνα, ἔσται σχήματα[1] τὸ μὲν ἐγγεγραμμένον ἐν τῇ σφαίρᾳ, τὸ δὲ περιγεγραμμένον, καὶ ἕξει τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον τριπλασίονα λόγον ἤπερ ἡ πλευρὰ τοῦ περιγεγραμμένου πρὸς τὴν τοῦ ἐγγεγραμμένου εἰς τὸν ΑΒΓΔ κύκλον. ἡ δὲ πλευρὰ πρὸς τὴν πλευρὰν ἐλάσσονα λόγον ἔχει ἤπερ ἡ Κ πρὸς τὴν Ι· ὥστε τὸ σχῆμα τὸ περιγεγραμμένον ἐλάσσονα λόγον ἔχει ἢ τριπλασίονα τοῦ Κ πρὸς Ι.

[1] σχήματα Heiberg, τὸ σχῆμα codd.

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[a] Eutocius supplies a proof on these lines. Let the lengths of K, I, Θ, H be a, b, c, d. Then $a-b=b-c=c-d$, and it is required to prove that $a : d > a^3 : b^3$.

Take x such that $\quad a : b = b : x$.
Then $\quad a-b : a = b-x : b$,
and since $a>b$, $\quad a-b > b-x$.
But, by hypothesis, $\quad a-b = b-c$.
Therefore $\quad b-c > b-x$,
and so $\quad x > c$.

the greater will have to the less a less ratio than that which the sphere has to the cone Ξ. Therefore let the straight lines K, H, and the straight lines I, Θ, be so taken that K exceeds I, and I exceeds Θ and Θ exceeds H by an equal quantity; let there be imagined inscribed in the circle ΑΒΓΔ a polygon the number of whose sides is divisible by four; let another be circumscribed similar to that inscribed so that, as before, the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that K : I; and let ΑΓ, ΒΔ be diameters at right angles. Then if, while the diameter ΑΓ remains stationary, the surface in which the polygons lie be revolved, there will result two [solid] figures, one inscribed in the sphere and the other circumscribed, and the circumscribed figure will have to the inscribed the triplicate ratio of that which the side of the circumscribed figure has to the side of the figure inscribed in the circle ΑΒΓΔ [Prop. 32]. But the ratio of the one side to the other is less than K : I [*ex hypothesi*]; and so the circumscribed figure has [to the inscribed] a ratio less than $K^3 : I^3$. But [a] $K : H > K^3 : I^3$; by much more there-

Again, take y such that $b : x = x : y$.
Then, as before $b - x > x - y$.
Therefore, *a fortiori*, $b - e > x - y$.
But, by hypothesis, $b - e = e - d$.
Therefore $e - d > x - y$.
But $x > e$,
and so $y > d$.
But, by hypothesis, $a : b = b : x = x : y$.
$a : y = a^3 : b^3$ [Eucl. v. Def. 10, also vol. i. p. 258 n. *b*.
Therefore $a : d > a^3 : b^3$.

ἔχει δὲ καὶ ἡ Κ πρὸς Η μείζονα λόγον ἢ τριπλάσιον τοῦ, ὃν ἔχει ἡ Κ πρὸς Ι [τοῦτο γὰρ φανερὸν διὰ λημμάτων][1]· πολλῷ ἄρα τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχει τοῦ, ὃν ἔχει ἡ Κ πρὸς Η. ἡ δὲ Κ πρὸς Η ἐλάσσονα λόγον ἔχει ἤπερ ἡ σφαῖρα πρὸς τὸν Ξ κῶνον· καὶ ἐναλλάξ· ὅπερ ἀδύνατον· τὸ γὰρ σχῆμα τὸ περιγεγραμμένον μεῖζόν ἐστι τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον ἔλασσον τοῦ Ξ κώνου [διότι ὁ μὲν Ξ κῶνος τετραπλάσιός ἐστι τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος ἴσην τῷ ΑΒΓΔ κύκλῳ, ὕψος δὲ ἴσον τῇ ἐκ τοῦ κέντρου τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον σχῆμα ἔλασσον τοῦ εἰρημένου κώνου ἢ τετραπλάσιον].[2] οὐκ ἄρα μείζων ἢ τετραπλασία ἡ σφαῖρα τοῦ εἰρημένου.

Ἔστω, εἰ δυνατόν, ἐλάσσων ἢ τετραπλασία· ὥστε ἐλάσσων ἐστὶν ἡ σφαῖρα τοῦ Ξ κώνου. εἰλήφθωσαν δὴ αἱ Κ, Η εὐθεῖαι, ὥστε τὴν Κ μείζονα εἶναι τῆς Η καὶ ἐλάσσονα λόγον ἔχειν πρὸς αὐτὴν τοῦ, ὃν ἔχει ὁ Ξ κῶνος πρὸς τὴν σφαῖραν, καὶ αἱ Θ, Ι ἐκκείσθωσαν, καθὼς πρότερον, καὶ εἰς τὸν ΑΒΓΔ κύκλον νοείσθω πολύγωνον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὴν πλευρὰν τοῦ περιγεγραμμένου πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἔχειν ἤπερ ἡ Κ πρὸς Ι, καὶ τὰ ἄλλα κατεσκευασμένα τὸν αὐτὸν τρόπον τοῖς πρότερον· ἕξει ἄρα καὶ τὸ περιγεγραμμένον στερεὸν σχῆμα πρὸς τὸ ἐγγεγραμμένον τριπλασίονα λόγον ἤπερ ἡ πλευρὰ τοῦ περιγεγραμμένου περὶ τὸν ΑΒΓΔ κύκλον πρὸς τὴν τοῦ ἐγγεγραμμένου. ἡ δὲ πλευρὰ πρὸς τὴν πλευρὰν ἐλάσσονα λόγον ἔχει

fore the circumscribed figure has to the inscribed a ratio less than K : H. But K : H is a ratio less than that which the sphere has to the cone Ξ [*ex hypothesi*]; [therefore the circumscribed figure has to the inscribed a ratio less than that which the sphere has to the cone Ξ]; and *permutando*, [the circumscribed figure has to the sphere a ratio less than that which the inscribed figure has to the cone][a]; which is impossible; for the circumscribed figure is greater than the sphere [Prop. 28], but the inscribed figure is less than the cone Ξ [Prop. 27]. Therefore the sphere is not greater than four times the aforesaid cone.

Let it be, if possible, less than four times, so that the sphere is less than the cone Ξ. Let the straight lines K, H be so taken that K is greater than H and K : H is a ratio less than that which the cone Ξ has to the sphere [Prop. 2]; let the straight lines Θ, I be placed as before; let there be imagined in the circle ΑΒΓΔ one polygon inscribed and another circumscribed, so that the side of the circumscribed figure has to the side of the inscribed a ratio less than K : I; and let the other details in the construction be done as before. Then the circumscribed solid figure will have to the inscribed the triplicate ratio of that which the side of the figure circumscribed about the circle ΑΒΓΔ has to the side of the inscribed figure [Prop. 32]. But the ratio of the sides

[a] A marginal note in one MS. gives these words, which Archimedes would not have omitted.

[1] τοῦτο . . . λημμάτων om. Heiberg.
[2] διότι . . . τετραπλάσιον om. Heiberg.

ἥπερ ἡ Κ πρὸς Ι· ἕξει οὖν τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἢ τριπλάσιον τοῦ, ὃν ἔχει ἡ Κ πρὸς τὴν Ι. ἡ δὲ Κ πρὸς τὴν Η μείζονα λόγον ἔχει ἢ τριπλάσιον τοῦ, ὃν ἔχει ἡ Κ πρὸς τὴν Ι· ὥστε ἐλάσσονα λόγον ἔχει τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἢ ἡ Κ πρὸς τὴν Η. ἡ δὲ Κ πρὸς τὴν Η ἐλάσσονα λόγον ἔχει ἢ ὁ Ξ κῶνος πρὸς τὴν σφαῖραν· ὅπερ ἀδύνατον· τὸ μὲν γὰρ ἐγγεγραμμένον ἔλασσόν ἐστι τῆς σφαίρας, τὸ δὲ περιγεγραμμένον μεῖζον τοῦ Ξ κώνου. οὐκ ἄρα οὐδὲ ἐλάσσων ἐστὶν ἢ τετραπλασία ἡ σφαῖρα τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος ἴσην τῷ ΑΒΓΔ κύκλῳ, ὕψος δὲ τὴν ἴσην τῇ ἐκ τοῦ κέντρου τῆς σφαίρας. ἐδείχθη δέ, ὅτι οὐδὲ μείζων· τετραπλασία ἄρα.

[Πόρισμα][1]

Προδεδειγμένων δὲ τούτων φανερόν, ὅτι πᾶς κύλινδρος βάσιν μὲν ἔχων τὸν μέγιστον κύκλον τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ ἴσον τῇ διαμέτρῳ τῆς σφαίρας, ἡμιόλιός ἐστι τῆς σφαίρας καὶ ἡ ἐπιφάνεια αὐτοῦ μετὰ τῶν βάσεων ἡμιολία τῆς ἐπιφανείας τῆς σφαίρας.

Ὁ μὲν γὰρ κύλινδρος ὁ προειρημένος ἑξαπλάσιός ἐστι τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος τὴν αὐτήν, ὕψος δὲ ἴσον τῇ ἐκ τοῦ κέντρου, ἡ δὲ σφαῖρα δέδεικται τοῦ αὐτοῦ κώνου τετραπλασία οὖσα· δῆλον οὖν, ὅτι ὁ κύλινδρος ἡμιόλιός ἐστι τῆς σφαίρας. πάλιν, ἐπεὶ ἡ ἐπιφάνεια τοῦ κυλίνδρου χωρὶς τῶν βάσεων ἴση δέδεικται κύκλῳ, οὗ ἡ ἐκ

[1] πόρισμα. The title is not found in some mss.

is less than K : I [*ex hypothesi*]; therefore the circumscribed figure has to the inscribed a ratio less than $K^3 : I^3$. But $K : H > K^3 : I^3$; and so the circumscribed figure has to the inscribed a ratio less than K : H. But K : H is a ratio less than that which the cone Ξ has to the sphere [*ex hypothesi*]; [therefore the circumscribed figure has to the inscribed a ratio less than that which the cone Ξ has to the sphere] [a]; which is impossible; for the inscribed figure is less than the sphere [Prop. 28], but the circumscribed figure is greater than the cone Ξ [Prop. 31, coroll.]. Therefore the sphere is not less than four times the cone having its base equal to the circle ΑΒΓΔ, and height equal to the radius of the sphere. But it was proved that it cannot be greater; therefore it is four times as great.

[COROLLARY]

From what has been proved above it is clear that *any cylinder having for its base the greatest of the circles in the sphere, and having its height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface including the bases is one-and-a-half times the surface of the sphere.*

For the aforesaid cylinder is six times the cone having the same basis and height equal to the radius [from Eucl. xii. 10], while the sphere was proved to be four times the same cone [Prop. 34]. It is obvious therefore that the cylinder is one-and-a-half times the sphere. Again, since the surface of the cylinder excluding the bases has been proved equal to a circle

[a] These words, which Archimedes would not have omitted, are given in a marginal note to one MS.

τοῦ κέντρου μέση ἀνάλογόν ἐστι τῆς τοῦ κυλίνδρου πλευρᾶς καὶ τῆς διαμέτρου τῆς βάσεως, τοῦ δὲ εἰρημένου κυλίνδρου τοῦ περὶ τὴν σφαῖραν ἡ πλευρὰ ἴση ἐστὶ τῇ διαμέτρῳ τῆς βάσεως [δῆλον, ὅτι ἡ μέση αὐτῶν ἀνάλογον ἴση γίνεται τῇ διαμέτρῳ τῆς βάσεως],[1] ὁ δὲ κύκλος ὁ τὴν ἐκ τοῦ κέντρου ἔχων ἴσην τῇ διαμέτρῳ τῆς βάσεως τετραπλάσιός ἐστι τῆς βάσεως, τουτέστι τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ, ἔσται ἄρα καὶ ἡ ἐπιφάνεια τοῦ κυλίνδρου χωρὶς τῶν βάσεων τετραπλασία τοῦ μεγίστου κύκλου· ὅλη ἄρα μετὰ τῶν βάσεων ἡ ἐπιφάνεια τοῦ κυλίνδρου ἑξαπλασία ἔσται τοῦ μεγίστου κύκλου. ἔστιν δὲ καὶ ἡ τῆς σφαίρας ἐπιφάνεια τετραπλασία τοῦ μεγίστου κύκλου. ὅλη ἄρα ἡ ἐπιφάνεια τοῦ κυλίνδρου ἡμιολία ἐστὶ τῆς ἐπιφανείας τῆς σφαίρας.

(c) Solution of a Cubic Equation

Archim. *De Sphaera et Cyl.* ii., Prop. 4, Archim. ed. Heiberg i. 186. 15–192. 6

Τὴν δοθεῖσαν σφαῖραν τεμεῖν, ὥστε τὰ τμήματα τῆς σφαίρας πρὸς ἄλληλα λόγον ἔχειν τὸν αὐτὸν τῷ δοθέντι.

[1] δῆλον . . . βάσεως om. Heiberg.

[a] As the geometrical form of proof is rather diffuse, and may conceal from the casual reader the underlying nature of the operation, it may be as well to state at the outset the various stages of the proof. The problem is to cut a given sphere by a plane so that the segments shall have a given ratio, and the stages are :

(*a*) Analysis of this main problem in which it is reduced to a particular case of the general problem, " so to cut a given straight line ΔZ at X that XZ bears to the given

whose radius is a mean proportional between the side of the cylinder and the diameter of the base [Prop. 13], and the side of the aforementioned cylinder circumscribing the sphere is equal to the diameter of the base, while the circle having its radius equal to the diameter of the base is four times the base [Eucl. xii. 2], that is to say, four times the greatest of the circles in the sphere, therefore the surface of the cylinder excluding the bases is four times the greatest circle; therefore the whole surface of the cylinder, including the bases, is six times the greatest circle. But the surface of the sphere is four times the greatest circle. Therefore the whole surface of the cylinder is one-and-a-half times the surface of the sphere.

(*c*) Solution of a Cubic Equation

Archimedes, *On the Sphere and Cylinder* ii., Prop. 4, Archim. ed. Heiberg i. 186. 15–192. 6

To cut a given sphere, so that the segments of the sphere shall have, one towards the other, a given ratio.[a]

straight line the same ratio as a given area bears to the square on ΔX "; in algebraical notation, to solve the equation

$$\frac{a-x}{b}=\frac{c^2}{x^2}, \text{ or } x^2(a-x)=bc^2.$$

(*b*) Analysis of this general problem, in which it is shown that the required point can be found as the intersection of a parabola [$ax^2=c^2y$] and a hyperbola [$(a-x)y=ab$]. It is stated, for the time being without proof, that $x^2(a-x)$ is greatest when $x=\frac{2}{3}a$; in other words, that for a real solution $bc^2 \ngtr \frac{4}{27}a^3$.

(*c*) Synthesis of this general problem, according as bc^2 is greater than, equal to, or less than $\frac{4}{27}a^3$. If it be greater, there is no real solution; if equal, there is one real solution; if less, there are two real solutions.

(*d*) Proof that $x^2(a-x)$ is greatest when $x=\frac{2}{3}a$, deferred

Ἔστω ἡ δοθεῖσα σφαῖρα ἡ ΑΒΓΔ· δεῖ δὴ αὐτὴν τεμεῖν ἐπιπέδῳ, ὥστε τὰ τμήματα τῆς σφαίρας πρὸς ἄλληλα λόγον ἔχειν τὸν δοθέντα.

Τετμήσθω διὰ τῆς ΑΓ ἐπιπέδῳ· λόγος ἄρα τοῦ ΑΔΓ τμήματος τῆς σφαίρας πρὸς τὸ ΑΒΓ τμῆμα τῆς σφαίρας δοθείς. τετμήσθω δὲ ἡ σφαῖρα διὰ τοῦ κέντρου, καὶ ἔστω ἡ τομὴ μέγιστος κύκλος ὁ ΑΒΓΔ, κέντρον δὲ τὸ Κ καὶ διάμετρος ἡ ΔΒ, καὶ πεποιήσθω, ὡς μὲν συναμφότερος ἡ ΚΔΧ πρὸς ΔΧ, οὕτως ἡ ΡΧ πρὸς ΧΒ, ὡς δὲ συναμφότερος ἡ ΚΒΧ πρὸς ΒΧ, οὕτως ἡ ΛΧ πρὸς ΧΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΛ, ΛΓ, ΑΡ, ΡΓ· ἴσος ἄρα ἐστὶν ὁ μὲν ΑΛΓ κῶνος τῷ ΑΔΓ τμήματι τῆς σφαίρας, ὁ δὲ ΑΡΓ τῷ ΑΒΓ· λόγος ἄρα καὶ τοῦ ΑΛΓ κώνου πρὸς τὸν ΑΡΓ κῶνον δοθείς. ὡς δὲ ὁ κῶνος πρὸς τὸν κῶνον, οὕτως ἡ ΛΧ πρὸς ΧΡ [ἐπείπερ τὴν αὐτὴν βάσιν ἔχουσιν τὸν περὶ διάμετρον τὴν ΑΓ κύκλον][1]· λόγος ἄρα καὶ τῆς ΛΧ πρὸς ΧΡ δοθείς. καὶ διὰ ταὐτὰ τοῖς πρό-

[1] ἐπείπερ . . . κύκλον om. Heiberg.

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in (*b*). This is done in two parts, by showing that (1) if x has any value less than $\frac{2}{3}a$, (2) if x has any value greater than $\frac{2}{3}a$, then $x^2(a-x)$ has a smaller value than when $x=\frac{2}{3}a$.

(*e*) Proof that, if $bc^2 < \frac{4}{27}a^3$, there are always two real solutions.

(*f*) Proof that, in the particular case of the general problem to which Archimedes has reduced his original problem, there is always a real solution.

(*g*) Synthesis of the original problem.

Of these stages, (*a*) and (*g*) alone are found in our texts of Archimedes; but Eutocius found stages (*b*)-(*d*) in an old book, which he took to be the work of Archimedes; and he added stages (*e*) and (*f*) himself. When it is considered that all these stages are traversed by rigorous geometrical

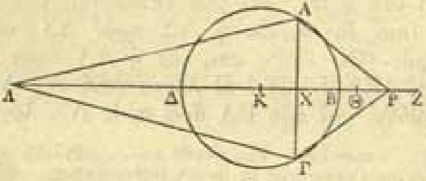
Let ΑΒΓΔ be the given sphere; it is required so to cut it by a plane that the segments of the sphere shall have, one towards the other, the given ratio.

Let it be cut by the plane ΑΓ; then the ratio of the segment ΑΔΓ of the sphere to the segment ΑΒΓ of the sphere is given. Now let the sphere be cut through the centre [by a plane perpendicular to the plane through ΑΓ], and let the section be the great circle ΑΒΓΔ of centre Κ and diameter ΔΒ, and let [Λ, Ρ be taken on ΒΔ produced in either direction so that]

$$\text{Κ}\Delta + \Delta\text{Χ} : \Delta\text{Χ} = \text{ΡΧ} : \text{ΧΒ},$$

$$\text{ΚΒ} + \text{ΒΧ} : \text{ΒΧ} = \Lambda\text{Χ} : \text{Χ}\Delta,$$

and let ΑΛ, ΛΓ, ΑΡ, ΡΓ be joined; then the cone ΛΑΓ is equal to the segment ΑΔΓ of the sphere, and



the cone ΑΡΓ to the segment ΑΒΓ [Prop. 2]; therefore the ratio of the cone ΛΑΓ to the cone ΑΡΓ is given. But cone ΛΑΓ : cone ΑΡΓ = ΛΧ : ΧΡ.[a] Therefore the ratio ΛΧ : ΧΡ is given. And in the

methods, the solution must be admitted a veritable *tour de force.* It is strictly analogous to the modern method of solving a cubic equation, but the concept of a cubic equation did not, of course, come within the purview of the ancient mathematicians.

[a] Since they have the same base.

τερον διὰ τῆς κατασκευῆς, ὡς ἡ ΛΔ πρὸς ΚΔ, ἡ ΚΒ πρὸς ΒΡ καὶ ἡ ΔΧ πρὸς ΧΒ. καὶ ἐπεί ἐστιν, ὡς ἡ ΡΒ πρὸς ΒΚ, ἡ ΚΔ πρὸς ΛΔ, συνθέντι, ὡς ἡ ΡΚ πρὸς ΚΒ, τουτέστι πρὸς ΚΔ, οὕτως ἡ ΚΛ πρὸς ΛΔ· καὶ ὅλη ἄρα ἡ ΡΛ πρὸς ὅλην τὴν ΚΛ ἐστιν, ὡς ἡ ΚΛ πρὸς ΛΔ. ἴσον ἄρα τὸ ὑπὸ τῶν ΡΛΔ τῷ ἀπὸ ΛΚ. ὡς ἄρα ἡ ΡΛ πρὸς ΛΔ, τὸ ἀπὸ ΚΛ πρὸς τὸ ἀπὸ ΛΔ. καὶ ἐπεί ἐστιν, ὡς ἡ ΛΔ πρὸς ΔΚ, οὕτως ἡ ΔΧ πρὸς ΧΒ, ἔσται ἀνάπαλιν καὶ συνθέντι, ὡς ἡ ΚΛ πρὸς ΛΔ, οὕτως ἡ ΒΔ πρὸς ΔΧ [καὶ ὡς ἄρα τὸ ἀπὸ ΚΛ πρὸς τὸ ἀπὸ ΛΔ, οὕτως τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ. πάλιν, ἐπεί ἐστιν, ὡς ἡ ΛΧ πρὸς ΔΧ, συναμφότερος ἡ ΚΒ, ΒΧ πρὸς ΒΧ, διελόντι, ὡς ἡ ΛΔ πρὸς ΔΧ, οὕτως ἡ ΚΒ πρὸς ΒΧ].[1] καὶ κείσθω τῇ ΚΒ ἴση ἡ ΒΖ· ὅτι γὰρ ἐκτὸς τοῦ Ρ πεσεῖται, δῆλον [καὶ ἔσται, ὡς ἡ ΛΔ πρὸς ΔΧ, οὕτως ἡ ΖΒ πρὸς ΒΧ· ὥστε καί, ὡς ἡ ΔΛ πρὸς ΛΧ, ἡ ΒΖ πρὸς ΖΧ].[2] ἐπεὶ δὲ λόγος ἐστὶ τῆς ΔΛ πρὸς ΛΧ δοθείς, καὶ τῆς ΡΛ ἄρα πρὸς ΛΧ λόγος ἐστὶ

[1] *καὶ . . . πρὸς* ΒΧ. The words *καὶ . . . ἀπὸ* ΔΧ are shown by Eutocius's comment to be an interpolation. The words *πάλιν . . . πρὸς* ΒΧ and *καὶ . . . πρὸς* ΖΧ must also be interpolated, as, in order to prove that ΔΛ : ΛΧ is given, Eutocius first proves that ΒΖ : ΖΧ = ΛΔ : ΛΧ, which he would hardly have done if Archimedes had himself provided the proof.

[2] *καὶ . . . πρὸς* ΖΧ ; *v.* preceding note.

[a] This is proved by Eutocius thus :

Since $\mathrm{K}\Delta + \Delta\mathrm{X} : \Delta\mathrm{X} = \mathrm{PX} : \mathrm{XB}$,

dirimendo, $\mathrm{K}\Delta : \Delta\mathrm{X} = \mathrm{PB} : \mathrm{BX}$,

and *permutando*, $\mathrm{K}\Delta : \mathrm{BP} = \Delta\mathrm{X} : \mathrm{XB}$,

i.e., $\mathrm{KB} : \mathrm{BP} = \Delta\mathrm{X} : \mathrm{XB}$.

Again, since $\mathrm{KB} + \mathrm{BX} : \mathrm{XB} = \Lambda\mathrm{X} : \mathrm{X}\Delta$,

same way as in a previous proposition [Prop. 2], by construction,

$$\Lambda\Delta : K\Delta = KB : BP = \Delta X : XB.$$[a]

And since $PB : BK = K\Delta : \Lambda\Delta$, [Eucl. v. 7, coroll.

componendo, $PK : KB = K\Lambda : \Lambda\Delta$, [Eucl. v. 18

i.e., $PK : K\Delta = K\Lambda : \Lambda\Delta$.

$\therefore$ $P\Lambda : K\Lambda = K\Lambda : \Lambda\Delta$. [Eucl. v. 12

$\therefore$ $P\Lambda \cdot \Lambda\Delta = \Lambda K^2$. [Eucl. vi. 17

$\therefore$ $P\Lambda : \Lambda\Delta = K\Lambda^2 : \Lambda\Delta^2$.

And since $\Lambda\Delta : \Delta K = \Delta X : XB$,

invertendo et componendo, $K\Lambda : \Lambda\Delta = B\Delta : \Delta X$. [Eucl. v. 7, coroll. and v. 18

Let BZ be placed equal to KB. It is plain that [Z] will fall beyond P.[b] Since the ratio $\Delta\Lambda : \Lambda X$ is given, therefore the ratio $P\Lambda : \Lambda X$ is given.[c] Then,

dirimendo et permutando $\Delta X : XB = \Lambda\Delta : \Delta K$.

Now $\Delta X : XB = KB : BP$.

Therefore $\Lambda\Delta : \Delta K = \Delta X : XB = KB : BP$.

[b] Since $X\Delta : XB = KB : BP$, and $\Delta X > XB$, $\therefore$ $KB > BP$. $\therefore$ $BZ > BP$.

[c] As Eutocius's note shows, what Archimedes wrote was: "Since the ratio $\Delta\Lambda : \Lambda X$ is given, and the ratio $P\Lambda : \Lambda X$, therefore the ratio $P\Lambda : \Lambda\Delta$ is also given." Eutocius's proof is:

Since $KB + BX : BX = \Lambda X : X\Delta$,

$\therefore$ $ZX : XB = \Lambda X : X\Delta$;

$\therefore$ $XZ : ZB = X\Lambda : \Lambda\Delta$:

$\therefore$ $BZ : ZX = \Lambda\Delta : \Lambda X$.

But the ratio $BZ : ZX$ is given because ZB is equal to the radius of the given sphere and BX is given. Therefore $\Lambda\Delta : \Lambda X$ is given.

Again, since the ratio of the segments is given, the ratio of

δοθείς. ἐπεὶ οὖν ὁ τῆς ΡΛ πρὸς ΛΧ λόγος συνῆπται ἔκ τε τοῦ, ὃν ἔχει ἡ ΡΛ πρὸς ΛΔ, καὶ ἡ ΔΛ πρὸς ΛΧ, ἀλλ' ὡς μὲν ἡ ΡΛ πρὸς ΛΔ, τὸ ἀπὸ ΔΒ πρὸς τὸ ἀπὸ ΔΧ, ὡς δὲ ἡ ΔΛ πρὸς ΛΧ, οὕτως ἡ ΒΖ πρὸς ΖΧ, ὁ ἄρα τῆς ΡΛ πρὸς ΛΧ λόγος συνῆπται ἔκ τε τοῦ, ὃν ἔχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ΖΧ. πεποιήσθω δέ, ὡς ἡ ΡΛ πρὸς ΛΧ, ἡ ΒΖ πρὸς ΖΘ· λόγος δὲ τῆς ΡΛ πρὸς ΛΧ δοθείς· λόγος ἄρα καὶ τῆς ΖΒ πρὸς ΖΘ δοθείς. δοθεῖσα δὲ ἡ ΒΖ—ἴση γάρ ἐστι τῇ ἐκ τοῦ κέντρου· δοθεῖσα ἄρα καὶ ἡ ΖΘ. καὶ ὁ τῆς ΒΖ ἄρα λόγος πρὸς ΖΘ συνῆπται ἔκ τε τοῦ, ὃν ἔχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ΖΧ. ἀλλ' ὁ ΒΖ πρὸς ΖΘ λόγος συνῆπται ἔκ τε τοῦ τῆς ΒΖ πρὸς ΖΧ καὶ τοῦ τῆς ΖΧ πρὸς ΖΘ [κοινὸς ἀφῃρήσθω ὁ τῆς ΒΖ πρὸς ΖΧ][1]· λοιπὸν ἄρα ἐστίν, ὡς τὸ ἀπὸ ΒΔ, τουτέστι δοθέν, πρὸς τὸ ἀπὸ ΔΧ, οὕτως ἡ ΧΖ πρὸς ΖΘ, τουτέστι πρὸς δοθέν. καί ἐστιν δοθεῖσα ἡ ΖΔ εὐθεῖα· εὐθεῖαν ἄρα δοθεῖσαν τὴν ΔΖ τεμεῖν δεῖ κατὰ τὸ Χ καὶ ποιεῖν, ὡς τὴν ΧΖ πρὸς δοθεῖσαν [τὴν ΖΘ],[2] οὕτως τὸ δοθὲν [τὸ ἀπὸ ΒΔ][2] πρὸς τὸ ἀπὸ ΔΧ. τοῦτο οὕτως ἁπλῶς μὲν λεγόμενον ἔχει διορισμόν,

[1] κοινὸς . . . πρὸς ΖΧ. Eutocius's comment shows that these words are interpolated.

[2] τὴν ΖΘ, τὸ ἀπὸ ΒΔ. Eutocius's comments show these words to be glosses.

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the cones ΑΛΓ, ΑΡΓ is also given, and therefore the ratio ΛΧ : ΧΡ. Therefore the ratio ΡΛ : ΛΧ is given. Since the ratios ΡΛ : ΛΧ and ΛΔ : ΛΧ are given, it follows that the ratio ΡΛ : ΛΔ is given.

since the ratio $P\Lambda : \Lambda X$ is composed of the ratios $P\Lambda : \Lambda\Delta$ and $\Delta\Lambda : \Lambda X$,

and since $$P\Lambda : \Lambda\Delta = \Delta B^2 : \Delta X^2,^a$$
$$\Delta\Lambda : \Lambda X = BZ : ZX,$$

therefore the ratio $P\Lambda : \Lambda X$ is composed of the ratios $B\Delta^2 : \Delta X^2$ and $BZ : ZX$. Let [Θ be chosen so that]

$$P\Lambda : \Lambda X = BZ : Z\Theta.$$

Now the ratio $P\Lambda : \Lambda X$ is given; therefore the ratio $ZB : Z\Theta$ is given. Now BZ is given—for it is equal to the radius; therefore $Z\Theta$ is also given. Therefore[b] the ratio $BZ : Z\Theta$ is composed of the ratios $B\Delta^2 : \Delta X^2$ and $BZ : ZX$. But the ratio $BZ : Z\Theta$ is composed of the ratios $BZ : ZX$ and $ZX : Z\Theta$. Therefore, the remainder[c] $B\Delta^2 : \Delta X^2 = XZ : Z\Theta$, in which $B\Delta^2$ and $Z\Theta$ are given. And the straight line $Z\Delta$ is given; therefore *it is required so to cut the given straight line ΔZ at X that XZ bears to a given straight line the same ratio as a given area bears to the square on ΔX.* When the problem is stated in this general form,[d] it is necessary to investigate the limits of possibility,

[a] For $$P\Lambda : \Lambda\Delta = \Lambda K^2 : \Delta\Lambda^2$$
$$= B\Delta^2 : \Delta X^2.$$

[b] "Therefore" refers to the last equation.

[c] *i.e.* the remainder in the process given fully by Eutocius as follows:

$$(B\Delta^2 : \Delta X^2) \, . \, (BZ : ZX) = BZ : \Theta Z = (BZ : ZX) \, . \, (XZ : Z\Theta).$$

Removing the common element $BZ : ZX$ from the extreme terms, we find that the remainder $B\Delta^2 : \Delta X^2 = XZ : Z\Theta$.

[d] In algebraic notation, if $\Delta X = x$ and $\Delta Z = a$, while the given straight line is b and the given area is c^2, then

$$\frac{a-x}{b} = \frac{c^2}{x^2},$$

or $$x^2(a-x) = bc^2.$$

προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε ὑπαρχόντων [τουτέστι τοῦ τε διπλασίαν εἶναι τὴν ΔΒ τῆς ΒΖ καὶ τοῦ μείζονα τῆς ΖΘ τὴν ΖΒ, ὡς κατὰ τὴν ἀνάλυσιν][1] οὐκ ἔχει διορισμόν· καὶ ἔσται τὸ πρόβλημα τοιοῦτον· δύο δοθεισῶν εὐθειῶν τῶν ΒΔ, ΒΖ καὶ διπλασίας οὔσης τῆς ΒΔ τῆς ΒΖ καὶ σημείου ἐπὶ τῆς ΒΖ τοῦ Θ τεμεῖν τὴν ΔΒ κατὰ τὸ Χ καὶ ποιεῖν, ὡς τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, τὴν ΧΖ πρὸς ΖΘ· ἑκάτερα δὲ ταῦτα ἐπὶ τέλει ἀναλυθήσεταί τε καὶ συντεθήσεται.

Eutoc. *Comm. in Archim. De Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 130. 17–150. 22

Ἐπὶ τέλει μὲν τὸ προρηθὲν ἐπηγγείλατο δεῖξαι, ἐν οὐδενὶ δὲ τῶν ἀντιγράφων εὑρεῖν ἔνεστι τὸ ἐπάγγελμα. ὅθεν καὶ Διονυσόδωρον μὲν εὑρίσκομεν μὴ τῶν αὐτῶν ἐπιτυχόντα, ἀδυνατήσαντα δὲ ἐπιβαλεῖν τῷ καταλειφθέντι λήμματι, ἐφ' ἑτέραν ὁδὸν τοῦ ὅλου προβλήματος ἐλθεῖν, ἥντινα ἑξῆς γράψομεν· Διοκλῆς μέντοι καὶ αὐτὸς ἐν τῷ Περὶ πυρίων αὐτῷ συγγεγραμμένῳ βιβλίῳ ἐπηγγέλθαι νομίζων τὸν Ἀρχιμήδη, μὴ πεποιηκέναι δὲ τὸ ἐπάγγελμα, αὐτὸς ἀναπληροῦν ἐπεχείρησεν, καὶ τὸ ἐπιχείρημα ἑξῆς γράψομεν· ἔστιν γὰρ καὶ αὐτὸ οὐδένα μὲν ἔχον πρὸς τὰ παραλελειμμένα λόγον, ὁμοίως δὲ τῷ Διονυσοδώρῳ δι' ἑτέρας ἀποδείξεως κατασκευάζον τὸ πρόβλημα. ἔν τινι μέντοι παλαιῷ

[1] τουτέστι . . . ἀνάλυσιν. Eutocius's notes make it seem likely that these words are interpolated.

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[a] In the technical language of Greek mathematics, the

but under the conditions of the present case no such investigation is necessary.[a] In the present case the problem will be of this nature: *Given two straight lines* BΔ, BZ, *in which* BΔ = 2BZ, *and a point* Θ *upon* BZ, *so to cut* ΔB *at* X *that*

$$B\Delta^2 : \Delta X^2 = XZ : Z\Theta ;$$

and the analysis and synthesis of both problems will be given at the end.[b]

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 130. 17–150. 22

He promised that he would give at the end a proof of what is stated, but the fulfilment of the promise cannot be found in any of his extant writings. Dionysodorus also failed to light on it, and, being unable to tackle the omitted lemma, he approached the whole problem in an altogether different way, which I shall describe in due course. Diocles, indeed, in his work *On Burning Mirrors* maintained that Archimedes made the promise but had not fulfilled it, and he undertook to supply the omission himself, which attempt I shall also describe in its turn; it bears, however, no relation to the missing discussion, but, like that of Dionysodorus, it solves the problem by a construction reached by a different proof. But

general problem requires a *diorismos*, for which v. vol. i. p. 151 n. *b* and p. 396 n. *a*. In algebraic notation, there must be limiting conditions if the equation

$$x^2(a - x) = bc^2$$

is to have a real root lying between 0 and *a*.

[b] Having made this promise, Archimedes proceeded to give the formal synthesis of the problem which he had thus reduced.

βιβλίῳ—οὐδὲ γὰρ τῆς εἰς πολλὰ ζητήσεως ἀπέστημεν—ἐντετύχαμεν θεωρήμασι γεγραμμένοις οὐκ ὀλίγην μὲν τὴν ἐκ τῶν πταισμάτων ἔχουσιν ἀσάφειαν περί τε τὰς καταγραφὰς πολυτρόπως ἡμαρτημένοις, τῶν μέντοι ζητουμένων εἶχον τὴν ὑπόστασιν, ἐν μέρει δὲ τὴν Ἀρχιμήδει φίλην Δωρίδα γλῶσσαν ἀπέσωζον καὶ τοῖς συνήθεσι τῷ ἀρχαίῳ τῶν πραγμάτων ὀνόμασιν ἐγέγραπτο τῆς μὲν παραβολῆς ὀρθογωνίου κώνου τομῆς ὀνομαζομένης, τῆς δὲ ὑπερβολῆς ἀμβλυγωνίου κώνου τομῆς, ὡς ἐξ αὐτῶν διανοεῖσθαι, μὴ ἄρα καὶ αὐτὰ εἴη τὰ ἐν τῷ τέλει ἐπηγγελμένα γράφεσθαι. ὅθεν σπουδαιότερον ἐντυγχάνοντες αὐτὸ μὲν τὸ ῥητόν, ὡς γέγραπται, διὰ πλῆθος, ὡς εἴρηται, τῶν πταισμάτων δυσχερὲς εὑρόντες τὰς ἐννοίας κατὰ μικρὸν ἀποσυλήσαντες κοινοτέρᾳ καὶ σαφεστέρᾳ κατὰ τὸ δυνατὸν λέξει γράφομεν. καθόλου δὲ πρῶτον τὸ θεώρημα γραφήσεται, ἵνα τὸ λεγόμενον ὑπ' αὐτοῦ σαφηνισθῇ περὶ τῶν διορισμῶν· εἶτα καὶ τοῖς ἀναλελυμένοις ἐν τῷ προβλήματι προσαρμοσθήσεται.

"Εὐθείας δοθείσης τῆς ΑΒ καὶ ἑτέρας τῆς ΑΓ καὶ χωρίου τοῦ Δ προκείσθω λαβεῖν ἐπὶ τῆς ΑΒ σημεῖον ὡς τὸ Ε, ὥστε εἶναι, ὡς τὴν ΑΕ πρὸς ΑΓ, οὕτω τὸ Δ χωρίον πρὸς τὸ ἀπὸ ΕΒ.

"Γεγονέτω, καὶ κείσθω ἡ ΑΓ πρὸς ὀρθὰς τῇ ΑΒ, καὶ ἐπιζευχθεῖσα ἡ ΓΕ διήχθω ἐπὶ τὸ Ζ, καὶ ἤχθω διὰ τοῦ Γ τῇ ΑΒ παράλληλος ἡ ΓΗ, διὰ δὲ τοῦ Β τῇ ΑΓ παράλληλος ἡ ΖΒΗ συμπίπτουσα

in a certain ancient book—for I pursued the inquiry thoroughly—I came upon some theorems which, though far from clear owing to errors and to manifold faults in the diagrams, nevertheless gave the substance of what I sought, and furthermore preserved in part the Doric dialect beloved by Archimedes, while they kept the names favoured by ancient custom, the parabola being called a section of a right-angled cone and the hyperbola a section of an obtuse-angled cone; in short, I felt bound to consider whether these theorems might not be what he had promised to give at the end. For this reason I applied myself with closer attention, and, although it was difficult to get at the true text owing to the multitude of the mistakes already mentioned, gradually I routed out the meaning and now set it out, so far as I can, in more familiar and clearer language. In the first place the theorem will be treated generally, in order to make clear what he says about the limits of possibility; then will follow the special form it takes under the conditions of his analysis of the problem.

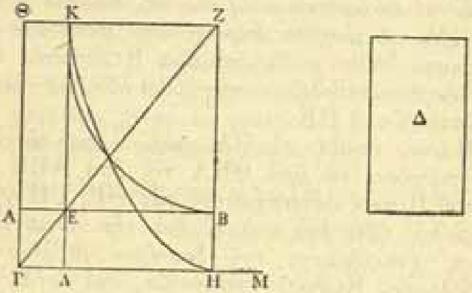
"Given a straight line AB and another straight line AΓ and an area Δ, let it be required to find a point E on AB such that $AE : A\Gamma = \Delta : EB^2$.

"Suppose it found, and let AΓ be at right angles to AB, and let ΓE be joined and produced to Z, and through Γ let ΓH be drawn parallel to AB, and through B let ZBH be drawn parallel to AΓ, meeting

ἑκατέρᾳ τῶν ΓΕ, ΓΗ, καὶ συμπεπληρώσθω τὸ ΗΘ παραλληλόγραμμον, καὶ διὰ τοῦ Ε ὁποτέρᾳ τῶν ΓΘ, ΗΖ παράλληλος ἤχθω ἡ ΚΕΛ, καὶ τῷ Δ ἴσον ἔστω τὸ ὑπὸ ΓΗΜ.

"Ἐπεὶ οὖν ἐστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, ὡς δὲ ἡ ΕΑ πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΖ, ὡς δὲ ἡ ΓΗ πρὸς ΗΖ, οὕτως τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗΖ, ὡς ἄρα τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗΖ, οὕτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, τουτέστι πρὸς τὸ ἀπὸ ΚΖ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ Δ, τουτέστι πρὸς τὸ ὑπὸ ΓΗΜ, οὕτως τὸ ὑπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗΜ, οὕτως ἡ ΓΗ πρὸς ΗΜ· καὶ ὡς ἄρα ἡ ΓΗ πρὸς ΗΜ, οὕτως τὸ ὑπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΜ, τῆς ΗΖ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΖ πρὸς τὸ ὑπὸ ΜΗΖ· ὡς ἄρα τὸ ὑπὸ ΓΗΖ πρὸς τὸ ὑπὸ ΜΗΖ, οὕτως τὸ ὑπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ· ἴσον ἄρα τὸ ὑπὸ ΜΗΖ τῷ ἀπὸ ΖΚ. ἐὰν ἄρα περὶ ἄξονα τὴν ΖΗ

both ΓE and ΓH, and let the parallelogram HΘ be completed, and through E let KEΛ be drawn parallel



to either ΓΘ or HZ, and let [M be taken so that]

$$\Gamma H \,.\, HM = \Delta.$$

"Then, since $EA : A\Gamma = \Delta : EB^2$ [*ex hyp.*

and $EA : A\Gamma = \Gamma H : HZ,$

and $\Gamma H : HZ = \Gamma H^2 : \Gamma H \,.\, HZ,$

$\therefore$ $\Gamma H^2 : \Gamma H \,.\, HZ = \Delta : EB^2$

$= \Delta : KZ^2;$

and, *permutando*, $\Gamma H^2 : \Delta$ $[= \Gamma H \,.\, HZ : ZK^2,]$

i.e., $\Gamma H^2 : \Gamma H \,.\, HM = \Gamma H \,.\, HZ : ZK^2.$

But $\Gamma H^2 : \Gamma H \,.\, HM = \Gamma H : HM;$

$\therefore$ $\Gamma H : HM = \Gamma H \,.\, HZ : ZK^2.$

But, by taking a common altitude HZ,

$$\Gamma H : HM = \Gamma H \,.\, HZ : MH \,.\, HZ;$$

$\therefore$ $\Gamma H \,.\, HZ : MH \,.\, HZ = \Gamma H \,.\, HZ : ZK^2;$

$\therefore$ $MH \,.\, HZ = ZK^2.$

γραφῇ διὰ τοῦ Η παραβολή, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΜ, ἥξει διὰ τοῦ Κ, καὶ ἔσται θέσει δεδομένη διὰ τὸ δεδομένην εἶναι τὴν ΗΜ τῷ μεγέθει περιέχουσαν μετὰ τῆς ΗΓ δεδομένης δοθὲν τὸ Δ· τὸ ἄρα Κ ἅπτεται θέσει δεδομένης παραβολῆς. γεγράφθω οὖν, ὡς εἴρηται, καὶ ἔστω ὡς ἡ ΗΚ.

"Πάλιν, ἐπειδὴ τὸ ΘΛ χωρίον ἴσον ἐστὶ τῷ ΓΒ, τουτέστι τὸ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΗ, ἐὰν διὰ τοῦ Β περὶ ἀσυμπτώτους τὰς ΘΓ, ΓΗ γραφῇ ὑπερβολή, ἥξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν τοῦ η' θεωρήματος τοῦ δευτέρου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων, καὶ ἔσται θέσει δεδομένη διὰ τὸ καὶ ἑκατέραν τῶν ΘΓ, ΓΗ, ἔτι μὴν καὶ τὸ Β τῇ θέσει δεδόσθαι. γεγράφθω, ὡς εἴρηται, καὶ ἔστω ὡς ἡ ΚΒ· τὸ ἄρα Κ ἅπτεται θέσει δεδομένης ὑπερβολῆς. ἥπτετο δὲ καὶ θέσει δεδομένης παραβολῆς· δέδοται ἄρα τὸ Κ. καί ἐστιν ἀπ' αὐτοῦ κάθετος ἡ ΚΕ ἐπὶ θέσει δεδομένην τὴν ΑΒ· δέδοται ἄρα τὸ Ε. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΕΑ πρὸς τὴν δοθεῖσαν τὴν ΑΓ, οὕτως δοθὲν τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, δύο στερεῶν, ὧν βάσεις τὸ ἀπὸ ΕΒ καὶ τὸ Δ, ὕψη δὲ αἱ ΕΑ, ΑΓ, ἀντιπεπόν-

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[a] Let ΑΒ $= a$, ΑΓ $= b$, and Δ $=$ ΓΗ . ΗΜ $= c^2$, so that ΗΜ $= \frac{c^2}{a}$. Then if ΗΓ be taken as the axis of x and ΗΖ as the axis of y, the equation of the parabola is

$$x^2 = \frac{c^2}{a}y,$$

and the equation of the hyperbola is

$$(a - x)y = ab.$$

Their points of intersection give solutions of the equation

$$x^2(a - x) = bc^2.$$

If, therefore, a parabola be drawn through H about the axis ZH with the parameter HM, it will pass through K [Apoll. *Con.* i. 11, converse], and it will be given in position because HM is given in magnitude [Eucl. *Data* 57], comprehending with the given straight line HΓ the given area Δ; therefore K lies on a parabola given in position. Let it then be drawn, as described, and let it be HK.

"Again, since the area $\Theta\Lambda = \Gamma B$ [Eucl. i. 43

i.e., $$\Theta K \,.\, K\Lambda = AB \,.\, BH,$$

if a hyperbola be drawn through B having ΘΓ, ΓH for asymptotes, it will pass through K by the converse to the eighth theorem of the second book of Apollonius's *Elements of Conics*, and it will be given in position because both the straight lines ΘΓ, ΓH, and also the point B, are given in position. Let it be drawn, as described, and let it be KB; therefore K lies on a hyperbola given in position. But it lies also on a parabola given in position; therefore K is given.[a] And KE is the perpendicular drawn from it to the straight line AB given in position; therefore E is given. Now since the ratio of EA to the given straight line AΓ is equal to the ratio of the given area Δ to the square on EB, we have two solids, whose bases are the square on EB and Δ and whose altitudes are EA, AΓ, and the bases are inversely pro-

to which, as already noted, Archimedes had reduced his problem. (*N.B.*—The axis of x is for convenience taken in a direction contrary to that which is usual; with the usual conventions, we should get slightly different equations.)

θασιν αἱ βάσεις τοῖς ὕψεσιν· ὥστε ἴσα ἐστὶ τὰ στερεά· τὸ ἄρα ἀπὸ ΕΒ ἐπὶ τὴν ΕΑ ἴσον ἐστὶ τῷ δοθέντι τῷ Δ ἐπὶ δοθεῖσαν τὴν ΓΑ. ἀλλὰ τὸ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ μέγιστόν ἐστι πάντων τῶν ὁμοίως λαμβανομένων ἐπὶ τῆς ΒΑ, ὅταν ᾖ διπλασία ἡ ΒΕ τῆς ΕΑ, ὡς δειχθήσεται· δεῖ ἄρα τὸ δοθὲν ἐπὶ τὴν δοθεῖσαν μὴ μεῖζον εἶναι τοῦ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ.

"Συντεθήσεται δὲ οὕτως· ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, ἄλλη δέ τις δοθεῖσα ἡ ΑΓ, τὸ δὲ δοθὲν χωρίον τὸ Δ, καὶ δέον ἔστω τεμεῖν τὴν ΑΒ, ὥστε εἶναι, ὡς τὸ ἓν τμῆμα πρὸς τὴν δοθεῖσαν τὴν ΑΓ, οὕτως τὸ δοθὲν τὸ Δ πρὸς τὸ ἀπὸ τοῦ λοιποῦ τμήματος.

"Εἰλήφθω τῆς ΑΒ τρίτον μέρος ἡ ΑΕ· τὸ ἄρα Δ ἐπὶ τὴν ΑΓ ἤτοι μεῖζόν ἐστι τοῦ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ ἢ ἴσον ἢ ἔλασσον.

"Εἰ μὲν οὖν μεῖζόν ἐστιν, οὐ συντεθήσεται, ὡς ἐν τῇ ἀναλύσει δέδεικται· εἰ δὲ ἴσον ἐστί, τὸ Ε σημεῖον ποιήσει τὸ πρόβλημα. ἴσων γὰρ ὄντων τῶν στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ Δ πρὸς τὸ ἀπὸ ΒΕ.

"Εἰ δὲ ἔλασσόν ἐστι τὸ Δ ἐπὶ τὴν ΑΓ τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, συντεθήσεται οὕτως· κείσθω ἡ ΑΓ πρὸς ὀρθὰς τῇ ΑΒ, καὶ διὰ τοῦ Γ τῇ ΑΒ παρ-

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[a] In our algebraical notation, $x^2(a-x)$ is a maximum when $x=\frac{2}{3}a$. We can easily prove this by the calculus. For, by differentiating and equating to zero, we see that $x^2(a-x)$ has

portional to the altitudes; therefore the solids are equal [Eucl. xi. 34]; therefore

$$EB^2 . EA = \Delta . \Gamma A,$$

in which both Δ and ΓA are given. But, of all the figures similarly taken upon BA, $BE^2 . BA$ is greatest when $BE = 2EA$,[a] as will be proved; it is therefore necessary that the product of the given area and the given straight line should not be greater than

$$BE^2 . EA.$$[b]

"The synthesis is as follows: Let AB be the given straight line,[c] let $A\Gamma$ be any other given straight line, let Δ be the given area, and let it be required to cut AB so that the ratio of one segment to the given straight line $A\Gamma$ shall be equal to the ratio of the given area Δ to the square on the remaining segment.

"Let AE be taken, the third part of AB; then $\Delta . A\Gamma$ is greater than, equal to or less than $BE^2 . EA$.

"If it is greater, no synthesis is possible, as was shown in the analysis; if it is equal, the point E satisfies the conditions of the problem. For in equal solids the bases are inversely proportional to the altitudes, and $EA : A\Gamma = \Delta : BE^2$.

"If $\Delta . A\Gamma$ is less than $BE^2 . EA$, the synthesis is thus accomplished: let $A\Gamma$ be placed at right angles to AB, and through Γ let ΓZ be drawn parallel to

a stationary value when $2ax - 3x^2 = 0$, *i.e.*, when $x = 0$ (which gives a minimum value) or $x = \frac{2}{3}a$ (which gives a maximum). No such easy course was open to Archimedes.

[b] *Sc.* "not greater than $BE^2 . EA$ when $BE = 2EA$."

[c] Figure on p. 146.

ἀλλήλος ἤχθω ἡ ΓΖ, διὰ δὲ τοῦ Β τῇ ΑΓ παράλληλος ἤχθω ἡ ΒΖ καὶ συμπιπτέτω τῇ ΓΕ ἐκβληθείσῃ κατὰ τὸ Η, καὶ συμπεπληρώσθω τὸ ΖΘ παραλληλόγραμμον, καὶ διὰ τοῦ Ε τῇ ΖΗ παράλληλος ἤχθω ἡ ΚΕΛ. ἐπεὶ οὖν τὸ Δ ἐπὶ τὴν ΑΓ ἔλασσόν ἐστι τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, ἔστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ Δ πρὸς ἔλασσόν τι τοῦ ἀπὸ τῆς ΒΕ, τουτέστι τοῦ ἀπὸ τῆς ΗΚ. ἔστω οὖν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ Δ πρὸς τὸ ἀπὸ ΗΜ, καὶ τῷ Δ ἴσον ἔστω τὸ ὑπὸ ΓΖΝ. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ Δ, τουτέστι τὸ ὑπὸ ΓΖΝ, πρὸς τὸ ἀπὸ ΗΜ, ἀλλ' ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως ἡ ΓΖ πρὸς ΖΗ, ὡς δὲ ἡ ΓΖ πρὸς ΖΗ, οὕτως τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, καὶ ὡς ἄρα τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, οὕτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ἀπὸ ΗΜ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, οὕτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ. ἀλλ' ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, ἡ ΓΖ πρὸς ΖΝ, ὡς δὲ ἡ ΓΖ πρὸς ΖΝ, τῆς ΖΗ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ· καὶ ὡς ἄρα τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ, οὕτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ· ἴσον ἄρα ἐστὶ τὸ ἀπὸ ΗΜ τῷ ὑπὸ ΗΖΝ.

"Ἐὰν ἄρα διὰ τοῦ Ζ περὶ ἄξονα τὴν ΖΗ γράψωμεν παραβολήν, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΖΝ, ἥξει διὰ τοῦ Μ. γεγράφθω, καὶ ἔστω ὡς ἡ ΜΞΖ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΘΛ τῷ ΑΖ, τουτέστι τὸ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΖ, ἐὰν διὰ

AB, and through B let BZ be drawn parallel to AΓ, and let it meet ΓE produced at H, and let the parallelogram ZΘ be completed, and through E let KEΛ be drawn parallel to ZH. Now

since	$\Delta . A\Gamma$	$< BE^2 . EA$,
$\therefore$	$EA : A\Gamma$	$= \Delta :$ (the square of some quantity less than BE)
		$= \Delta :$ (the square of some quantity less than HK).
Let	$EA : A\Gamma$	$= \Delta : HM^2$,
and let	Δ	$= \Gamma Z . ZN$.
Then	$EA : A\Gamma$	$= \Delta : HM^2$
		$= \Gamma Z . ZN : HM^2$.
But	$EA : A\Gamma$	$= \Gamma Z : ZH$,
and	$\Gamma Z : ZH$	$= \Gamma Z^2 : \Gamma Z . ZH$;

$\therefore$ $\quad \Gamma Z^2 : \Gamma Z . ZH = \Gamma Z . ZN : HM^2$;

and *permutando*, $\Gamma Z^2 : \Gamma Z . ZN = \Gamma Z . ZH : HM^2$.

But $\quad \Gamma Z^2 : \Gamma Z . ZN = \Gamma Z : ZN$,

and $\quad \Gamma Z : ZN \quad = \Gamma Z . ZH : NZ . ZH$,

by taking a common altitude ZH ;

and, $\therefore$ $\quad \Gamma Z . ZH : NZ . ZH = \Gamma Z . ZH : HM^2$;

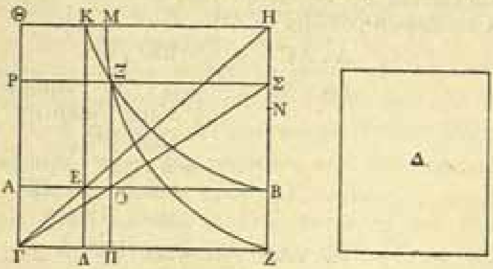
$\therefore$ $\quad HM^2 = HZ . ZN$.

"Therefore if we describe through Z a parabola about the axis ZH and with parameter ZN, it will pass through M. Let it be described, and let it be as MΞZ. Then since

$$\Theta\Lambda = AZ, \qquad \text{[Eucl. i. 43}$$

i.e. $\quad \Theta K . K\Lambda = AB . BZ$,

τοῦ Β περὶ ἀσυμπτώτους τὰς ΘΓ, ΓΖ γράψωμεν ὑπερβολήν, ἥξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν



τοῦ η′ θεωρήματος τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. γεγράφθω, καὶ ἔστω ὡς ἡ ΒΚ τέμνουσα τὴν παραβολὴν κατὰ τὸ Ξ, καὶ ἀπὸ τοῦ Ξ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΞΟΠ, καὶ διὰ τοῦ Ξ τῇ ΑΒ παράλληλος ἤχθω ἡ ΡΞΣ. ἐπεὶ οὖν ὑπερβολή ἐστιν ἡ ΒΞΚ, ἀσύμπτωτοι δὲ αἱ ΘΓ, ΓΖ, καὶ παράλληλοι ἠγμέναι εἰσὶν αἱ ΡΞΠ ταῖς ΑΒΖ, ἴσον ἐστὶ τὸ ὑπὸ ΡΞΠ τῷ ὑπὸ ΑΒΖ· ὥστε καὶ τὸ ΡΟ τῷ ΟΖ. ἐὰν ἄρα ἀπὸ τοῦ Γ ἐπὶ τὸ Σ ἐπιζευχθῇ εὐθεῖα, ἥξει διὰ τοῦ Ο. ἐρχέσθω, καὶ ἔστω ὡς ἡ ΓΟΣ. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΟΑ πρὸς ΑΓ, οὕτως ἡ ΟΒ πρὸς ΒΣ, τουτέστιν ἡ ΓΖ πρὸς ΖΣ, ὡς δὲ ἡ ΓΖ πρὸς ΖΣ, τῆς ΖΝ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΣΖΝ, καὶ ὡς ἄρα ἡ ΟΑ πρὸς ΑΓ, οὕτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΣΖΝ. καί ἐστι τῷ μὲν ὑπὸ ΓΖΝ ἴσον τὸ Δ χωρίον, τῷ δὲ ὑπὸ ΣΖΝ ἴσον τὸ ἀπὸ ΣΞ, τουτέστι τὸ ἀπὸ ΒΟ, διὰ τὴν παραβολήν· ὡς ἄρα ἡ ΟΑ πρὸς ΑΓ, οὕτως τὸ Δ

if we describe through B a hyperbola in the asymptotes ΘΓ, ΓΖ, it will pass through K by the converse of the eighth theorem [of the second book] of Apollonius's *Elements of Conics*. Let it be described, and let it be as BK cutting the parabola in Ξ, and from Ξ let ΞΟΠ be drawn perpendicular to AB, and through Ξ let ΡΞΣ be drawn parallel to AB. Then since BΞK is a hyperbola and ΘΓ, ΓΖ are its asymptotes, while ΡΞ, ΞΠ are parallel to AB, BZ,

$$P\Xi \,.\, \Xi\Pi = AB \,.\, BZ\,; \qquad \text{[Apoll. ii. 12}$$

$$\therefore \qquad PO = OZ.$$

Therefore if a straight line be drawn from Γ to Σ it will pass through O [Eucl. i. 43, converse]. Let it be drawn, and let it be as ΓΟΣ. Then since

$$OA : A\Gamma = OB : B\Sigma \qquad \text{[Eucl. vi. 4}$$

$$= \Gamma Z : Z\Sigma,$$

and $\Gamma Z : Z\Sigma = \Gamma Z \,.\, ZN : \Sigma Z \,.\, ZN,$

by taking a common altitude ZN,

$$\therefore \qquad OA : A\Gamma = \Gamma Z \,.\, ZN : \Sigma Z \,.\, ZN.$$

And $\Gamma Z \,.\, ZN = \Delta$, $\Sigma Z \,.\, ZN = \Sigma\Xi^2 = BO^2$, by the property of the parabola [Apoll. i. 11].

$$\therefore \qquad OA : A\Gamma = \Delta : BO^2\,;$$

χωρίον πρὸς τὸ ἀπὸ τῆς ΒΟ. εἴληπται ἄρα τὸ Ο σημεῖον ποιοῦν τὸ πρόβλημα.

"Ὅτι δὲ διπλασίας οὔσης τῆς ΒΕ τῆς ΕΑ τὸ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ μέγιστόν ἐστι πάντων τῶν ὁμοίως λαμβανομένων ἐπὶ τῆς ΒΑ, δειχθήσεται οὕτως. ἔστω γάρ, ὡς ἐν τῇ ἀναλύσει, πάλιν δοθεῖσα εὐθεῖα πρὸς ὀρθὰς τῇ ΑΒ ἡ ΑΓ, καὶ ἐπιζευχθεῖσα ἡ ΓΕ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ διὰ τοῦ Β παραλλήλῳ ἠγμένῃ τῇ ΑΓ κατὰ τὸ Ζ, καὶ διὰ τῶν Γ, Ζ παράλληλοι τῇ ΑΒ ἤχθωσαν αἱ ΘΖ, ΓΗ, καὶ ἐκβεβλήσθω ἡ ΓΑ ἐπὶ τὸ Θ, καὶ ταύτῃ παράλληλος διὰ τοῦ Ε ἤχθω ἡ ΚΕΛ, καὶ γεγονέτω, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ ὑπὸ ΓΗΜ πρὸς τὸ ἀπὸ ΕΒ· τὸ ἄρα ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ ἴσον ἐστὶ τῷ ὑπὸ ΓΗΜ ἐπὶ τὴν ΑΓ διὰ τὸ δύο στερεῶν ἀντιπεπονθέναι τὰς βάσεις τοῖς ὕψεσιν. λέγω οὖν, ὅτι τὸ ὑπὸ ΓΗΜ ἐπὶ τὴν ΑΓ μέγιστόν ἐστι πάντων τῶν ὁμοίως ἐπὶ τῆς ΒΑ λαμβανομένων.

"Γεγράφθω γὰρ διὰ τοῦ Η περὶ ἄξονα τὴν ΖΗ παραβολή, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΜ· ἥξει δὴ διὰ τοῦ Κ, ὡς ἐν τῇ ἀναλύσει δέδεικται, καὶ συμπεσεῖται ἐκβαλλομένη τῇ ΘΓ παραλλήλῳ οὔσῃ τῇ διαμέτρῳ τῆς τομῆς διὰ τὸ ἕβδομον καὶ εἰκοστὸν θεώρημα τοῦ πρώτου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. ἐκβεβλήσθω καὶ συμπιπτέτω κατὰ τὸ Ν, καὶ διὰ τοῦ Β περὶ ἀσυμπτώτους τὰς ΝΓΗ γεγράφθω ὑπερβολή· ἥξει ἄρα διὰ τοῦ Κ, ὡς ἐν τῇ ἀναλύσει εἴρηται. ἐρχέσθω οὖν ὡς ἡ ΒΚ, καὶ ἐκβληθείσῃ τῇ ΖΗ ἴση κείσθω ἡ ΗΞ, καὶ ἐπεζεύχθω ἡ ΞΚ

therefore the point O has been found satisfying the conditions of the problem.

"That $BE^2 . EA$ is the greatest of all the figures similarly taken upon BA when $BE = 2EA$ will be thus proved. Let there again be, as in the analysis, a given straight line $A\Gamma$ at right angles to AB,[a] and let ΓE be joined and let it, when produced, meet at Z the line through B drawn parallel to $A\Gamma$, and through Γ, Z let ΘZ, ΓH be drawn parallel to AB, and let ΓA be produced to Θ, and through E let $KE\Lambda$ be drawn parallel to it, and let

$$EA : A\Gamma = \Gamma H . HM : EB^2 ;$$

then $$BE^2 . EA = (\Gamma H . HM) . A\Gamma,$$

owing to the fact that in two [equal] solids the bases are inversely proportional to the altitudes. I assert, then, that $(\Gamma H . HM) . A\Gamma$ is the greatest of all the figures similarly taken upon BA.

"For let there be described through H a parabola about the axis ZH and with parameter HM; it will pass through K, as was proved in the analysis, and, if produced, it will meet $\Theta\Gamma$, being parallel to the axis[b] of the parabola, by the twenty-seventh theorem of the first book of Apollonius's *Elements of Conics*.[c] Let it be produced, and let it meet at N, and through B let a hyperbola be drawn in the asymptotes $N\Gamma$, ΓH; it will pass through K, as was shown in the analysis. Let it be described as BK, and let ZH be produced to Ξ so that $ZH = H\Xi$, and let ΞK be joined

[a] Figure on p. 151.
[b] Lit. "diameter," in accordance with Archimedes' usage.
[c] Apoll. i. 26 in our texts.

καὶ ἐκβεβλήσθω ἐπὶ τὸ Ο· φανερὸν ἄρα, ὅτι ἐφάπτεται τῆς παραβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τετάρτου καὶ τριακοστοῦ θεωρήματος τοῦ πρώτου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. ἐπεὶ οὖν διπλῆ ἐστιν ἡ ΒΕ τῆς ΕΑ—οὕτως γὰρ ὑπόκειται—τουτέστιν ἡ ΖΚ τῆς ΚΘ,

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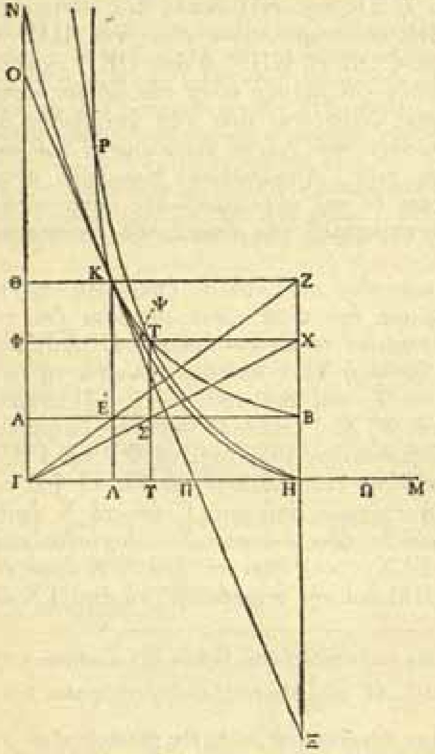
[a] Apoll. i. 33 in our texts.

and produced to O; it is clear that it will touch the parabola by the converse of the thirty-fourth



theorem of the first book of Apollonius's *Elements of Conics*.[a] Then since ΒΕ = 2ΕΑ—for this hypothesis has been made—therefore ΖΚ = 2ΚΘ, and the triangle

καί ἐστιν ὅμοιον τὸ ΟΘΚ τρίγωνον τῷ ΞΖΚ τριγώνῳ, διπλασία ἐστὶ καὶ ἡ ΞΚ τῆς ΚΟ. ἔστιν δὲ καὶ ἡ ΞΚ τῆς ΚΠ διπλῆ διὰ τὸ καὶ τὴν ΞΖ τῆς ΞΗ καὶ παράλληλον εἶναι τὴν ΠΗ τῇ ΚΖ· ἴση ἄρα ἡ ΟΚ τῇ ΚΠ. ἡ ἄρα ΟΚΠ ψαύουσα τῆς ὑπερβολῆς καὶ μεταξὺ οὖσα τῶν ἀσυμπτώτων δίχα τέμνεται· ἐφάπτεται ἄρα τῆς ὑπερβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τρίτου θεωρήματος τοῦ δευτέρου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. ἐφήπτετο δὲ καὶ τῆς παραβολῆς κατὰ τὸ αὐτὸ Κ· ἡ ἄρα παραβολὴ τῆς ὑπερβολῆς ἐφάπτεται κατὰ τὸ Κ.

"Νενοήσθω οὖν καὶ ἡ ὑπερβολὴ προσεκβαλλομένη ὡς ἐπὶ τὸ Ρ, καὶ εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Σ, καὶ διὰ τοῦ Σ τῇ ΚΛ παράλληλος ἤχθω ἡ ΤΣΥ καὶ συμβαλλέτω τῇ ὑπερβολῇ κατὰ τὸ Τ, καὶ διὰ τοῦ Τ τῇ ΓΗ παράλληλος ἤχθω ἡ ΦΤΧ. ἐπεὶ οὖν διὰ τὴν ὑπερβολὴν καὶ τὰς ἀσυμπτώτους ἴσον ἐστὶ τὸ ΦΥ τῷ ΓΒ, κοινοῦ ἀφαιρεθέντος τοῦ ΓΣ ἴσον γίνεται τὸ ΦΣ τῷ ΣΗ, καὶ διὰ τοῦτο ἡ ἀπὸ τοῦ Γ ἐπὶ τὸ Χ ἐπιζευγνυμένη εὐθεῖα ἥξει διὰ τοῦ Σ. ἐρχέσθω καὶ ἔστω ὡς ἡ ΓΣΧ. καὶ ἐπεὶ τὸ ἀπὸ ΨΧ ἴσον ἐστὶ τῷ ὑπὸ ΧΗΜ διὰ τὴν παραβολήν, τὸ ἀπὸ ΤΧ ἔλασσόν

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[a] In the same notation as before, the condition $BE^2 . EA = (\Gamma H . HM) . A\Gamma$ is $\frac{4}{27}a^3 = bc^2$: and Archimedes has proved that, when this condition holds, the parabola $x^2 = \frac{c^2}{a}y$ touches the hyperbola $(a-x)y = ab$ at the point $\left(\frac{2}{3}a, 3b\right)$ because they both touch at this point the same straight line, that is the

$O\Theta K$ is similar to the triangle ΞZK, so that $\Xi K = 2KO$. But $\Xi K = 2K\Pi$ because $\Xi Z = 2\Xi H$ and ΠH is parallel to KZ; therefore $OK = K\Pi$. Therefore $OK\Pi$, which meets the hyperbola and lies between the asymptotes, is bisected; therefore, by the converse of the third theorem of the second book of Apollonius's *Elements of Conics*, it is a tangent to the hyperbola. But it touches the parabola at the same point K. Therefore the parabola touches the hyperbola at K.[a]

"Let the hyperbola be therefore conceived as produced to P, and upon AB let any point Σ be taken, and through Σ let $T\Sigma Y$ be drawn parallel to $K\Lambda$ and let it meet the hyperbola at T, and through T let ΦTX be drawn parallel to ΓH. Now by virtue of the property of the hyperbola and its asymptotes, $\Phi Y = \Gamma B$, and, the common element $\Gamma\Sigma$ being subtracted, $\Phi\Sigma = \Sigma H$, and therefore the straight line drawn from Γ to X will pass through Σ [Eucl. i. 43, conv.]. Let it be drawn, and let it be as $\Gamma\Sigma X$. Then since, in virtue of the property of the parabola,

$$\Psi X^2 = XH \cdot HM, \qquad \text{[Apoll. i. 11}$$

line $9bx - ay - 3ab = 0$, as may easily be shown. We may prove this fact in the following simple manner. Their points of intersection are given by the equation

$$x^2(a-x) = bc^2,$$

which may be written

$$x^3 - ax^2 + \frac{4}{27}a^3 = \frac{4}{27}a^3 - bc^2,$$

or

$$\left(x - \frac{2}{3}a\right)^2\left(x + \frac{a}{3}\right) = \frac{4}{27}a^3 - bc^2.$$

Therefore, when $bc^2 = \frac{4}{27}a^3$ there are two coincident solutions, $x = \frac{2}{3}a$, lying between 0 and a, and a third solution $x = -\frac{a}{3}$, outside that range.

ἐστι τοῦ ὑπὸ ΧΗΜ. γεγονέτω οὖν τῷ ἀπὸ ΤΧ ἴσον τὸ ὑπὸ ΧΗΩ. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΣΑ πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΧ, ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΧ, τῆς ΗΩ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ ΧΗΩ καὶ πρὸς τὸ ἴσον αὐτῷ τὸ ἀπὸ ΧΤ, τουτέστι τὸ ἀπὸ ΒΣ, τὸ ἄρα ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἴσον ἐστὶ τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ. τὸ δὲ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ ἔλασσόν ἐστι τοῦ ὑπὸ ΓΗΜ ἐπὶ τὴν ΓΑ· τὸ ἄρα ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἔλαττόν ἐστι τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ. ὁμοίως δὴ δειχθήσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ λαμβανομένων τῶν Ε, Β.

"Ἀλλὰ δὴ εἰλήφθω μεταξὺ τῶν Ε, Α σημεῖον τὸ ϛ. λέγω, ὅτι καὶ οὕτως τὸ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ μεῖζόν ἐστι τοῦ ἀπὸ Βϛ ἐπὶ τὴν ϛΑ.

"Τῶν γὰρ αὐτῶν κατεσκευασμένων ἤχθω διὰ τοῦ ϛ τῇ ΚΛ παράλληλος ἡ ϛϛΡ καὶ συμβαλλέτω τῇ ὑπερβολῇ κατὰ τὸ Ρ· συμβαλεῖ γὰρ αὐτῇ διὰ τὸ παράλληλος εἶναι τῇ ἀσυμπτώτῳ· καὶ διὰ τοῦ Ρ παράλληλος ἀχθεῖσα τῇ ΑΒ ἡ Α'ΡΒ' συμβαλλέτω τῇ ΗΖ ἐκβαλλομένῃ κατὰ τὸ Β'. καὶ ἐπεὶ πάλιν διὰ τὴν ὑπερβολὴν ἴσον ἐστὶ τὸ Γ'ϛ τῷ ΑΗ, ἡ ἀπὸ τοῦ Γ ἐπὶ τὸ Β' ἐπιζευγνυμένη εὐθεῖα ἥξει διὰ τοῦ ϛ. ἐρχέσθω καὶ ἔστω ὡς ἡ ΓϛΒ'. καὶ ἐπεὶ πάλιν διὰ τὴν παραβολὴν ἴσον ἐστὶ τὸ ἀπὸ Α'Β' τῷ ὑπὸ Β'ΗΜ, τὸ ἄρα ἀπὸ ΡΒ' ἔλασσόν ἐστι τοῦ ὑπὸ Β'ΗΜ. γεγονέτω τὸ ἀπὸ ΡΒ' ἴσον τῷ ὑπὸ Β'ΗΩ. ἐπεὶ οὖν ἐστιν, ὡς ἡ ϛΑ πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΒ', ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΒ', τῆς

* Figure on p. 156.

$\therefore \qquad TX^2 < XH \cdot HM.$

Let $\qquad TX^2 = XH \cdot H\Omega.$

Then since $\Sigma A : A\Gamma = \Gamma H : HX,$

while $\qquad \Gamma H : HX = \Gamma H \cdot H\Omega : XH \cdot H\Omega,$

by taking a common altitude $H\Omega$,

$$= \Gamma H \cdot H\Omega : XT^2$$

$$= \Gamma H \cdot H\Omega : B\Sigma^2,$$

$\therefore \qquad B\Sigma^2 \cdot \Sigma A = (\Gamma H \cdot H\Omega) \cdot \Gamma A.$

But $\quad (\Gamma H \cdot H\Omega) \cdot \Gamma A < (\Gamma H \cdot HM) \cdot \Gamma A;$

$\therefore \qquad B\Sigma^2 \cdot \Sigma A < BE^2 \cdot EA.$

This can be proved similarly for all points taken between E, B.

"Now let there be taken a point ϛ between E, A. I assert that in this case also $BE^2 \cdot EA > B\text{ϛ} \cdot \text{ϛ}A$.

"With the same construction, let ϚϛP be drawn [a] through ϛ parallel to KA and let it meet the hyperbola at P; it will meet the hyperbola because it is parallel to an asymptote [Apoll. ii. 13]; and through P let A′PB′ be drawn parallel to AB and let it meet HZ produced in B′. Since, in virtue of the property of the hyperbola, Γ′Ϛ = AH, the straight line drawn from Γ to B′ will pass through ϛ [Eucl. i. 43, conv.]. Let it be drawn and let it be as Γϛ B′. Again, since, in virtue of the property of the parabola,

$$A'B'^2 = B'H \cdot HM,$$

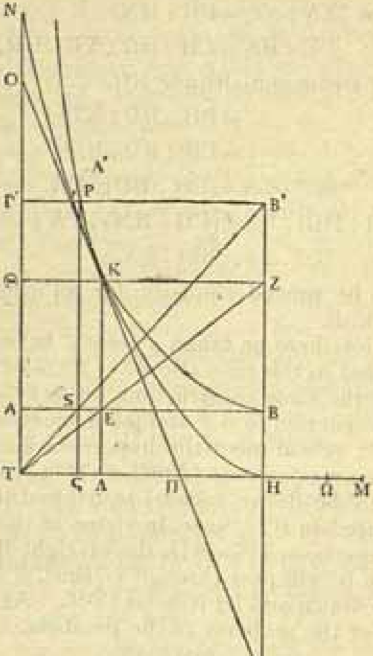
$\therefore \qquad PB'^2 < B'H \cdot HM.$

Let $\qquad PB'^2 = B'H \cdot H\Omega.$

Then since $\text{ϛ}A : A\Gamma = \Gamma H : HB',$

while $\qquad \Gamma H : HB' = \Gamma H \cdot H\Omega : B'H \cdot H\Omega,$

ΗΩ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ Β΄ΗΩ, τουτέστι πρὸς τὸ ἀπὸ



ΡΒ΄, τουτέστι πρὸς τὸ ἀπὸ Βϛ, τὸ ἄρα ἀπὸ Βϛ ἐπὶ τὴν ϛΑ ἴσον ἐστὶ τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ. καὶ μεῖζον τὸ ὑπὸ ΓΗΜ τοῦ ὑπὸ ΓΗΩ· μεῖζον ἄρα καὶ τὸ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ τοῦ ἀπὸ Βϛ ἐπὶ

by taking a common altitude HΩ,

$$= \Gamma H \,.\, H\Omega : PB'^2$$

$$= \Gamma H \,.\, H\Omega : B\digamma^2,$$

$$\therefore \qquad B\digamma^2 \,.\, \digamma A = (\Gamma H \,.\, H\Omega) \,.\, \Gamma A.$$

And $\qquad \Gamma H \,.\, HM > \Gamma H \,.\, H\Omega;$

$$\therefore \qquad BE^2 \,.\, EA > B\digamma^2 \,.\, \digamma A.$$

τὴν ϛΑ. ὁμοίως δὴ δειχθήσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ τῶν Ε, Α λαμβανομένων. ἐδείχθη δὲ καὶ ἐπὶ πάντων τῶν μεταξὺ τῶν Ε, Β· πάντων ἄρα τῶν ἐπὶ τῆς ΑΒ ὁμοίως λαμβανομένων μέγιστόν ἐστιν τὸ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ, ὅταν ᾖ διπλασία ἡ ΒΕ τῆς ΕΑ."

Ἐπιστῆσαι δὲ χρὴ καὶ τοῖς ἀκολουθοῦσιν κατὰ τὴν εἰρημένην καταγραφήν. ἐπεὶ γὰρ δέδεικται τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ καὶ τὸ ἀπὸ Βϛ ἐπὶ τὴν ϛΑ ἔλασσον τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, δυνατόν ἐστι καὶ τοῦ δοθέντος χωρίου ἐπὶ τὴν δοθεῖσαν ἐλάσσονος ὄντος τοῦ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ κατὰ δύο σημεῖα τὴν ΑΒ τεμνομένην ποιεῖν τὸ ἐξ ἀρχῆς πρόβλημα. τοῦτο δὲ γίνεται, εἰ νοήσαιμεν περὶ διάμετρον τὴν ΧΗ γραφομένην παραβολήν, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΩ· ἡ γὰρ τοιαύτη παραβολὴ πάντως ἔρχεται διὰ τοῦ Τ. καὶ ἐπειδὴ ἀνάγκη αὐτὴν συμπίπτειν τῇ ΓΝ παραλλήλῳ οὔσῃ τῇ διαμέτρῳ, δῆλον, ὅτι τέμνει τὴν ὑπερβολὴν καὶ κατ' ἄλλο σημεῖον ἀνωτέρω τοῦ Κ, ὡς ἐνταῦθα κατὰ τὸ Ρ, καὶ ἀπὸ τοῦ Ρ ἐπὶ τὴν ΑΒ κάθετος ἀγομένη, ὡς ἐνταῦθα ἡ Ρϛ, τέμνει τὴν ΑΒ κατὰ τὸ ϛ, ὥστε τὸ ϛ σημεῖον ποιεῖν τὸ πρό-

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[a] There is some uncertainty where the quotation from Archimedes ends and Eutocius's comments are resumed. Heiberg, with some reason, makes Eutocius resume his comments at this point.

[b] In the mss. the figures on pp. 150 and 156 are com-

This can be proved similarly for all points taken between E, A. And it was proved for all points between E, B; therefore for all figures similarly taken upon AB, $BE^2 . EA$ is greatest when $BE = 2EA$."

The following consequences[a] should also be noticed in the aforementioned figure.[b] Inasmuch as it has been proved that

$$B\Sigma^2 . \Sigma A < BE^2 .$$

and $$B\digamma^2 . \digamma A < BE^2 . EA,$$

if the product of the given space and the given straight line is less than $BE^2 . EA$, it is possible to cut AB in two points satisfying the conditions of the original problem.[c] This comes about if we conceive a parabola described about the axis XH with parameter HΩ; for such a parabola will necessarily pass through T.[d] And since it must necessarily meet ΓN, being parallel to a diameter [Apoll. *Con.* i. 26], it is clear that it cuts the hyperbola in another point above K, as at P in this case, and a perpendicular drawn from P to AB, as P$\digamma$ in this case, will cut AB in $\digamma$, so that the point $\digamma$ satisfies the conditions of the

bined; in this edition it is convenient, for the sake of clarity, to give separate figures.

[c] With the same notation as before this may be stated: when $be^2 < \frac{4}{27}a^3$, there are always two real solutions of the cubic equation $x^2(a-x) = be^2$ lying between 0 and a. If the cubic has two real roots it must, of course, have a third real root as well, but the Greeks did not recognize negative solutions.

[d] By Apoll. i. 11, since $TX^2 = XH . H\Omega$.

βλημα, καὶ ἴσον γίνεσθαι τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ τῷ ἀπὸ Βϛ ἐπὶ τὴν ϛΑ, ὡς ἐστι διὰ τῶν προειρημένων ἀποδείξεων ἐμφανές. ὥστε δυνατοῦ ὄντος ἐπὶ τῆς ΒΑ δύο σημεῖα λαμβάνειν ποιοῦντα τὸ ζητούμενον, ἔξεστιν, ὁπότερόν τις βούλοιτο, λαμβάνειν ἢ τὸ μεταξὺ τῶν Ε, Β ἢ τὸ μεταξὺ τῶν Ε, Α. εἰ μὲν γὰρ τὸ μεταξὺ τῶν Ε, Β, ὡς εἴρηται, τῆς διὰ τῶν Η, Τ σημείων γραφομένης παραβολῆς κατὰ δύο σημεῖα τεμνούσης τὴν ὑπερβολὴν τὸ μὲν ἐγγύτερον τοῦ Η, τουτέστι τοῦ ἄξονος τῆς παραβολῆς, εὑρήσει τὸ μεταξὺ τῶν Ε, Β, ὡς ἐνταῦθα τὸ Τ εὑρίσκει τὸ Σ, τὸ δὲ ἀπωτέρω τὸ μεταξὺ τῶν Ε, Α, ὡς ἐνταῦθα τὸ Ρ εὑρίσκει τὸ ϛ.

Καθόλου μὲν οὖν οὕτως ἀναλέλυται καὶ συντέθειται τὸ πρόβλημα· ἵνα δὲ καὶ τοῖς Ἀρχιμηδείοις ῥήμασιν ἐφαρμοσθῇ, νενοήσθω ὡς ἐν αὐτῇ τῇ τοῦ ῥητοῦ καταγραφῇ διάμετρος μὲν τῆς σφαίρας ἡ ΔΒ, ἡ δὲ ἐκ τοῦ κέντρου ἡ ΒΖ, καὶ ἡ δεδομένη ἡ ΖΘ. κατηντήσαμεν ἄρα, φησίν, εἰς τὸ " τὴν ΔΖ τεμεῖν κατὰ τὸ Χ, ὥστε εἶναι, ὡς τὴν ΧΖ πρὸς τὴν δοθεῖσαν, οὕτως τὸ δοθὲν πρὸς τὸ ἀπὸ τῆς ΔΧ. τοῦτο δὲ ἁπλῶς μὲν λεγόμενον ἔχει διορισμόν." εἰ γὰρ τὸ δοθὲν ἐπὶ τὴν δοθεῖσαν μεῖζον ἐτύγχανεν τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ, ἀδύνατον ἦν τὸ πρόβλημα, ὡς δέδεικται, εἰ δὲ ἴσον, τὸ Β σημεῖον ἐποίει τὸ πρόβλημα, καὶ οὕτως δὲ οὐδὲν ἦν πρὸς τὴν ἐξ ἀρχῆς Ἀρχιμήδους πρόθεσιν· ἡ γὰρ σφαῖρα οὐκ ἐτέμνετο εἰς τὸν δοθέντα

[a] Archimedes' figure is re-drawn (v. page 162) so that B, Z come on the left of the figure and Δ on the right, instead of B, Z on the right and Δ on the left.

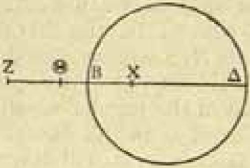
[b] *v. supra*, p. 133.

problem, and $B\Sigma^2 . \Sigma A = B\varsigma^2 . \varsigma A$, as is clear from the above proof. Inasmuch as it is possible to take on BA two points satisfying what is sought, it is permissible to take whichever one wills, either the point between E, B or that between E, A. If we choose the point between E, B, the parabola described through the points H, T will, as stated, cut the hyperbola in two points; of these the one nearer to H, that is to the axis of the parabola, will determine the point between E, B, as in this case T determines Σ, while the point farther away will determine the point between E, A, as in this case P determines ϛ.

The analysis and synthesis of the general problem have thus been completed; but in order that it may be harmonized with Archimedes' words, let there be conceived, as in Archimedes' own figure,[a] a diameter ΔB of the sphere, with radius [equal to] BZ, and a given straight line ZΘ. We are therefore faced with the problem, he says, "so to cut ΔZ at X that XZ bears to the given straight line the same ratio as the given area bears to the square on ΔX. When the problem is stated in this general form, it is necessary to investigate the limits of possibility."[b] If therefore the product of the given area and the given straight line chanced to be greater than $\Delta B^2 . BZ$,[c] the problem would not admit a solution, as was proved, and if it were equal the point B would satisfy the conditions of the problem, which also would be of no avail for the purpose Archimedes set himself at the outset; for the sphere would not be

[a] For $\Delta B = \frac{2}{3}\Delta Z$ [*ex hyp.*], and so ΔB in the figure on p. 162 corresponds with BE in the figure on p. 146, while BZ in the figure on p. 162 corresponds with EA in the figure on p. 146.

λόγον. ἁπλῶς ἄρα λεγόμενον εἶχεν προσδιορισμόν· "προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε



ὑπαρχόντων," τουτέστι τοῦ τε διπλασίαν εἶναι τὴν ΔΒ τῆς ΖΒ καὶ τοῦ μείζονα εἶναι τὴν ΒΖ τῆς ΖΘ, "οὐκ ἔχει διορισμόν." τὸ γὰρ ἀπὸ ΔΒ τὸ δοθὲν ἐπὶ τὴν ΖΘ τὴν δοθεῖσαν ἔλαττόν ἐστι τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ διὰ τὸ τὴν ΒΖ τῆς ΖΘ μείζονα εἶναι, οὗπερ ὑπάρχοντος ἐδείξαμεν δυνατόν, καὶ ὅπως προβαίνει τὸ πρόβλημα.

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[a] Eutocius proceeds to give solutions of the problem by Dionysodorus and Diocles, by whose time, as he has explained, Archimedes' own solution had already disappeared. Dionysodorus solves the less general equation by means of the intersection of a parabola and a rectangular hyperbola; Diocles solves the general problem by the intersection of an ellipse with a rectangular hyperbola, and his proof is both ingenious and intricate. Details may be consulted in Heath, *H.G.M.* ii. 46-49 and more fully in Heath,

cut in the given ratio. Therefore when the problem was stated generally, an investigation of the limits of possibility was necessary as well; "but under the conditions of the present case," that is, if $\Delta B = 2ZB$ and $BZ > Z\Theta$, "no such investigation is necessary." For the product of the given area ΔB^2 into the given straight line $Z\Theta$ is less than the product of ΔB^2 into BZ by reason of the fact that BZ is greater than $Z\Theta$, and we have shown that in this case there is a solution, and how it can be effected.[a]

The Works of Archimedes, pp. cxxiii-cxli, which deals with the whole subject of cubic equations in Greek mathematical history. It is there pointed out that the problem of finding mean proportionals is equivalent to the solution of a pure cubic equation, $\frac{a^3}{x^3}=\frac{a}{b}$, and that Menaechmus's solution, by the intersection of two conic sections (*v.* vol. i. pp. 278-283), is the precursor of the method adopted by Archimedes, Dionysodorus and Diocles. The solution of cubic equations by means of conics was, no doubt, found easier than a solution by the manipulation of parallelepipeds, which would have been analogous to the solution of quadratic equations by the application of areas (*v.* vol. i. pp. 186-215). No other examples of the solution of cubic equations have survived, but in his preface to the book *On Conoids and Spheroids* Archimedes says the results there obtained can be used to solve other problems, including the following, "*from a given spheroidal figure or conoid to cut off, by a plane drawn parallel to a given plane, a segment which shall be equal to a given cone or cylinder, or to a given sphere*" (Archim. ed. Heiberg i. 258. 11-15): the case of the paraboloid of revolution does not lead to a cubic equation, but those of the spheroid and hyperboloid of revolution do lead to cubics, which Archimedes may be presumed to have solved. The conclusion reached by Heath is that Archimedes solved completely, so far as the real roots are concerned, a cubic equation in which the term in x is absent: and as all cubic equations can be reduced to this form, he may be regarded as having solved geometrically the general cubic.

(d) Conoids and Spheroids

(i.) *Preface*

Archim. *De Con. et Sphaer.*, Praef., Archim. ed. Heiberg i. 246. 1-14

Ἀρχιμήδης Δοσιθέῳ εὖ πράττειν.

Ἀποστέλλω τοι γράψας ἐν τῷδε τῷ βιβλίῳ τῶν τε λοιπῶν θεωρημάτων τὰς ἀποδείξιας, ὧν οὐκ εἶχες ἐν τοῖς πρότερον ἀπεσταλμένοις, καὶ ἄλλων ὕστερον ποτεξευρημένων, ἃ πρότερον μὲν ἤδη πολλάκις ἐγχειρήσας ἐπισκέπτεσθαι δύσκολον ἔχειν τι φανείσας μοι τᾶς εὑρέσιος αὐτῶν ἀπόρησα· διόπερ οὐδὲ συνεξεδόθεν τοῖς ἄλλοις αὐτὰ τὰ προβεβλημένα. ὕστερον δὲ ἐπιμελέστερον ποτ' αὐτοῖς γενόμενος ἐξεῦρον τὰ ἀπορηθέντα. ἦν δὲ τὰ μὲν λοιπὰ τῶν προτέρων θεωρημάτων περὶ τοῦ ὀρθογωνίου κωνοειδέος προβεβλημένα, τὰ δὲ νῦν ἐντι ποτεξευρημένα περί τε ἀμβλυγωνίου κωνοειδέος καὶ περὶ σφαιροειδέων σχημάτων, ὧν τὰ μὲν παραμάκεα, τὰ δὲ ἐπιπλατέα καλέω.

(ii.) *Two Lemmas*

Ibid., Lemma ad Prop. 1, Archim. ed. Heiberg i. 260. 17-24

Εἴ κα ἔωντι μεγέθεα ὁποσαοῦν τῷ ἴσῳ ἀλλάλων

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[a] In the books *On the Sphere and Cylinder*, *On Spirals* and on the *Quadrature of a Parabola*.

[b] *i.e.*, the paraboloid of revolution.

[c] *i.e.*, the hyperboloid of revolution.

[d] An *oblong spheroid* is formed by the revolution of an

(d) Conoids and Spheroids

(i.) Preface

Archimedes, *On Conoids and Spheroids*, Preface, Archim. ed. Heiberg i. 246. 1-14

Archimedes to Dositheus greeting.

I have written out and now send you in this book the proofs of the remaining theorems, which you did not have among those sent to you before,[a] and also of some others discovered later, which I had often tried to investigate previously but their discovery was attended with some difficulty and I was at a loss over them ; and for this reason not even the propositions themselves were forwarded with the rest. But later, when I had studied them more carefully, I discovered what had left me at a loss before. Now the remainder of the earlier theorems were propositions about the *right-angled conoid* [b] ; but the discoveries now added relate to an *obtuse-angled conoid* [c] and to *spheroidal figures*, of which I call some *oblong* and others *flat*.[d]

(ii.) Two Lemmas

Ibid., Lemma to Prop. 1, Archim. ed. Heiberg i. 260. 17-24

If there be a series of magnitudes, as many as you please, in which each term exceeds the previous term by an

ellipse about its major axis, a *flat spheroid* by the revolution of an ellipse about its minor axis.

In the remainder of our preface Archimedes gives a number of definitions connected with those solids. They are of importance in studying the development of Greek mathematical terminology.

ὑπερέχοντα, ἢ δὲ ἁ ὑπεροχὰ ἴσα τῷ ἐλαχίστῳ, καὶ ἄλλα μεγέθεα τῷ μὲν πλήθει ἴσα τούτοις, τῷ δὲ μεγέθει ἕκαστον ἴσον τῷ μεγίστῳ, πάντα τὰ μεγέθεα, ὧν ἐστιν ἕκαστον ἴσον τῷ μεγίστῳ, πάντων μὲν τῶν τῷ ἴσῳ ὑπερεχόντων ἐλάσσονα ἐσσοῦνται ἢ διπλάσια, τῶν δὲ λοιπῶν χωρὶς τοῦ μεγίστου μείζονα ἢ διπλάσια. ἁ δὲ ἀπόδειξις τούτου φανερά.

Ibid., Prop. 1, Archim. ed. Heiberg i. 260. 26–261. 22

Εἴ κα μεγέθεα ὁποσαοῦν τῷ πλήθει ἄλλοις μεγέθεσιν ἴσοις τῷ πλήθει κατὰ δύο τὸν αὐτὸν λόγον ἔχωντι τὰ ὁμοίως τεταγμένα, λέγηται δὲ τά τε πρῶτα μεγέθεα ποτ' ἄλλα μεγέθεα ἢ πάντα ἤ τινα αὐτῶν ἐν λόγοις ὁποιοισοῦν, καὶ τὰ ὕστερον ποτ' ἄλλα μεγέθεα τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, πάντα τὰ πρῶτα μεγέθεα ποτὶ πάντα, ἃ λέγονται, τὸν αὐτὸν ἑξοῦντι λόγον, ὃν ἔχοντι πάντα τὰ ὕστερον μεγέθεα ποτὶ πάντα, ἃ λέγονται.

Ἔστω τινὰ μεγέθεα τὰ Α, Β, Γ, Δ, Ε, Ζ ἄλλοις μεγέθεσιν ἴσοις τῷ πλήθει τοῖς, Η, Θ, Ι, Κ, Λ, Μ

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[a] If h is the common difference, the first series is h, $2h$, $3h$ . . . nh, and the second series is nh, nh . . . to n terms, its sum obviously being n^2h. The lemma asserts that

$$2(h+2h+3h+\ .\ .\ .\ \overline{n-1}h)<n^2h<2(h+2h+3h+\ .\ .\ .\ nh).$$

It is proved in the book *On Spirals*, Prop. 11. The proof is geometrical, lines being placed side by side to represent the

equal quantity, which common difference is equal to the least term, and if there be a second series of magnitudes, equal to the first in number, in which each term is equal to the greatest term [in the first series], the sum of the magnitudes in the series in which each term is equal to the greatest term is less than double of the sum of the magnitudes differing by an equal quantity, but greater than double of the sum of those magnitudes less the greatest term. The proof of this is clear.[a]

Ibid., Prop. 1, Archim. ed. Heiberg i. 260. 26–261. 22

If there be a series of magnitudes, as many as you please, and it be equal in number to another series of magnitudes, and the terms have the same ratio two by two, and if any or all of the first series of magnitudes form any proportion with another series of magnitudes, and if the second series of magnitudes form the same proportion with the corresponding terms of another series of magnitudes, the sum of the first series of magnitudes bears to the sum of those with which they are in proportion the same ratio as the sum of the second series of magnitudes bears to the sums of the terms with which they are in proportion.

Let the series of magnitudes Α, Β, Γ, Δ, Ε, Ζ be equal in number to the series of magnitudes Η, Θ, Ι,

terms of the arithmetical progression and produced until each is equal to the greatest term. It is equivalent to this algebraical proof:

Let $S_n = h + 2h + 3h + \ldots + nh.$

Then $S_n = nh + (n-1)h + (n-2)h + \ldots + h.$

Adding, $2S_n = n(n+1)h,$

and so $2S_{n-1} = (n-1)nh.$

Therefore $2S_{n-1} < n^2h < 2S_n.$

κατὰ δύο τὸν αὐτὸν ἔχοντα λόγον, καὶ ἐχέτω τὸ μὲν Α ποτὶ τὸ Β τὸν αὐτὸν λόγον, ὃν τὸ Η ποτὶ τὸ Θ, τὸ δὲ Β ποτὶ τὸ Γ, ὃν τὸ Θ ποτὶ τὸ Ι, καὶ τὰ ἄλλα ὁμοίως τούτοις, λεγέσθω δὲ τὰ μὲν Α, Β, Γ, Δ, Ε, Ζ μεγέθεα ποτ' ἄλλα μεγέθεα τὰ Ν, Ξ, Ο, Π, Ρ, Σ ἐν λόγοις ὁποιοισοῦν, τὰ δὲ Η, Θ, Ι, Κ, Λ, Μ ποτ' ἄλλα τὰ Τ, Υ, Φ, Χ, Ψ, Ω, τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, καὶ ὃν μὲν ἔχει λόγον τὸ Α ποτὶ τὸ Ν, τὸ Η ἐχέτω ποτὶ τὸ Τ, ὃν δὲ λόγον ἔχει τὸ Β ποτὶ τὸ Ξ, τὸ Θ ἐχέτω ποτὶ τὸ Υ, καὶ τὰ ἄλλα ὁμοίως τούτοις· δεικτέον, ὅτι πάντα τὰ Α, Β, Γ, Δ, Ε, Ζ ποτὶ πάντα τὰ Ν, Ξ, Ο, Π, Ρ, Σ τὸν αὐτὸν ἔχοντι λόγον, ὃν πάντα τὰ Η, Θ, Ι, Κ, Λ, Μ ποτὶ πάντα τὰ Τ, Υ, Φ, Χ, Ψ, Ω.

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[a] Since $\mathrm{N} : \mathrm{A} = \mathrm{T} : \mathrm{H}$, $\mathrm{A} : \mathrm{B} = \mathrm{H} : \Theta$, [*ex hyp.*

∴ *ex aequo* $\mathrm{N} : \mathrm{B} = \mathrm{T} : \Theta$. [Eucl. v. 22

But $\mathrm{B} : \Xi = \Theta : \Upsilon$; [*ex hyp.*

∴ *ex aequo* $\mathrm{N} : \Xi = \mathrm{T} : \Upsilon$. [Eucl. v. 22

Similarly

$$\Xi : \mathrm{O} = \Upsilon : \Phi,\ \mathrm{O} : \Pi = \Phi : \mathrm{X},\ \Pi : \mathrm{P} = \mathrm{X} : \Psi,\ \mathrm{P} : \Sigma = \Psi : \Omega.$$

Now since $\mathrm{A} : \mathrm{B} = \mathrm{H} : \Theta$, [*ex hyp.*

∴ *componendo* $\mathrm{A} + \mathrm{B} : \mathrm{A} = \mathrm{H} + \Theta : \mathrm{H}$. [Eucl. v. 18

i.e., *permutando* $\mathrm{A} + \mathrm{B} : \mathrm{H} + \Theta = \mathrm{A} : \mathrm{H}$. [Eucl. v. 16

But since $\mathrm{N} : \mathrm{A} = \mathrm{T} : \mathrm{H}$, [*ex hyp.*

∴ $\mathrm{A} : \mathrm{H} = \mathrm{N} : \mathrm{T}$ [Eucl. v. 16

$= \Xi : \Upsilon$ [*ibid.*

$= \mathrm{O} : \Phi$ [*ibid.*

$= \Gamma : \mathrm{I}$. [*ibid.*

∴ $\mathrm{A} + \mathrm{B} : \mathrm{H} + \Theta = \Gamma : \mathrm{I}$.

∴ $\mathrm{A} + \mathrm{B} + \Gamma : \mathrm{H} + \Theta + \mathrm{I} = \Gamma : \mathrm{I}$ [Eucl. v. 18

$= \mathrm{O} : \Phi$ [Eucl. v. 16

K, Λ, M, and let them have the same ratio two by two, so that

$$A : B = H : \Theta,\ B : \Gamma = \Theta : I,$$

and so on, and let the series of magnitudes A, B, Γ, Δ, E, Z form any proportion with another series of magnitudes N, Ξ, O, Π, P, Σ, and let H, Θ, I, K, Λ, M form the same proportion with the corresponding terms of another series, T, Υ, Φ, X, Ψ, Ω so that

$$A : N = H : T,\ B : \Xi = \Theta : \Upsilon,$$

and so on ; it is required to prove that

$$\frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}.$$[a]

$$= \Pi : X \qquad [ibid.$$
$$= \Delta : K. \qquad [ibid.$$

By pursuing this method it may be proved that

$$A + B + \Gamma + \Delta + E + Z : H + \Theta + I + K + \Lambda + M = A : H,$$

or, *permutando,*

$$A + B + \Gamma + \Delta + E + Z : A = H + \Theta + I + K + \Lambda + M : H \quad . \ (1)$$

Now $N : \Xi = T : \Upsilon$;

∴ *componendo et permutando,*

$$N + \Xi : T + \Upsilon = \Xi : \Upsilon \qquad \text{[Eucl. v. 18, v. 16}$$
$$= O : \Phi; \qquad \text{[Eucl. v. 16}$$

whence $N + \Xi + O : T + \Upsilon + \Phi = O : \Phi$, [Eucl. v. 18

and so on until we obtain

$$N + \Xi + O + \Pi + P + \Sigma : T + \Upsilon + \Phi + X + \Psi + \Omega = N : T \quad . \quad . \ (2)$$

But $A : N = H : T$; [*ex hyp.*

∴ by (1) and (2),

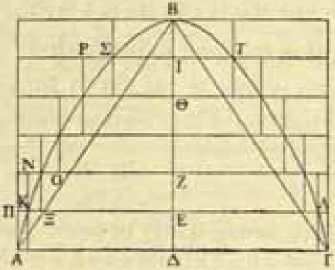
$$\frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}.$$

Q.E.D.

(iii.) *Volume of a Segment of a Paraboloid of Revolution*

Ibid., Prop. 21, Archim. ed. Heiberg i. 344. 21–354. 20

Πᾶν τμᾶμα ὀρθογωνίου κωνοειδέος ἀποτετμαμένον ἐπιπέδῳ ὀρθῷ ποτὶ τὸν ἄξονα ἡμιόλιόν ἐστι τοῦ κώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν τῷ τμάματι καὶ ἄξονα.



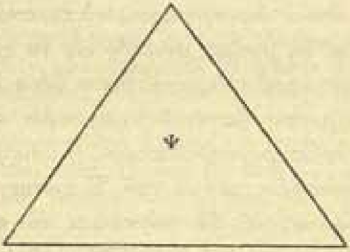
Ἔστω γὰρ τμᾶμα ὀρθογωνίου κωνοειδέος ἀποτετμαμένον ὀρθῷ ἐπιπέδῳ ποτὶ τὸν ἄξονα, καὶ τμαθέντος αὐτοῦ ἐπιπέδῳ ἄλλῳ διὰ τοῦ ἄξονος τᾶς μὲν ἐπιφανείας τομὰ ἔστω ἁ ΑΒΓ ὀρθογωνίου κώνου τομά, τοῦ δὲ ἐπιπέδου τοῦ ἀποτέμνοντος τὸ τμᾶμα ἁ ΓΑ εὐθεῖα, ἄξων δὲ ἔστω τοῦ τμάματος ἁ ΒΔ, ἔστω δὲ καὶ κῶνος τὰν αὐτὰν βάσιν ἔχων τῷ τμάματι καὶ ἄξονα τὸν αὐτόν, οὗ κορυφὰ τὸ Β. δεικτέον, ὅτι τὸ τμᾶμα τοῦ κωνοειδέος ἡμιόλιόν ἐστι τοῦ κώνου τούτου.

Ἐκκείσθω γὰρ κῶνος ὁ Ψ ἡμιόλιος ἐὼν τοῦ κώνου, οὗ βάσις ὁ περὶ διάμετρον τὰν ΑΓ, ἄξων δὲ ἁ ΒΔ, ἔστω δὲ καὶ κύλινδρος βάσιν μὲν ἔχων

(iii.) *Volume of a Segment of a Paraboloid of Revolution*

Ibid., Prop. 21, Archim. ed. Heiberg i. 344. 21–354. 20

Any segment of a right-angled conoid cut off by a plane perpendicular to the axis is one-and-a-half times the cone having the same base as the segment and the same axis.



For let there be a segment of a right-angled conoid cut off by a plane perpendicular to the axis, and let it be cut by another plane through the axis, and let the section be the section of a right-angled cone ΑΒΓ,[a] and let ΓΑ be a straight line in the plane cutting off the segment, and let ΒΔ be the axis of the segment, and let there be a cone, with vertex Β, having the same base and the same axis as the segment. It is required to prove that the segment of the conoid is one-and-a-half times this cone.

For let there be set out a cone Ψ one-and-a-half times as great as the cone with base about the diameter ΑΓ and with axis ΒΔ, and let there be a

[a] It is proved in Prop. 11 that the section will be a parabola.

τὸν κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ τὰν ΒΔ· ἐσσεῖται οὖν ὁ Ψ κῶνος ἡμίσεος τοῦ κυλίνδρου [ἐπείπερ ἡμιόλιός ἐστιν ὁ Ψ κῶνος τοῦ αὐτοῦ κώνου].[1] λέγω, ὅτι τὸ τμᾶμα τοῦ κωνοειδέος ἴσον ἐστὶ τῷ Ψ κώνῳ.

Εἰ γὰρ μή ἐστιν ἴσον, ἤτοι μεῖζόν ἐντι ἢ ἔλασσον. ἔστω δὴ πρότερον, εἰ δυνατόν, μεῖζον. ἐγγεγράφθω δὴ σχῆμα στερεὸν εἰς τὸ τμᾶμα, καὶ ἄλλο περιγεγράφθω ἐκ κυλίνδρων ὕψος ἴσον ἐχόντων συγκείμενον, ὥστε τὸ περιγραφὲν σχῆμα τοῦ ἐγγραφέντος ὑπερέχειν ἐλάσσονι, ἢ ἁλίκῳ ὑπερέχει τὸ τοῦ κωνοειδέος τμᾶμα τοῦ Ψ κώνου, καὶ ἔστω τῶν κυλίνδρων, ἐξ ὧν σύγκειται τὸ περιγραφὲν σχῆμα, μέγιστος μὲν ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ τὰν ΕΔ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἄξονα δὲ τὰν ΒΙ, τῶν δὲ κυλίνδρων, ἐξ ὧν σύγκειται τὸ ἐγγραφὲν σχῆμα, μέγιστος μὲν ἔστω ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΚΛ, ἄξονα δὲ τὰν ΔΕ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἄξονα δὲ τὰν ΘΙ, ἐκβεβλήσθω δὲ τὰ ἐπίπεδα πάντων τῶν κυλίνδρων ποτὶ τὰν

[1] ἐπείπερ . . . κώνου om. Heiberg.

[a] For the cylinder is three times, and the cone Ψ one-and-a-

cylinder having for its base the circle about the diameter ΑΓ and for its axis ΒΔ; then the cone Ψ is one-half of the cylinder [a]; I say that the segment of the conoid is equal to the cone Ψ.

If it be not equal, it is either greater or less. Let it first be, if possible, greater. Then let there be inscribed in the segment a solid figure and let there be circumscribed another solid figure made up of cylinders having an equal altitude,[b] in such a way that the circumscribed figure exceeds the inscribed figure by a quantity less than that by which the segment of the conoid exceeds the cone Ψ [Prop. 19]; and let the greatest of the cylinders composing the circumscribed figure be that having for its base the circle about the diameter ΑΓ and for axis ΕΔ, and let the least be that having for its base the circle about the diameter ΣΤ and for axis ΒΙ; and let the greatest of the cylinders composing the inscribed figure be that having for its base the circle about the diameter ΚΛ and for axis ΔΕ, and let the least be that having for its base the circle about the diameter ΣΤ and for axis ΘΙ; and let the planes of all the cylinders be

half times, as great as the same cone; but because *τοῦ αὐτοῦ κώνου* is obscure and *ἐπείπερ* often introduces an interpolation, Heiberg rejects the explanation to this effect in the text.

[b] Archimedes has used those inscribed and circumscribed figures in previous propositions. The paraboloid is generated by the revolution of the parabola ΑΒΓ about its axis ΒΔ. Chords ΚΛ . . . ΣΤ are drawn in the parabola at right angles to the axis and at equal intervals from each other. From the points where they meet the parabola, perpendiculars are drawn to the next chords. In this way there are built up inside and outside the parabola "staggered" figures consisting of decreasing rectangles. When the parabola revolves, the rectangles become cylinders, and the segment of the paraboloid lies between the inscribed set of cylinders and the circumscribed set of cylinders.

ἐπιφάνειαν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τὸν κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ τὰν ΒΔ· ἐσσεῖται δὴ ὁ ὅλος κύλινδρος διῃρημένος εἰς κυλίνδρους τῷ μὲν πλήθει ἴσους τοῖς κυλίνδροις τοῖς ἐν τῷ περιγεγραμμένῳ σχήματι, τῷ δὲ μεγέθει ἴσους τῷ μεγίστῳ αὐτῶν. καὶ ἐπεὶ τὸ περιγεγραμμένον σχῆμα περὶ τὸ τμᾶμα ἐλάσσονι ὑπερέχει τοῦ ἐγγεγραμμένου σχήματος ἢ τὸ τμᾶμα τοῦ κώνου, δῆλον, ὅτι καὶ τὸ ἐγγεγραμμένον σχῆμα ἐν τῷ τμάματι μεῖζόν ἐστι τοῦ Ψ κώνου. ὁ δὴ πρῶτος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κύλινδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν ἔχοντα ἄξονα τὰν ΔΕ τὸν αὐτὸν ἔχει λόγον, ὃν ἁ ΔΑ ποτὶ τὰν ΚΕ δυνάμει· οὗτος δέ ἐστιν ὁ αὐτὸς τῷ, ὃν ἔχει ἁ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, ὃν ἔχει ἁ ΔΑ ποτὶ τὰν ΕΞ. ὁμοίως δὲ δειχθήσεται καὶ ὁ δεύτερος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὸν ΕΖ ποτὶ τὸν δεύτερον κύλινδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν αὐτὸν ἔχειν λόγον, ὃν ἁ ΠΕ, τουτέστιν ἁ ΔΑ, ποτὶ τὰν ΖΟ, καὶ τῶν ἄλλων κυλίνδρων ἕκαστος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ἄξονα ἐχόντων ἴσον τᾷ ΔΕ ποτὶ ἕκαστον τῶν κυλίνδρων τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι ἄξονα ἐχόντων τὸν αὐτὸν ἕξει τοῦτον τὸν λόγον, ὃν ἁ ἡμίσεια τᾶς διαμέτρου τᾶς βάσιος αὐτοῦ ποτὶ τὰν ἀπολελαμμέναν ἀπ' αὐτᾶς μεταξὺ τᾶν ΑΒ, ΒΔ εὐθειᾶν· καὶ πάντες οὖν οἱ κυλίνδροι οἱ ἐν τῷ κυλίνδρῳ, οὗ βάσις μέν ἐστιν ὁ κύκλος ὁ περὶ διάμετρον τὰν ΑΓ, ἄξων δέ [ἐστιν][1] ἁ ΔΙ εὐθεῖα, ποτὶ πάντας τοὺς κυλίνδρους τοὺς ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν αὐτὸν ἑξοῦντι λόγον, ὃν

produced to the surface of the cylinder having for its base the circle about the diameter ΑΓ and for axis ΒΔ; then the whole cylinder is divided into cylinders equal in number to the cylinders in the circumscribed figure and in magnitude equal to the greatest of them. And since the figure circumscribed about the segment exceeds the inscribed figure by a quantity less than that by which the segment exceeds the cone, it is clear that the figure inscribed in the segment is greater than the cone Ψ.[a] Now the first cylinder of those in the whole cylinder, that having ΔΕ for its axis, bears to the first cylinder in the inscribed figure, which also has ΔΕ for its axis, the ratio $ΔΑ^2 : ΚΕ^2$ [Eucl. xii. 11 and xii. 2]; but $ΔΑ^2 : ΚΕ^2 = ΒΔ : ΒΕ$ [b] $= ΔΑ : ΕΞ$. Similarly it may be proved that the second cylinder of those in the whole cylinder, that having ΕΖ for its axis, bears to the second cylinder in the inscribed figure the ratio ΠΕ : ΖΟ, that is, ΔΑ : ΖΟ, and each of the other cylinders in the whole cylinder, having its axis equal to ΔΕ, bears to each of the cylinders in the inscribed figure, having the same axis in order, the same ratio as half the diameter of the base bears to the part cut off between the straight lines ΑΒ, ΒΔ; and therefore the sum of the cylinders in the cylinder having for its base the circle about the diameter ΑΓ and for axis the straight line ΔΙ bears to the sum of the cylinders in the inscribed figure the same ratio as the sum of

[a] Because the circumscribed figure is greater than the segment.

[b] By the property of the parabola; v. *Quadr. parab.* 3.

[1] ἐστιν om. Heiberg.

πᾶσαι αἱ εὐθεῖαι αἱ ἐκ τῶν κέντρων τῶν κύκλων, οἵ ἐντι βάσιες τῶν εἰρημένων κυλίνδρων, ποτὶ πάσας τὰς εὐθείας τὰς ἀπολελαμμένας ἀπ' αὐτᾶν μεταξὺ τᾶν ΑΒ, ΒΔ. αἱ δὲ εἰρημέναι εὐθεῖαι τῶν εἰρημένων χωρὶς τᾶς ΑΔ μείζονές ἐντι ἢ διπλάσιαι· ὥστε καὶ οἱ κύλινδροι πάντες οἱ ἐν τῷ κυλίνδρῳ, οὗ ἄξων ὁ ΔΙ, μείζονές ἐντι ἢ διπλάσιοι τοῦ ἐγγεγραμμένου σχήματος· πολλῷ ἄρα καὶ ὁ ὅλος κύλινδρος, οὗ ἄξων ἁ ΔΒ, μείζων ἐντὶ ἢ διπλασίων τοῦ ἐγγεγραμμένου σχήματος. τοῦ δὲ Ψ κώνου ἦν διπλασίων· ἔλασσον ἄρα τὸ ἐγγεγραμμένον σχῆμα τοῦ Ψ κώνου· ὅπερ ἀδύνατον· ἐδείχθη γὰρ μεῖζον. οὐκ ἄρα ἐστὶν μεῖζον τὸ κωνοειδὲς τοῦ Ψ κώνου.

Ὁμοίως δὲ οὐδὲ ἔλασσον· πάλιν γὰρ ἐγγεγράφθω τὸ σχῆμα καὶ περιγεγράφθω, ὥστε ὑπερέχειν [ἕκαστον][1] ἐλάσσονι, ᾗ ἀλίκῳ ὑπερέχει ὁ Ψ κῶνος τοῦ κωνοειδέος, καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρότερον κατεσκευάσθω. ἐπεὶ οὖν ἔλασσόν ἐστι τὸ ἐγγεγραμμένον σχῆμα τοῦ τμάματος, καὶ τὸ ἐγγραφὲν τοῦ περιγραφέντος ἐλάσσονι λείπεται ἢ τὸ τμᾶμα τοῦ Ψ κώνου, δῆλον, ὡς ἔλασσόν ἐστι τὸ περιγραφὲν σχῆμα τοῦ Ψ κώνου. πάλιν δὲ ὁ

[1] ἕκαστον om. Heiberg, ἕκαστον ἑκάστου Torelli (for ἑκάτερον ἑκατέρου).

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[a] *i.e.*, $\frac{\text{First cylinder in whole cylinder}}{\text{First cylinder in inscribed figure}} = \frac{\Delta\Lambda}{\mathrm{E}\Xi}$,

$\frac{\text{Second cylinder in whole cylinder}}{\text{Second cylinder in inscribed figure}} = \frac{\mathrm{E}\Pi}{\mathrm{ZO}}$,

and so on.

$$\therefore \quad \frac{\text{Whole cylinder}}{\text{Inscribed figure}} = \frac{\Delta\Lambda + \mathrm{E}\Pi + \dots}{\mathrm{E}\Xi + \mathrm{ZO} + \dots}.$$

the radii of the circles, which are the bases of the aforesaid cylinders, bears to the sum of the straight lines cut off from them between AB, BΔ.[a] But the sum of the aforesaid straight lines is greater than double of the aforesaid straight lines without AΔ[b]; so that the sum of the cylinders in the cylinder whose axis is ΔI is greater than double of the inscribed figure; therefore the whole cylinder, whose axis is ΔB, is greater by far than double of the inscribed figure. But it was double of the cone Ψ; therefore the inscribed figure is less than the cone Ψ; which is impossible, for it was proved to be greater. Therefore the conoid is not greater than the cone Ψ.

Similarly [it can be shown] not to be less; for let the figure be again inscribed and another circumscribed so that the excess is less than that by which the cone Ψ exceeds the conoid, and let the rest of the construction be as before. Then because the inscribed figure is less than the segment, and the inscribed figure is less than the circumscribed by some quantity less than the difference between the segment and the cone Ψ, it is clear that the circumscribed figure is less than the cone Ψ. Again, the first

[a] This follows from Prop. 1, for

$$\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}}=1=\frac{\Delta\mathrm{A}}{\mathrm{E\Pi}},$$

and so on, and thus the other condition of the theorem is satisfied.

[b] For ΔA, EΞ, ZO . . . is a series diminishing in arithmetical progression, and ΔA, EΠ . . . is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression. Therefore, by the Lemma to Prop. 1,

$$\Delta\mathrm{A}+\mathrm{E\Pi}+\ldots>2(\mathrm{E\Xi}+\mathrm{ZO}+\ldots).$$

πρῶτος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχήματι τὸν τὸν αὐτὸν ἔχοντα ἄξονα τὰν ΕΔ τὸν αὐτὸν ἔχει λόγον, ὃν τὸ ἀπὸ τᾶς ΑΔ τετράγωνον ποτὶ τὸ αὐτό, ὁ δὲ δεύτερος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὰν ΕΖ ποτὶ τὸν δεύτερον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχήματι τὸν ἔχοντα ἄξονα τὰν ΕΖ τὸν αὐτὸν ἔχει λόγον, ὃν ἁ ΔΑ ποτὶ τὰν ΚΕ δυνάμει· οὗτος δέ ἐστιν ὁ αὐτὸς τῷ, ὃν ἔχει ἁ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, ὃν ἔχει ἁ ΔΑ ποτὶ τὰν ΕΞ· καὶ τῶν ἄλλων κυλίνδρων ἕκαστος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ἄξονα ἐχόντων ἴσον τᾷ ΔΕ ποτὶ ἕκαστον τῶν κυλίνδρων τῶν ἐν τῷ περιγεγραμμένῳ σχήματι ἄξονα ἐχόντων τὸν αὐτόν, ἕξει τοῦτον τὸν λόγον, ὃν ἁ ἡμίσεια τᾶς διαμέτρου τᾶς βάσιος αὐτοῦ ποτὶ τὰν ἀπολελαμμέναν ἀπ' αὐτᾶς μεταξὺ τᾶν ΑΒ, ΒΔ εὐθειᾶν· καὶ πάντες οὖν οἱ κυλίνδροι οἱ ἐν τῷ ὅλῳ κυλίνδρῳ, οὗ ἄξων ἐστὶν ἁ ΒΔ εὐθεῖα, ποτὶ πάντας τοὺς κυλίνδρους τοὺς ἐν τῷ περιγεγραμμένῳ σχήματι τὸν αὐτὸν ἑξοῦντι λόγον, ὃν πᾶσαι αἱ εὐθεῖαι ποτὶ πάσας τὰς εὐθείας. αἱ δὲ εὐθεῖαι πᾶσαι αἱ ἐκ τῶν κέντρων τῶν κύκλων, οἳ βάσιές ἐντι τῶν κυλίνδρων, τᾶν εὐθειᾶν πασᾶν τᾶν ἀπολελαμμενᾶν ἀπ' αὐτᾶν σὺν τᾷ ΑΔ ἐλάσσονές ἐντι

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[a] As before,

$$\frac{\text{First cylinder in whole cylinder}}{\text{First cylinder in circumscribed figure}} = \frac{\Delta\mathrm{A}}{\Delta\mathrm{A}}$$

$$\frac{\text{Second cylinder in whole cylinder}}{\text{Second cylinder in circumscribed figure}} = \frac{\Delta\mathrm{A}}{\mathrm{E}\Xi} = \frac{\mathrm{EH}}{\mathrm{E}\Xi},$$

and so on.

cylinder of those in the whole cylinder, having ΔE for its axis, bears to the first cylinder of those in the circumscribed figure, having the same axis EΔ, the ratio $A\Delta^2 : A\Delta^2$; the second cylinder in the whole cylinder, having EZ for its axis, bears to the second cylinder in the circumscribed figure, having EZ also for its axis, the ratio $\Delta A^2 : KE^2$; this is the same as $B\Delta : BE$, and this is the same as $\Delta A : E\Xi$; and each of the other cylinders in the whole cylinder, having its axis equal to ΔE, will bear to the corresponding cylinder in the circumscribed figure, having the same axis, the same ratio as half the diameter of the base bears to the portion cut off from it between the straight lines AB, BΔ; and therefore the sum of the cylinders in the whole cylinder, whose axis is the straight line BΔ, bears to the sum of the cylinders in the circumscribed figure the same ratio as the sum of the one set of straight lines bears to the sum of the other set of straight lines.[a] But the sum of the radii of the circles which are the bases of the cylinders is less than double of the sum of the straight lines cut off from them together with AΔ [b]; it is therefore clear

And

$$\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}}=1=\frac{\Delta A}{E\Pi},$$

and so on.

Therefore the conditions of Prop. 1 are satisfied and

$$\frac{\text{Whole cylinder}}{\text{Circumscribed figure}}=\frac{\Delta A+E\Pi+\ldots}{\Delta A+E\Xi+\ldots}.$$

[a] As before, $\Delta A, E\Xi \ldots$ is a series diminishing in arithmetical progression, and $\Delta A, E\Pi \ldots$ is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression.

Therefore, by the Lemma to Prop. 1,

$$\Delta A+E\Pi+\ldots<2(\Delta A+E\Xi+\ldots).$$

ἢ διπλάσιαι· δῆλον οὖν, ὅτι καὶ οἱ κυλίνδροι πάντες οἱ ἐν τῷ ὅλῳ κυλίνδρῳ ἐλάσσονές ἐντι ἢ διπλάσιοι τῶν κυλίνδρων τῶν ἐν τῷ περιγεγραμμένῳ σχήματι· ὁ ἄρα κύλινδρος ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ τὰν ΒΔ, ἐλάσσων ἐστὶν ἢ διπλασίων τοῦ περιγεγραμμένου σχήματος. οὐκ ἔστι δέ, ἀλλὰ μείζων ἢ διπλάσιος· τοῦ γὰρ Ψ κώνου διπλασίων ἐστί, τὸ δὲ περιγεγραμμένον σχῆμα ἔλαττον ἐδείχθη τοῦ Ψ κώνου. οὐκ ἄρα ἐστὶν οὐδὲ ἔλασσον τὸ τοῦ κωνοειδέος τμᾶμα τοῦ Ψ κώνου. ἐδείχθη δέ, ὅτι οὐδὲ μεῖζον· ἡμιόλιον ἄρα ἐστὶν τοῦ κώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν τῷ τμάματι καὶ ἄξονα τὸν αὐτόν.

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[a] Archimedes' proof may be shown to be equivalent to an integration, as Heath has done (*The Works of Archimedes*, cxlvii-cxlviii).

For, if n be the number of cylinders in the whole cylinder, and $A\Delta = nh$, Archimedes has shown that

$$\frac{\text{Whole cylinder}}{\text{Inscribed figure}} = \frac{n^2h}{h+2h+3h+\ldots(n-1)h} > 2, \qquad \text{[Lemma to Prop. 1}$$

and

$$\frac{\text{Whole cylinder}}{\text{Circumscribed figure}} = \frac{n^2h}{n+2h+3h+\ldots+nh} < 2. \qquad [\textit{ibid.}$$

In Props. 19 and 20 he has meanwhile shown that, by increasing n sufficiently, the inscribed and circumscribed figures can be made to differ by less than any assigned volume.

that the sum of all the cylinders in the whole cylinder is less than double of the cylinders in the circumscribed figure; therefore the cylinder having for its base the circle about the diameter ΑΓ and for axis ΒΔ is less than double of the circumscribed figure. But it is not, for it is greater than double; for it is double of the cone Ψ, and the circumscribed figure was proved to be less than the cone Ψ. Therefore the segment of the conoid is not less than the cone Ψ. But it was proved not to be greater; therefore it is one-and-a-half times the cone having the same base as the segment and the same axis.[a]

When n is increased, h is diminished, but their product remains constant; let $nh=c$.

Then the proof is equivalent to an assertion that, when n is indefinitely increased,

limit of $\quad h[h+2h+3h+\ .\ .\quad +(n-1)h]=\frac{1}{2}c^2$,

which, in the notation of the integral calculus reads,

$$\int_0^c xdx=\tfrac{1}{2}c^2.$$

If the paraboloid is formed by the revolution of the parabola $y^2=ax$ about its axis, we should express the volume of a segment as

$$\int_0^c \pi y^2dx,$$

or

$$\pi a.\int_0^c xdx.$$

The constant does not appear in Archimedes' proof because he merely compares the volume of the segment with the cone, and does not give its absolute value. But his method is seen to be equivalent to a genuine integration.

As in other cases, Archimedes refrains from the final step of making the divisions in his circumscribed and inscribed figures indefinitely large; he proceeds by the orthodox method of *reductio ad absurdum*.

(e) The Spiral of Archimedes

(i.) *Definitions*

Archim. *De Lin. Spir.*, Deff., Archim. ed. Heiberg ii. 44. 17–46. 21

α΄. Εἴ κα εὐθεῖα ἐπιζευχθῇ γραμμὰ ἐν ἐπιπέδῳ καὶ μένοντος τοῦ ἑτέρου πέρατος αὐτᾶς ἰσοταχέως περιενεχθεῖσα ὁσακισοῦν ἀποκατασταθῇ πάλιν, ὅθεν ὥρμασεν, ἅμα δὲ τᾷ γραμμᾷ περιαγομένᾳ φέρηταί τι σαμεῖον ἰσοταχέως αὐτὸ ἑαυτῷ κατὰ τᾶς εὐθείας ἀρξάμενον ἀπὸ τοῦ μένοντος πέρατος, τὸ σαμεῖον ἕλικα γράψει ἐν τῷ ἐπιπέδῳ.

β΄. Καλείσθω οὖν τὸ μὲν πέρας τᾶς εὐθείας τὸ μένον περιαγομένας αὐτᾶς ἀρχὰ τᾶς ἕλικος.

γ΄. Ἁ δὲ θέσις τᾶς γραμμᾶς, ἀφ᾿ ἇς ἄρξατο ἁ εὐθεῖα περιφέρεσθαι, ἀρχὰ τῆς περιφορᾶς.

δ΄. Εὐθεῖα, ἃν μὲν ἐν τᾷ πρώτᾳ περιφορᾷ διαπορευθῇ τὸ σαμεῖον τὸ κατὰ τᾶς εὐθείας φερόμενον, πρώτα καλείσθω, ἃν δ᾿ ἐν τᾷ δευτέρᾳ περιφορᾷ τὸ αὐτὸ σαμεῖον διανύσῃ, δευτέρα, καὶ αἱ ἄλλαι ὁμοίως ταύταις ὁμωνύμως ταῖς περιφοραῖς καλείσθωσαν.

ε΄. Τὸ δὲ χωρίον τὸ περιλαφθὲν ὑπό τε τᾶς ἕλικος τᾶς ἐν τᾷ πρώτᾳ περιφορᾷ γραφείσας καὶ τᾶς εὐθείας, ἅ ἐστιν πρώτα, πρῶτον καλείσθω, τὸ δὲ περιλαφθὲν ὑπό τε τᾶς ἕλικος τᾶς ἐν τᾷ δευτέρᾳ περιφορᾷ γραφείσας καὶ τᾶς εὐθείας τᾶς δευτέρας δεύτερον καλείσθω, καὶ τὰ ἄλλα ἑξῆς οὕτω καλείσθω.

ϛ΄. Καὶ εἴ κα ἀπὸ τοῦ σαμείου, ὅ ἐστιν ἀρχὰ τᾶς ἕλικος, ἀχθῇ τις εὐθεῖα γραμμά, τᾶς εὐθείας ταύτας

(e) The Spiral of Archimedes

(i.) *Definitions*

Archimedes, *On Spirals*, Definitions, Archim. ed. Heiberg ii. 44. 17–46. 21

1. If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a *spiral* in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral.

3. Let the position of the line, from which the straight line began to revolve, be called the *initial line* of the revolution.

4. Let the distance along the straight line which the point moving along the straight line traverses in the first turn be called the *first distance*, let the distance which the same point traverses in the second turn be called the *second distance*, and in the same way let the other distances be called according to the number of turns.

5. Let the area comprised between the first turn of the spiral and the first distance be called the *first area*, let the area comprised between the second turn of the spiral and the second distance be called the *second area*, and let the remaining areas be so called in order.

6. And if any straight line be drawn from the origin, let [points] on the side of this straight line in

τὰ ἐπὶ τὰ αὐτά, ἐφ' ἅ κα ἁ περιφορὰ γένηται, προαγούμενα καλείσθω, τὰ δὲ ἐπὶ θάτερα ἑπόμενα.

ζ'. Ὅ τε γραφεὶς κύκλος κέντρῳ μὲν τῷ σαμείῳ, ὅ ἐστιν ἀρχὰ τᾶς ἕλικος, διαστήματι δὲ τᾷ εὐθείᾳ, ἅ ἐστιν πρώτα, πρῶτος καλείσθω, ὁ δὲ γραφεὶς κέντρῳ μὲν τῷ αὐτῷ, διαστήματι δὲ τᾷ διπλασίᾳ εὐθείᾳ δεύτερος καλείσθω, καὶ οἱ ἄλλοι δὲ ἑξῆς τούτοις τὸν αὐτὸν τρόπον.

(ii.) *Fundamental Property*

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9–52. 15

Εἴ κα ποτὶ τὰν ἕλικα τὰν ἐν τᾷ πρώτᾳ περιφορᾷ γεγραμμέναν ποτιπεσῶντι δύο εὐθεῖαι ἀπὸ τοῦ σαμείου, ὅ ἐστιν ἀρχὰ τᾶς ἕλικος, καὶ ἐκβληθέωντι ποτὶ τὰν τοῦ πρώτου κύκλου περιφέρειαν, τὸν αὐτὸν ἑξοῦντι λόγον αἱ ποτὶ τὰν ἕλικα ποτιπίπτουσαι ποτ' ἀλλάλας, ὃν αἱ περιφέρειαι τοῦ κύκλου αἱ μεταξὺ τοῦ πέρατος τᾶς ἕλικος καὶ τῶν περάτων τᾶν ἐκβληθεισᾶν εὐθειᾶν τῶν ἐπὶ τᾶς περιφερείας γινομένων, ἐπὶ τὰ προαγούμενα λαμβανομενᾶν τᾶν περιφερειᾶν ἀπὸ τοῦ πέρατος τᾶς ἕλικος.

Ἔστω ἕλιξ ἁ ΑΒΓΔΕΘ ἐν τᾷ πρώτᾳ περιφορᾷ γεγραμμένα, ἀρχὰ δὲ τᾶς μὲν ἕλικος ἔστω τὸ Α σαμεῖον, ἁ δὲ ΘΑ εὐθεῖα ἀρχὰ τᾶς περιφορᾶς ἔστω, καὶ κύκλος ὁ ΘΚΗ ἔστω ὁ πρῶτος, ποτιπιπτόντων δὲ ἀπὸ τοῦ Α σαμείου ποτὶ τὰν ἕλικα αἱ ΑΕ, ΑΔ καὶ ἐκπιπτόντων ποτὶ τὰν τοῦ κύκλου περιφέρειαν ἐπὶ τὰ Ζ, Η. δεικτέον, ὅτι τὸν αὐτὸν ἔχοντι λόγον ἁ ΑΕ ποτὶ τὰν ΑΔ, ὃν ἁ ΘΚΖ περιφέρεια ποτὶ τὰν ΘΚΗ περιφέρειαν.

Περιαγομένας γὰρ τᾶς ΑΘ γραμμᾶς δῆλον, ὡς

the direction of the revolution be called *forward*, and let those on the other side be called *rearward*.

7. Let the circle described with the origin as centre and the first distance as radius be called the *first circle*, let the circle described with the same centre and double of the radius of the first circle [a] be called the *second circle*, and let the remaining circles in order be called after the same manner.

(ii.) *Fundamental Property*

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9–52. 15

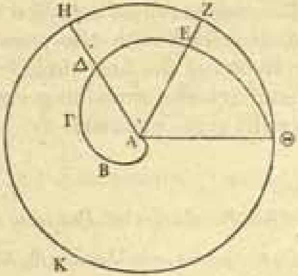
If, from the origin of the spiral, two straight lines be drawn to meet the first turn of the spiral and produced to meet the circumference of the first circle, the lines drawn to the spiral will have the same ratio one to the other as the arcs of the circle between the extremity of the spiral and the extremities of the straight lines produced to meet the circumference, the arcs being measured in a forward direction from the extremity of the spiral.

Let ΑΒΓΔΕΘ be the first turn of a spiral, let the point Α be the origin of the spiral, let ΘΑ be the initial line, let ΘΚΗ be the first circle, and from the point Α let ΑΕ, ΑΔ be drawn to meet the spiral and be produced to meet the circumference of the circle at Ζ, Η. It is required to prove that ΑΕ : ΑΔ = arc ΘΚΖ : arc ΘΚΗ.

When the line ΑΘ revolves it is clear that the point

[a] *i.e.*, with radius equal to the sum of the radii of the first and second circles.

τὸ μὲν Θ σαμεῖον κατὰ τᾶς τοῦ ΘΚΗ κύκλου περιφερείας ἐνηνεγμένον ἐστὶν ἰσοταχέως, τὸ δὲ



Α κατὰ τᾶς εὐθείας φερόμενον τὰν ΑΘ γραμμὰν πορεύεται, καὶ τὸ Θ σαμεῖον κατὰ τᾶς τοῦ κύκλου περιφερείας φερόμενον τὰν ΘΚΖ περιφέρειαν, τὸ δὲ Α τὰν ΑΕ εὐθεῖαν, καὶ πάλιν τό τε Α σαμεῖον τὰν ΑΔ γραμμὰν καὶ τὸ Θ τὰν ΘΚΗ περιφέρειαν, ἑκάτερον ἰσοταχέως αὐτὸ ἑαυτῷ φερόμενον· δῆλον οὖν, ὅτι τὸν αὐτὸν ἔχοντι λόγον ἁ ΑΕ ποτὶ τὰν ΑΔ, ὃν ἁ ΘΚΖ περιφέρεια ποτὶ τὰν ΘΚΗ περιφέρειαν [δέδεικται γὰρ τοῦτο ἔξω ἐν τοῖς πρώτοις].[1] Ὁμοίως δὲ δειχθήσεται, καὶ εἴ κα ἁ ἑτέρα τᾶν ποτιπιπτουσᾶν ἐπὶ τὸ πέρας τᾶς ἕλικος ποτιπίπτῃ, ὅτι τὸ αὐτὸ συμβαίνει.

(iii.) *A Verging*

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

Τῶν αὐτῶν δεδομένων καὶ τᾶς ἐν τῷ κύκλῳ εὐθείας ἐκβεβλημένας δυνατόν ἐστιν ἀπὸ τοῦ

Θ moves uniformly round the circumference ΘKH of the circle while the point A, which moves along the straight line, traverses the line AΘ; the point Θ which moves round the circumference of the circle traverses the arc ΘKZ while A traverses the straight line AE; and furthermore the point A traverses the line AΔ in the same time as Θ traverses the arc ΘKH, each moving uniformly; it is clear, therefore, that AE : AΔ = arc ΘKZ : arc ΘKH [Prop. 2].

Similarly it may be shown that if one of the straight lines be drawn to the extremity of the spiral the same conclusion follows.[a]

(iii.) *A Verging*[b]

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

With the same data and the chord in the circle produced,[c] it is possible to draw a line from the centre to meet

[a] In Prop. 15 Archimedes shows (using different letters, however) that if AE, AΔ are drawn to meet the *second* turn of the spiral, while AZ, AH are drawn, as before, to meet the circumference of the *first* circle, then

$$AE : A\Delta = \text{arc } \Theta KZ + \text{circumference of first circle} : \text{arc } \Theta KH + \text{circumference of first circle},$$

and so on for higher turns.

In general, if E, Δ lie on the nth turn of the spiral, and the circumference of the first circle is c, then

$$AE : A\Delta = \text{arc } \Theta KZ + \overline{n-1}c : \text{arc } \Theta KH + \overline{n-1}c.$$

These theorems correspond to the equation of the curve $r = a\theta$ in polar co-ordinates.

[b] This theorem is essential to the one that follows.

[c] See n. *a* on this page.

[1] δέδεικται . . . πρώτοις om. Heiberg.

κέντρου ποτιβαλεῖν ποτὶ τὰν ἐκβεβλημέναν, ὥστε τὰν μεταξὺ τᾶς περιφερείας καὶ τᾶς ἐκβεβλημένας ποτὶ τὰν ἐπιζευχθεῖσαν ἀπὸ τοῦ πέρατος τᾶς ἐναπολαφθείσας ποτὶ τὸ πέρας τᾶς ἐκβεβλημένας τὸν ταχθέντα λόγον ἔχειν, εἴ κα ὁ δοθεὶς λόγος μείζων ᾖ τοῦ, ὃν ἔχει ἁ ἡμίσεια τᾶς ἐν τῷ κύκλῳ δεδομένας ποτὶ τὰν ἀπὸ τοῦ κέντρου κάθετον ἐπ' αὐτὰν ἀγμέναν.

Δεδόσθω τὰ αὐτά, καὶ ἔστω ἁ ἐν τῷ κύκλῳ γραμμὰ ἐκβεβλημένα, ὁ δὲ δοθεὶς λόγος ἔστω, ὃν ἔχει ἁ Ζ ποτὶ τὰν Η, μείζων τοῦ, ὃν ἔχει ἁ ΓΘ ποτὶ τὰν ΘΚ· μείζων οὖν ἐσσεῖται καὶ τοῦ, ὃν ἔχε ἁ ΚΓ ποτὶ ΓΛ. ὃν δὴ λόγον ἔχει ἁ Ζ ποτὶ Η τοῦτον ἕξει ἁ ΚΓ ποτὶ ἐλάσσονα τᾶς ΓΛ. ἐχέτω ποτὶ ΙΝ νεύουσαν ἐπὶ τὸ Γ—δυνατὸν δέ ἐστιν οὕτως τέμνειν—καὶ πεσεῖται ἐντὸς τᾶς ΓΛ, ἐπειδὴ ἐλάσσων ἐστὶ τᾶς ΓΛ. ἐπεὶ οὖν τὸν αὐτὸν ἔχει λόγον ἁ ΚΓ ποτὶ ΙΝ, ὃν ἁ Ζ ποτὶ Η, καὶ ἁ ΕΙ ποτὶ ΙΓ τὸν αὐτὸν ἕξει λόγον, ὃν ἁ Ζ ποτὶ τὰν Η.

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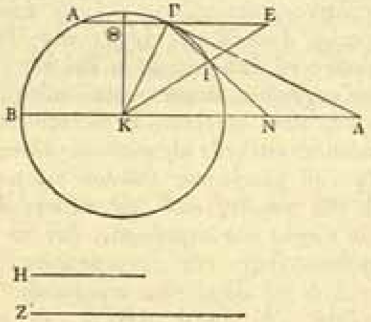
[a] ΑΓ is a chord in a circle of centre K, and BN is the diameter drawn parallel to ΑΓ and produced. From K, ΚΘ is drawn perpendicular to ΑΓ, and ΓΛ is drawn perpendicular to ΚΓ so as to meet the diameter in Λ. Archimedes asserts that it is possible to draw KE to meet the circle in I and ΑΓ produced in E so that ΕΙ : ΙΓ = Ζ : Η, an assigned ratio, provided that Ζ : Η > ΓΘ : ΘΚ. The straight line ΓΙ meets ΒΛ in N. In Prop. 5 Archimedes has proved a similar proposition when ΑΓ is a tangent, and in Prop. 6 he has proved the proposition for the case where the positions of I, Γ are reversed.

[b] For triangle ΓΙΕ is similar to triangle ΚΙΝ, and therefore ΚΙ : ΙΝ = ΕΙ : ΙΓ [Eucl. vi. 4] : and ΚΙ = ΚΓ.

[c] The type of problem known as νεύσεις, *vergings*, has already been encountered (vol. i. p. 244 n. *a*). In this proposition, as in Props. 5 and 6, Archimedes gives no hint how

the produced chord so that the distance between the circumference and the produced chord shall bear to the distance between the extremity of the line intercepted [by the circle] and the extremity of the produced chord an assigned ratio, provided that the given ratio is greater than that which half of the given chord in the circle bears to the perpendicular drawn to it from the centre.



Let the same things be given,[a] and let the chord in the circle be produced, and let the given ratio be Ζ : Η, and let it be greater than ΓΘ : ΘΚ; therefore it will be greater than ΚΓ : ΓΛ [Eucl. vi. 4]. Then Ζ : Η is equal to the ratio of ΚΓ to some line less than ΓΛ [Eucl. v. 10]. Let it be to ΙΝ verging upon Γ—for it is possible to make such an intercept—and ΙΝ will fall within ΓΛ, since it is less than ΓΛ.

Then since ΚΓ : ΙΝ = Ζ : Η,

therefore[b] ΕΙ : ΙΓ = Ζ : Η.[c]

the construction is to be accomplished, though he was presumably familiar with a solution.

In the figure of the text, let Τ be the foot of the perpen-

(iv.) *Property of the Subtangent*

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

Εἴ κα τᾶς ἕλικος τᾶς ἐν τᾷ πρώτᾳ περιφορᾷ γεγραμμένας εὐθεῖα γραμμὰ ἐπιψαύῃ μὴ κατὰ τὸ πέρας τᾶς ἕλικος, ἀπὸ δὲ τᾶς ἁφᾶς ἐπὶ τὰν ἀρχὰν τᾶς ἕλικος εὐθεῖα ἐπιζευχθῇ, καὶ κέντρῳ μὲν τᾷ ἀρχᾷ τᾶς ἕλικος, διαστήματι δὲ τᾷ ἐπιζευχθείσᾳ κύκλος γραφῇ, ἀπὸ δὲ τᾶς ἀρχᾶς τᾶς ἕλικος ἀχθῇ τις ποτ' ὀρθὰς τᾷ ἀπὸ τᾶς ἁφᾶς ἐπὶ τὰν ἀρχὰν τᾶς ἕλικος ἐπιζευχθείσᾳ, συμπεσεῖται αὕτα ποτὶ τὰν ἐπιψαύουσαν, καὶ ἐσσεῖται ἁ μεταξὺ εὐθεῖα τᾶς τε συμπτώσιος καὶ τᾶς ἀρχᾶς τᾶς ἕλικος ἴσα τᾷ περιφερείᾳ τοῦ γραφέντος κύκλου τᾷ μεταξὺ τᾶς ἁφᾶς καὶ τᾶς τομᾶς, καθ' ἃν τέμνει ὁ γραφεὶς κύκλος τὰν ἀρχὰν τᾶς περιφορᾶς, ἐπὶ τὰ προαγούμενα λαμβανομένας τᾶς περιφερείας ἀπὸ τοῦ σαμείου τοῦ ἐν τᾷ ἀρχᾷ τᾶς περιφορᾶς.

Ἔστω ἕλιξ, ἐφ' ἇς ἁ ΑΒΓΔ, ἐν τᾷ πρώτᾳ περιφορᾷ γεγραμμένα, καὶ ἐπιψαυέτω τις αὐτᾶς εὐθεῖα ἁ ΕΖ κατὰ τὸ Δ, ἀπὸ δὲ τοῦ Δ ποτὶ τὰν

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dicular from Γ to ΒΛ, and let Δ be the other extremity of the diameter through B. Let the unknown length $KN = x$, let $\Gamma T = a$, $KT = b$, $B\Delta = 2c$, and let $IN = k$, a given length.

Then $$NI \cdot N\Gamma = N\Delta \cdot NB,$$

i.e., $$k\sqrt{a^2 + (x-b)^2} = (x-c)(x+c),$$

which, after rationalization, is an equation of the fourth degree in x.

Alternatively, if we denote NΓ by y, we can determine x and y by the two equations

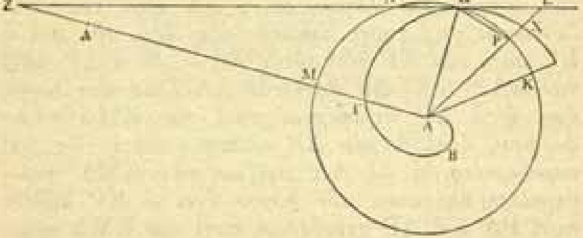
$$y^2 = a^2 + (x-b)^2,$$
$$ky = x^2 - c^2,$$

(iv.) *Property of the Subtangent*

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

If a straight line touch the first turn of the spiral other than at the extremity of the spiral, and from the point of contact a straight line be drawn to the origin, and with the origin as centre and this connecting line as radius a circle be drawn, and from the origin a straight line be drawn at right angles to the straight line joining the point of contact to the origin, it will meet the tangent, and the straight line between the point of meeting and the origin will be equal to the arc of the circle between the point of contact and the point in which the circle cuts the initial line, the arc being measured in the forward direction from the point on the initial line.

Let ΑΒΓΔ lie on the first turn of a spiral, and let



the straight line EZ touch it at Δ, and from Δ let ΑΔ

so that values of x and y satisfying the conditions of the problem are given by the points of intersection of a certain parabola and a certain hyperbola.

The whole question of *vergings*, including this problem, is admirably discussed by Heath, *The Works of Archimedes*, c–cxxii.

ἀρχὰν τᾶς ἕλικος ἐπεζεύχθω ἁ ΑΔ, καὶ κέντρῳ μὲν τῷ Α, διαστήματι δὲ τῷ ΑΔ κύκλος γεγράφθω ὁ ΔΜΝ, τεμνέτω δ' οὗτος τὰν ἀρχὰν τᾶς περιφορᾶς κατὰ τὸ Κ, ἄχθω δὲ ἁ ΖΑ ποτὶ τὰν ΑΔ ὀρθά. ὅτι μὲν οὖν αὕτα συμπίπτει, δῆλον· ὅτι δὲ καὶ ἴσα ἐστὶν ἁ ΖΑ εὐθεῖα τᾷ ΚΜΝΔ περιφερείᾳ, δεικτέον.

Εἰ γὰρ μή, ἤτοι μείζων ἐστὶν ἢ ἐλάσσων. ἔστω, εἰ δυνατόν, πρότερον μείζων, λελάφθω δέ τις ἁ ΛΑ τᾶς μὲν ΖΑ εὐθείας ἐλάσσων, τᾶς δὲ ΚΜΝΔ περιφερείας μείζων. πάλιν δὴ κύκλος ἐστὶν ὁ ΚΜΝ καὶ ἐν τῷ κύκλῳ γραμμὰ ἐλάσσων τᾶς διαμέτρου ἁ ΔΝ καὶ λόγος, ὃν ἔχει ἁ ΔΑ ποτὶ ΑΛ, μείζων τοῦ, ὃν ἔχει ἁ ἡμίσεια τᾶς ΔΝ ποτὶ τὰν ἀπὸ τοῦ Α κάθετον ἐπ' αὐτὰν ἀγμέναν· δυνατὸν οὖν ἐστιν ἀπὸ τοῦ Α ποτιβαλεῖν τὰν ΑΕ ποτὶ τὰν ΝΔ ἐκβεβλημέναν, ὥστε τὰν ΕΡ ποτὶ τὰν ΔΡ τὸν αὐτὸν ἔχειν λόγον, ὃν ἁ ΔΑ ποτὶ τὰν ΑΛ· δέδεικται γὰρ τοῦτο δυνατὸν ἐόν· ἕξει οὖν καὶ ἁ ΕΡ ποτὶ τὰν ΑΡ τὸν αὐτὸν λόγον, ὃν ἁ ΔΡ ποτὶ τὰν ΑΛ. ἁ δὲ ΔΡ ποτὶ τὰν ΑΛ ἐλάσσονα λόγον ἔχει ἢ ἁ ΔΡ περιφέρεια ποτὶ τὰν ΚΜΔ περιφέρειαν, ἐπεὶ ἁ μὲν ΔΡ ἐλάσσων ἐστὶ τᾶς ΔΡ περιφερείας, ἁ δὲ ΑΛ μείζων τᾶς ΚΜΔ περιφερείας· ἐλάσσονα οὖν λόγον ἔχει ἁ ΕΡ εὐθεῖα ποτὶ ΡΑ ἢ ἁ ΔΡ περιφέρεια ποτὶ τὰν ΚΜΔ περιφέρειαν· ὥστε καὶ ἁ ΑΕ ποτὶ ΑΡ ἐλάσσονα λόγον ἔχει ἢ ἁ ΚΜΡ περιφέρεια ποτὶ τὰν ΚΜΔ περι-

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[a] For in Prop. 16 the angle ΑΔΖ was shown to be acute.

[b] For ΔΝ touches the spiral and so can have no part within the spiral, and therefore cannot pass through Α; therefore it is a chord of the circle and less than the diameter.

[c] For, if a perpendicular be drawn from Α to ΔΝ, it bisects

be drawn to the origin, and with centre A and radius AΔ let the circle ΔMN be described, and let this circle cut the initial line at K, and let ZA be drawn at right angles to AΔ. That it will meet [ZΔ] is clear [a]; it is required to prove that the straight line ZA is equal to the arc KMNΔ.

If not, it is either greater or less. Let it first be, if possible, greater, and let ΛA be taken less than the straight line ZA, but greater than the arc KMNΔ [Prop. 4]. Again, KMN is a circle, and in this circle ΔN is a line less than the diameter,[b] and the ratio ΔA : AΛ is greater than the ratio of half ΔN to the perpendicular drawn to it from A [c]; it is therefore possible to draw from A a straight line AE meeting NΔ produced in such a way that

$$EP : \Delta P = \Delta A : A\Lambda ;$$

for this has been proved possible [Prop. 7]; therefore

$$EP : AP = \Delta P : A\Lambda.$$ [d]

But $\qquad \Delta P : A\Lambda < \text{arc } \Delta P : \text{arc } KM\Delta,$

since ΔP is less than the arc ΔP, and AΛ is greater than the arc KMΔ;

$\therefore \qquad EP : P\Lambda < \text{arc } \Delta P : \text{arc } KM\Delta ;$

$\therefore \qquad AE : AP < \text{arc } KMP : \text{arc } KM\Delta.$

[Eucl. v. 18

ΔN [Eucl. iii. 3] and divides triangle ΔAZ into two triangles of which one is similar to triangle ΔAZ [Eucl. vi. 8]; therefore

$$\Delta A : AZ = \tfrac{1}{2}N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).$$

[Eucl. vi. 4

But $\qquad AZ > A\Lambda ;$

$\therefore \quad \Delta A : A\Lambda > \tfrac{1}{2}N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).$

[d] For ΔA = AP, being a radius of the same circle; and the proportion follows *permutando*.

φέρειαν. ὃν δὲ λόγον ἔχει ἁ ΚΜΡ ποτὶ τὰν ΚΜΔ περιφέρειαν, τοῦτον ἔχει ἁ ΧΑ ποτὶ ΑΔ· ἐλάσσονα ἄρα λόγον ἔχει ἁ ΕΑ ποτὶ ΑΡ ἢ ἁ ΑΧ ποτὶ ΔΑ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μείζων ἁ ΖΑ τᾶς ΚΜΔ περιφερείας. ὁμοίως δὲ τοῖς πρότερον δειχθήσεται, ὅτι οὐδὲ ἐλάσσων ἐστίν· ἴσα ἄρα.

(*f*) Semi-Regular Solids

Papp. *Coll.* v. 19, ed. Hultsch i. 352. 7–354. 10

Πολλὰ γὰρ ἐπινοῆσαι δυνατὸν στερεὰ σχήματα παντοίας ἐπιφανείας ἔχοντα, μᾶλλον δ' ἂν τις ἀξιώσειε λόγου τὰ τετάχθαι δοκοῦντα [καὶ τούτων πολὺ πλέον τούς τε κώνους καὶ κυλίνδρους καὶ τὰ καλούμενα πολύεδρα].[1] ταῦτα δ' ἐστὶν οὐ μόνον τὰ παρὰ τῷ θειοτάτῳ Πλάτωνι πέντε σχήματα, τουτέστιν τετράεδρόν τε καὶ ἑξάεδρον, ὀκτάεδρόν τε καὶ δωδεκάεδρον, πέμπτον δ' εἰκοσάεδρον, ἀλλὰ καὶ τὰ ὑπὸ Ἀρχιμήδους εὑρεθέντα τρισκαίδεκα τὸν ἀριθμὸν ὑπὸ ἰσοπλεύρων μὲν καὶ ἰσογωνίων οὐχ ὁμοίων δὲ πολυγώνων περιεχόμενα.

[1] καὶ . . . πολύεδρα om. Hultsch.

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[a] This part of the proof involves a *verging* assumed in Prop. 8, just as the earlier part assumed the *verging* of Prop. 7. The verging of Prop. 8 has already been described (vol. i. p. 350 n. *b*) in connexion with Pappus's comments on it.

[b] Archimedes goes on to show that the theorem is true even if the tangent touches the spiral in its second or some higher turn, not at the extremity of the turn; and in Props. 18 and 19 he has shown that the theorem is true if the tangent should touch at an extremity of a turn.

Now arc KMP : arc KMΔ = XA : AΔ ; [Prop. 14

∴ EA : AP < AX : ΔA ;

which is impossible. Therefore ZA is not greater than the arc KMΔ. In the same way as above it may be shown to be not less [a]; therefore it is equal.[b]

(*f*) Semi-Regular Solids

Pappus, *Collection* v. 19, ed. Hultsch i. 352. 7–354. 10

Although many solid figures having all kinds of surfaces can be conceived, those which appear to be regularly formed are most deserving of attention. Those include not only the five figures found in the godlike Plato, that is, the tetrahedron and the cube, the octahedron and the dodecahedron, and fifthly the icosahedron,[c] but also the solids, thirteen in number, which were discovered by Archimedes [d] and are contained by equilateral and equiangular, but not similar, polygons.

As Pappus (ed. Hultsch 302. 14-18) notes, the theorem can be established without recourse to propositions involving *solid loci* (for the meaning of which see vol. i. pp. 348-349), and proofs involving only "plane" methods have been developed by Tannery, *Mémoires scientifiques*, i., 1912, pp. 300-316 and Heath, *H.G.M.* ii. 556-561. It must remain a puzzle why Archimedes chose his particular method of proof, especially as Heath's proof is suggested by the figures of Props. 6 and 9 ; Heath (*loc. cit.*, p. 557) says " it is scarcely possible to assign any reason except his definite predilection for the form of proof by *reductio ad absurdum* based ultimately on his famous 'Lemma' or Axiom."

[c] For the five regular solids, see vol. i. pp. 216-225.

[d] Heron (*Definitions* 104, ed. Heiberg 66. 1-9) asserts that two were known to Plato. One is that described as P_2 below, but the other, said to be bounded by eight squares and six triangles, is wrongly given.

Τὸ μὲν γὰρ πρῶτον ὀκτάεδρόν ἐστιν περιεχόμενον ὑπὸ τριγώνων δ̅ καὶ ἑξαγώνων δ̅.

Τρία δὲ μετὰ τοῦτο τεσσαρεσκαιδεκάεδρα, ὧν τὸ μὲν πρῶτον περιέχεται τριγώνοις η̅ καὶ τετραγώνοις ϛ̅, τὸ δὲ δεύτερον τετραγώνοις ϛ̅ καὶ ἑξαγώνοις η̅, τὸ δὲ τρίτον τριγώνοις η̅ καὶ ὀκταγώνοις ϛ̅.

Μετὰ δὲ ταῦτα ἑκκαιεικοσάεδρά ἐστιν δύο, ὧν τὸ μὲν πρῶτον περιέχεται τριγώνοις η̅ καὶ τετραγώνοις ι̅η̅, τὸ δὲ δεύτερον τετραγώνοις ι̅β̅, ἑξαγώνοις η̅ καὶ ὀκταγώνοις ϛ̅.

Μετὰ δὲ ταῦτα δυοκαιτριακοντάεδρά ἐστιν τρία, ὧν τὸ μὲν πρῶτον περιέχεται τριγώνοις κ̅ καὶ πενταγώνοις ι̅β̅, τὸ δὲ δεύτερον πενταγώνοις ι̅β̅ καὶ ἑξαγώνοις κ̅, τὸ δὲ τρίτον τριγώνοις κ̅ καὶ δεκαγώνοις ι̅β̅.

Μετὰ δὲ ταῦτα ἕν ἐστιν ὀκτωκαιτριακοντάεδρον περιεχόμενον ὑπὸ τριγώνων λ̅β̅ καὶ τετραγώνων ϛ̅.

Μετὰ δὲ τοῦτο δυοκαιεξηκοντάεδρά ἐστι δύο, ὧν τὸ μὲν πρῶτον περιέχεται τριγώνοις κ̅ καὶ τετραγώνοις λ̅ καὶ πενταγώνοις ι̅β̅, τὸ δὲ δεύτερον τετραγώνοις λ̅ καὶ ἑξαγώνοις κ̅ καὶ δεκαγώνοις ι̅β̅.

Μετὰ δὲ ταῦτα τελευταῖόν ἐστιν δυοκαιενενηκοντάεδρον, ὃ περιέχεται τριγώνοις π̅ καὶ πενταγώνοις ι̅β̅.

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[a] For the purposes of n. *b*, the thirteen polyhedra will be designated as $P_1, P_2 \ldots P_{13}$.

[b] Kepler, in his *Harmonice mundi* (*Opera*, 1864, v. 123-126), appears to have been the first to examine these figures systematically, though a method of obtaining some is given in a scholium to the Vatican ms. of Pappus. If a solid angle of a regular solid be cut by a plane so that the same length is cut off from each of the edges meeting at the solid angle,

The first is a figure of eight bases, being contained by four triangles and four hexagons [P_1].[a]

After this come three figures of fourteen bases, the first contained by eight triangles and six squares [P_2], the second by six squares and eight hexagons [P_3], and the third by eight triangles and six octagons [P_4].

After these come two figures of twenty-six bases, the first contained by eight triangles and eighteen squares [P_5], the second by twelve squares, eight hexagons and six octagons [P_6].

After these come three figures of thirty-two bases, the first contained by twenty triangles and twelve pentagons [P_7], the second by twelve pentagons and twenty hexagons [P_8], and the third by twenty triangles and twelve decagons [P_9].

After these comes one figure of thirty-eight bases, being contained by thirty-two triangles and six squares [P_{10}].

After this come two figures of sixty-two bases, the first contained by twenty triangles, thirty squares and twelve pentagons [P_{11}], the second by thirty squares, twenty hexagons and twelve decagons [P_{12}].

After these there comes lastly a figure of ninety-two bases, which is contained by eighty triangles and twelve pentagons [P_{13}].[b]

the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is composed of three, four or five plane angles. If certain equal lengths be cut off in this way from all the solid angles, regular polygons will also be left in the faces of the solid. This happens (i) obviously when the cutting planes bisect the edges of the solid, and (ii) when the cutting planes cut off a smaller length from each edge in such a way that a regular polygon is left in each face with double the number of sides. This method gives (1) from the tetrahedron, P_1; (2) from the

(g) System of expressing Large Numbers

Archim. *Aren.* 3, Archim. ed. Heiberg ii. 236. 17–240. 1

Ἃ μὲν οὖν ὑποτίθεμαι, ταῦτα· χρήσιμον δὲ εἶμεν ὑπολαμβάνω τὰν κατονόμαξιν τῶν ἀριθμῶν ῥηθῆμεν, ὅπως καὶ τῶν ἄλλων οἱ τῷ βιβλίῳ μὴ περιτετευχότες τῷ ποτὶ Ζεύξιππον γεγραμμένῳ μὴ πλανῶνται διὰ τὸ μηδὲν εἶμεν ὑπὲρ αὐτᾶς ἐν τῷδε τῷ βιβλίῳ προειρημένον. συμβαίνει δὴ τὰ ὀνόματα τῶν ἀριθμῶν ἐς τὸ μὲν τῶν μυρίων ὑπάρχειν ἁμῖν παραδεδομένα, καὶ ὑπὲρ τὸ τῶν μυρίων [μὲν][1] ἀποχρεόντως γιγνώσκομες μυριάδων ἀριθμὸν λέγοντες ἔστε ποτὶ τὰς μυρίας μυριάδας. ἔστων οὖν ἁμῖν οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ ἐς τὰς μυρίας μυριάδας πρῶτοι καλουμένοι, τῶν δὲ πρώτων ἀριθμῶν αἱ μύριαι μυριάδες μονὰς καλείσθω δευτέρων ἀριθμῶν, καὶ ἀριθμείσθων τῶν δευτέρων μονάδες καὶ ἐκ τᾶν μονάδων δεκάδες καὶ ἑκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐς τὰς μυρίας μυριάδας. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τῶν δευτέρων ἀριθμῶν μονὰς καλείσθω τρίτων ἀριθμῶν, καὶ ἀριθμείσθων τῶν τρίτων ἀριθμῶν μονάδες καὶ ἀπὸ τᾶν μονάδων δεκάδες καὶ ἑκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐς τὰς μυρίας μυριάδας. τὸν αὐτὸν δὲ τρόπον καὶ τῶν τρίτων ἀριθμῶν μύριαι μυριάδες μονὰς καλείσθω τετάρτων ἀριθμῶν,

[1] μὲν om. Heiberg.

cube, P_2 and P_4: (3) from the octahedron, P_3 and P_5: (4) from the icosahedron, P_7 and P_8: (5) from the dodecahedron, P_7 and P_9. It was probably the method used by Plato.

Four more of the semi-regular solids are obtained by first cutting all the edges symmetrically and equally by planes parallel to the edges, and then cutting off angles. This

(*g*) System of expressing Large Numbers

Archimedes, *Sand-Reckoner* 3, Archim. ed. Heiberg ii. 236. 17–240. 1

Such are then the assumptions I make; but I think it would be useful to explain the naming of the numbers, in order that, as in other matters, those who have not come across the book sent to Zeuxippus may not find themselves in difficulty through the fact that there had been no preliminary discussion of it in this book. Now we already have names for the numbers up to a myriad [10^4], and beyond a myriad we can count in myriads up to a myriad myriads [10^8]. Therefore, let the aforesaid numbers up to a myriad myriads be called *numbers of the first order* [numbers from 1 to 10^8], and let a myriad myriads of numbers of the first order be called a unit of *numbers of the second order* [numbers from 10^8 to 10^{16}], and let units of the numbers of the second order be enumerable, and out of the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. Again, let a myriad myriads of numbers of the second order be called a unit of *numbers of the third order* [numbers from 10^{16} to 10^{24}], and let units of numbers of the third order be enumerable, and from the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. In the same manner, let a myriad myriads of numbers of the third order be

gives (1) from the cube, P_5 and P_8; (2) from the icosahedron, P_{11}; (3) from the dodecahedron, P_{12}.

The two remaining solids are more difficult to obtain; P_{10} is the *snub cube* in which each solid angle is formed by the angles of four equilateral triangles and one square; P_{13} is the *snub dodecahedron* in which each solid angle is formed by the angles of four equilateral triangles and one regular pentagon.

καὶ αἱ τῶν τετάρτων ἀριθμῶν μύριαι μυριάδες μονὰς καλείσθω πέμπτων ἀριθμῶν, καὶ ἀεὶ οὕτως προάγοντες οἱ ἀριθμοὶ τὰ ὀνόματα ἐχόντων ἐς τὰς μυριακισμυριοστῶν ἀριθμῶν μυρίας μυριάδας.

Ἀποχρέοντι μὲν οὖν καὶ ἐπὶ τοσοῦτον οἱ ἀριθμοὶ γιγνωσκομένοι, ἔξεστι δὲ καὶ ἐπὶ πλέον προάγειν. ἔστων γὰρ οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ πρώτας περιόδου καλουμένοι, ὁ δὲ ἔσχατος ἀριθμὸς τᾶς πρώτας περιόδου μονὰς καλείσθω δευτέρας περιόδου πρώτων ἀριθμῶν. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τᾶς δευτέρας περιόδου πρώτων ἀριθμῶν μονὰς καλείσθω τᾶς δευτέρας περιόδου δευτέρων ἀριθμῶν. ὁμοίως δὲ καὶ τούτων ὁ ἔσχατος μονὰς καλείσθω δευτέρας περιόδου τρίτων ἀριθμῶν, καὶ ἀεὶ οὕτως οἱ ἀριθμοὶ προάγοντες τὰ ὀνόματα ἐχόντων τᾶς δευτέρας περιόδου ἐς τὰς μυριακισμυριοστῶν ἀριθμῶν μυρίας μυριάδας.

Πάλιν δὲ καὶ ὁ ἔσχατος ἀριθμὸς τᾶς δευτέρας περιόδου μονὰς καλείσθω τρίτας περιόδου πρώτων ἀριθμῶν, καὶ ἀεὶ οὕτως προαγόντων ἐς τὰς μυριακισμυριοστᾶς περιόδου μυριακισμυριοστῶν ἀριθμῶν μυρίας μυριάδας.

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[a] Expressed in full, the last number would be 1 followed by 80,000 million millions of ciphers. Archimedes uses this system to show that it is more than sufficient to express the number of grains of sand which it would take to fill the universe, basing his argument on estimates by astronomers of the sizes and distances of the sun and moon and their relation to the size of the universe and allowing a wide margin for safety. Assuming that a poppy-head (for so μήκων is here to be understood, not "poppy-seed," v. D'Arcy W. Thompson, *The Classical Review*, lvi. (1942), p. 75) would contain not more than 10,000 grains of sand, and that its diameter is not less than a finger's breadth, and having proved that the

called a unit of *numbers of the fourth order* [numbers from 10^{24} to 10^{32}], and let a myriad myriads of numbers of the fourth order be called a unit of *numbers of the fifth order* [numbers from 10^{32} to 10^{40}], and let the process continue in this way until the designations reach a myriad myriads taken a myriad myriad times [$10^{8 \cdot 10^8}$].

It is sufficient to know the numbers up to this point, but we may go beyond it. For let the numbers now mentioned be called *numbers of the first period* [1 to $10^{8 \cdot 10^8}$], and let the last number of the first period be called a unit of *numbers of the first order of the second period* [$10^{8 \cdot 10^8}$ to $10^{8 \cdot 10^8} . 10^8$]. And again, let a myriad myriads of numbers of the first order of the second period be called a unit of *numbers of the second order of the second period* [$10^{8 \cdot 10^8} . 10^8$ to $10^{8 \cdot 10^8} . 10^{16}$]. Similarly let the last of these numbers be called a unit of *numbers of the third order of the second period* [$10^{8 \cdot 10^8} . 10^{16}$ to $10^{8 \cdot 10^8} . 10^{24}$], and let the process continue in this way until the designations of numbers in the second period reach a myriad myriads taken a myriad myriad times [$10^{8 \cdot 10^8} . 10^{8 \cdot 10^8}$, or $(10^{8 \cdot 10^8})^2$].

Again, let the last number of the second period be called a unit of *numbers of the first order of the third period* [$(10^{8 \cdot 10^8})^2$ to $(10^{8 \cdot 10^8})^2 . 10^8$], and let the process continue in this way up to a myriad myriad units of *numbers of the myriad myriadth order of the myriad myriadth period* [$(10^{8 \cdot 10^8})^{10^8}$ or $10^{8 \cdot 10^{16}}$].[a]

sphere of the fixed stars is less than 10^7 times the sphere in which the sun's orbit is a great circle, Archimedes shows that the number of grains of sand which would fill the universe is less than " 10,000,000 units of the eighth order of numbers," or 10^{63}. The work contains several references important for the history of astronomy.

(*h*) Indeterminate Analysis: The Cattle Problem

Archim. (?) *Prob. Bov.*, Archim. ed. Heiberg ii. 528. 1–532. 9

Πρόβλημα

ὅπερ Ἀρχιμήδης ἐν ἐπιγράμμασιν εὑρὼν τοῖς ἐν Ἀλεξανδρείᾳ περὶ ταῦτα πραγματευομένοις ζητεῖν ἀπέστειλεν ἐν τῇ πρὸς Ἐρατοσθένην τὸν Κυρηναῖον ἐπιστολῇ.

Πληθὺν Ἠελίοιο βοῶν, ὦ ξεῖνε, μέτρησον
  φροντίδ' ἐπιστήσας, εἰ μετέχεις σοφίης,
πόσση ἄρ' ἐν πεδίοις Σικελῆς ποτ' ἐβόσκετο νήσου
  Θρινακίης τετραχῇ στίφεα δασσαμένη
χροιὴν ἀλλάσσοντα· τὸ μὲν λευκοῖο γάλακτος,
  κυανέῳ δ' ἕτερον χρώματι λαμπόμενον,
ἄλλο γε μὲν ξανθόν, τὸ δὲ ποικίλον. ἐν δὲ ἑκάστῳ
  στίφει ἔσαν ταῦροι πλήθεσι βριθόμενοι
συμμετρίης τοιῆσδε τετευχότες· ἀργότριχας μὲν
  κυανέων ταύρων ἡμίσει ἠδὲ τρίτῳ
καὶ ξανθοῖς σύμπασιν ἴσους, ὦ ξεῖνε, νόησον,
  αὐτὰρ κυανέους τῷ τετράτῳ τε μέρει
μικτοχρόων καὶ πέμπτῳ, ἔτι ξανθοῖσί τε πᾶσιν.
  τοὺς δ' ὑπολειπομένους ποικιλόχρωτας ἄθρει
ἀργεννῶν ταύρων ἕκτῳ μέρει ἑβδομάτῳ τε
  καὶ ξανθοῖς αὐτοὺς πᾶσιν ἰσαζομένους.
θηλείαισι δὲ βουσὶ τάδ' ἔπλετο· λευκότριχες μὲν
  ἦσαν συμπάσης κυανέης ἀγέλης
τῷ τριτάτῳ τε μέρει καὶ τετράτῳ ἀτρεκὲς ἶσαι·
  αὐτὰρ κυάνεαι τῷ τετράτῳ τε πάλιν
μικτοχρόων καὶ πέμπτῳ ὁμοῦ μέρει ἰσάζοντο
  σὺν ταύροις πάσαις εἰς νομὸν ἐρχομέναις.

(h) Indeterminate Analysis: The Cattle Problem

Archimedes (?), *Cattle Problem*,[a] Archim. ed. Heiberg ii. 528. 1–532. 9

A Problem

which Archimedes solved in epigrams, and which he communicated to students of such matters at Alexandria in a letter to Eratosthenes of Cyrene.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now

[a] It is unlikely that the epigram itself, first edited by G. E. Lessing in 1773, is the work of Archimedes, but there is ample evidence from antiquity that he studied the actual problem. The most important papers bearing on the subject have already been mentioned (vol. i. p. 16 n. *c*), and further references to the literature are given by Heiberg *ad loc.*

ξανθοτρίχων δ' ἀγέλης πέμπτῳ μέρει ἠδὲ καὶ ἕκτῳ
  ποικίλαι ἰσάριθμον πλῆθος ἔχον τετραχῇ.
ξανθαὶ δ' ἠριθμεῦντο μέρους τρίτου ἡμίσει ἶσαι
  ἀργεννῆς ἀγέλης ἑβδομάτῳ τε μέρει.
ξεῖνε, σὺ δ', Ἠελίοιο βόες πόσαι, ἀτρεκὲς εἰπών,
  χωρὶς μὲν ταύρων ζατρεφέων ἀριθμόν,
χωρὶς δ' αὖ, θήλειαι ὅσαι κατὰ χροιὰν ἕκασται,
  οὐκ ἄιδρίς κε λέγοι' οὐδ' ἀριθμῶν ἀδαής,
οὐ μὴν πώ γε σοφοῖς ἐναρίθμιος. ἀλλ' ἴθι φράζευ
  καὶ τάδε πάντα βοῶν Ἠελίοιο πάθη.
ἀργότριχες ταῦροι μὲν ἐπεὶ μιξαίατο πληθὺν
  κυανέοις, ἵσταντ' ἔμπεδον ἰσόμετροι
εἰς βάθος εἰς εὖρός τε, τὰ δ' αὖ περιμήκεα πάντῃ
  πίμπλαντο πλήθους[1] Θρινακίης πεδία.
ξανθοὶ δ' αὖτ' εἰς ἓν καὶ ποικίλοι ἀθροισθέντες
  ἵσταντ' ἀμβολάδην ἐξ ἑνὸς ἀρχόμενοι
σχῆμα τελειοῦντες τὸ τρικράσπεδον οὔτε προσόντων
  ἀλλοχρόων ταύρων οὔτ' ἐπιλειπομένων.
ταῦτα συνεξευρὼν καὶ ἐνὶ πραπίδεσσιν ἀθροίσας
  καὶ πληθέων ἀποδούς, ξεῖνε, τὰ πάντα μέτρα
ἔρχεο κυδιόων νικηφόρος ἴσθι τε πάντως
  κεκριμένος ταύτῃ γ' ὄμπνιος ἐν σοφίῃ.

[1] πλήθους Krumbiegel, πλίνθου cod.

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[a] *i.e.* a fifth and a sixth both of the males and of the females.

[b] At a first glance this would appear to mean that the sum of the number of white and black bulls is a square, but this makes the solution of the problem intolerably difficult. There is, however, an easier interpretation. If the bulls are packed together so as to form a square figure, their number need not be a square, since each bull is longer than it is broad. The simplified condition is that the sum of the number of white and black bulls shall be a rectangle.

the dappled in four parts[a] were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth,[b] and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.[c]

[c] If

X, x are the numbers of white bulls and cows respectively,
Y, y „ „ „ black „ „ „
Z, z „ „ „ yellow „ „ „
W, w „ „ „ dappled „ „ „

the first part of the epigram states that

(a)
$$X = (\tfrac{1}{2} + \tfrac{1}{3})Y + Z \quad . \quad . \quad . \quad . \quad (1)$$
$$Y = (\tfrac{1}{4} + \tfrac{1}{5})W + Z \quad . \quad . \quad . \quad . \quad (2)$$
$$W = (\tfrac{1}{6} + \tfrac{1}{7})X + Z \quad . \quad . \quad . \quad . \quad (3)$$

(i) Mechanics: Centres of Gravity

(i.) *Postulates*

Archim. *De Plan. Aequil.*, Deff., Archim. ed. Heiberg ii. 124. 3–126. 3

α'. Αἰτούμεθα τὰ ἴσα βάρεα ἀπὸ ἴσων μακέων ἰσορροπεῖν, τὰ δὲ ἴσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μείζονος μάκεος.

(*b*)

$$x=(\tfrac{1}{3}+\tfrac{1}{4})(Y+y) \quad . \quad . \quad . \quad . \quad (4)$$
$$y=(\tfrac{1}{4}+\tfrac{1}{5})(W+w) \quad . \quad . \quad . \quad . \quad (5)$$
$$w=(\tfrac{1}{5}+\tfrac{1}{6})(Z+z) \quad . \quad . \quad . \quad . \quad (6)$$
$$z=(\tfrac{1}{6}+\tfrac{1}{7})(X+x) \quad . \quad . \quad . \quad . \quad (7)$$

The second part of the epigram states that

$$X+Y=\text{a rectangular number} \quad . \quad . \quad (8)$$
$$Z+W=\text{a triangular number} \quad . \quad . \quad . \quad (9)$$

This was solved by J. F. Wurm, and the solution is given by A. Amthor, *Zeitschrift für Math. u. Physik.* (*Hist.-litt. Abtheilung*), xxv. (1880), pp. 153-171, and by Heath, *The Works of Archimedes*, pp. 319-326. For reasons of space, only the results can be noted here.

Equations (1) to (7) give the following as the values of the unknowns in terms of an unknown integer n:

$$\begin{array}{ll} X=10366482n & x=7206360n \\ Y=\ \ 7460514n & y=4893246n \\ Z=\ \ 4149387n & z=5439213n \\ W=\ \ 7358060n & w=3515820n. \end{array}$$

We have now to find a value of n such that equation (9) is also satisfied—equation (8) will then be simultaneously satisfied. Equation (9) means that

$$Z+W=\frac{p(p+1)}{2},$$

where p is some positive integer, or

$$(4149387+7358060)n=\frac{p(p+1)}{2},$$

(*i*) Mechanics : Centres of Gravity

(i.) *Postulates*

Archimedes, *On Plane Equilibriums*,[a] Definitions, Archim. ed. Heiberg ii. 124. 3–126. 3

1. I postulate that equal weights at equal distances balance, and equal weights at unequal distances do not balance, but incline towards the weight which is at the greater distance.

i.e.
$$2471 \,.\, 4657n = \frac{p(p+1)}{2}.$$

This is found to be satisfied by

$$n = 3^3 \,.\, 4349,$$

and the final solution is

$X = 1217263415886$	$x = 846192410280$
$Y = 876035935422$	$y = 574579625058$
$Z = 487233469701$	$z = 638688708099$
$W = 864005479380$	$w = 412838131860$

and the total is 5916837175686.

If equation (8) is taken to be that $X + Y =$ a square number, the solution is much more arduous ; Amthor found that in this case,

$$W = 1598 \,\langle \boxed{206541} \rangle,$$

where $\langle \boxed{206541} \rangle$ means that there are 206541 more digits to follow, and the whole number of cattle $= 7766 \,\langle \boxed{206541} \rangle$. Merely to write out the eight numbers, Amthor calculates, would require a volume of 660 pages, so we may reasonably doubt whether the problem was really framed in this more difficult form, or, if it were, whether Archimedes solved it.

[a] This is the earliest surviving treatise on mechanics ; it presumably had predecessors, but we may doubt whether mechanics had previously been developed by rigorous geometrical principles from a small number of assumptions. References to the principle of the lever and the parallelogram of velocities in the Aristotelian *Mechanics* have already been given (vol. i. pp. 430–433).

β'. εἴ κα βαρέων ἰσορροπεόντων ἀπό τινων μακέων ποτὶ τὸ ἕτερον τῶν βαρέων ποτιτεθῇ, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος ἐκεῖνο, ᾧ ποτετέθη.

γ'. Ὁμοίως δὲ καί, εἴ κα ἀπὸ τοῦ ἑτέρου τῶν βαρέων ἀφαιρεθῇ τι, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος, ἀφ' οὗ οὐκ ἀφῃρέθη.

δ'. Τῶν ἴσων καὶ ὁμοίων σχημάτων ἐπιπέδων ἐφαρμοζομένων ἐπ' ἄλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπ' ἄλλαλα.

ε'. Τῶν δὲ ἀνίσων, ὁμοίων δέ, τὰ κέντρα τῶν βαρέων ὁμοίως ἐσσεῖται κείμενα. ὁμοίως δὲ λέγομες σαμεῖα κέεσθαι ποτὶ τὰ ὁμοῖα σχήματα, ἀφ' ὧν ἐπὶ τὰς ἴσας γωνίας ἀγόμεναι εὐθεῖαι ποιέοντι γωνίας ἴσας ποτὶ τὰς ὁμολόγους πλευράς.

ϛ'. Εἴ κα μεγέθεα ἀπό τινων μακέων ἰσορροπέωντι, καὶ τὰ ἴσα αὐτοῖς ἀπὸ τῶν αὐτῶν μακέων ἰσορροπήσει.

ζ'. Παντὸς σχήματος, οὗ κα ἁ περίμετρος ἐπὶ τὰ αὐτὰ κοῖλα ᾖ, τὸ κέντρον τοῦ βάρεος ἐντὸς εἶμεν δεῖ τοῦ σχήματος.

(ii.) *Principle of the Lever*

Ibid., Props. 6 et 7, Archim. ed. Heiberg ii. 132. 13–138. 8

ϛ'

Τὰ σύμμετρα μεγέθεα ἰσορροπέοντι ἀπὸ μακέων ἀντιπεπονθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς βάρεσιν.

Ἔστω σύμμετρα μεγέθεα τὰ Α, Β, ὧν κέντρα τὰ Α, Β, καὶ μᾶκος ἔστω τι τὸ ΕΔ, καὶ ἔστω, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ μᾶκος ποτὶ τὸ ΓΕ

2. If weights at certain distances balance, and something is added to one of the weights, they will not remain in equilibrium, but will incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they will not remain in equilibrium, but will incline towards the weight from which nothing was subtracted.

4. When equal and similar plane figures are applied one to the other, their centres of gravity also coincide.

5. In unequal but similar figures, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances balance, magnitudes equal to them will also balance at the same distances.

7. In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

(ii.) *Principle of the Lever*

Ibid., Props. 6 and 7, Archim. ed. Heiberg ii. 132. 13–138. 8

Prop. 6

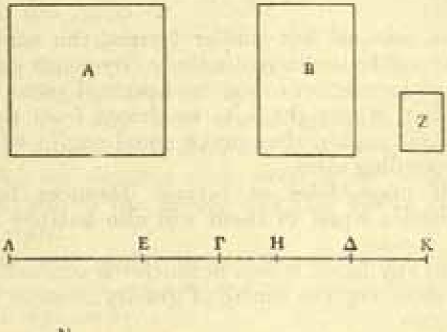
Commensurable magnitudes balance at distances reciprocally proportional to their weights.

Let A, B be commensurable magnitudes with centres [of gravity] A, B, and let EΔ be any distance, and let

$$A : B = \Delta\Gamma : \Gamma E\,;$$

μᾶκος· δεικτέον, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν Α, Β συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεος τὸ Γ.

Ἐπεὶ γάρ ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ ποτὶ τὸ ΓΕ, τὸ δὲ Α τῷ Β σύμμετρον, καὶ τὸ ΓΔ ἄρα τῷ ΓΕ σύμμετρον, τουτέστιν εὐθεῖα τᾷ εὐθείᾳ· ὥστε τῶν ΕΓ, ΓΔ ἐστι κοινὸν μέτρον. ἔστω δὴ τὸ Ν, καὶ κείσθω τᾷ μὲν ΕΓ ἴσα ἑκατέρα τᾶν ΔΗ, ΔΚ, τᾷ δὲ ΔΓ ἴσα ἁ ΕΛ. καὶ ἐπεὶ ἴσα



ἁ ΔΗ τᾷ ΓΕ, ἴσα καὶ ἁ ΔΓ τᾷ ΕΗ· ὥστε καὶ ἁ ΛΕ ἴσα τᾷ ΕΗ. διπλασία ἄρα ἁ μὲν ΛΗ τᾶς ΔΓ, ἁ δὲ ΗΚ τᾶς ΓΕ· ὥστε τὸ Ν καὶ ἑκατέραν τᾶν ΛΗ, ΗΚ μετρεῖ, ἐπειδήπερ καὶ τὰ ἡμίσεα αὐτᾶν. καὶ ἐπεί ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως ἁ ΔΓ ποτὶ ΓΕ, ὡς δὲ ἁ ΔΓ ποτὶ ΓΕ, οὕτως ἁ ΛΗ ποτὶ ΗΚ—διπλασία γὰρ ἑκατέρα ἑκατέρας—καὶ ὡς ἄρα τὸ Α ποτὶ τὸ Β, οὕτως ἁ ΛΗ ποτὶ ΗΚ. ὁσαπλασίων δέ ἐστιν ἁ ΛΗ τᾶς Ν, τοσαυ-

it is required to prove that the centre of gravity of the magnitude composed of both A, B is Γ.

Since $$\mathrm{A} : \mathrm{B} = \Delta\Gamma : \Gamma\mathrm{E},$$

and A is commensurate with B, therefore ΓΔ is commensurate with ΓE, that is, a straight line with a straight line [Eucl. x. 11]; so that EΓ, ΓΔ have a common measure. Let it be N, and let ΔH, ΔK be each equal to EΓ, and let EΛ be equal to ΔΓ. Then since ΔH = ΓE, it follows that ΔΓ = EH; so that ΛEE = H. Therefore ΛH = 2ΔΓ and HK = 2ΓE; so that N measures both ΛH and HK, since it measures their halves [Eucl. x. 12]. And since

$$\mathrm{A} : \mathrm{B} = \Delta\Gamma : \Gamma\mathrm{E},$$

while $$\Delta\Gamma : \Gamma\mathrm{E} = \Lambda\mathrm{H} : \mathrm{HK}—$$

for each is double of the other—

therefore $$\mathrm{A} : \mathrm{B} = \Lambda\mathrm{H} : \mathrm{HK}.$$

Now let Z be the same part of A as N is of ΛH;

ταπλασίων ἔστω καὶ τὸ Α τοῦ Ζ· ἔστιν ἄρα, ὡς ἁ ΛΗ ποτὶ Ν, οὕτως τὸ Α ποτὶ Ζ. ἔστι δὲ καί, ὡς ἁ ΚΗ ποτὶ ΛΗ, οὕτως τὸ Β ποτὶ Α· δι' ἴσου ἄρα ἐστίν, ὡς ἁ ΚΗ ποτὶ Ν, οὕτως τὸ Β ποτὶ Ζ· ἰσάκις ἄρα πολλαπλασίων ἐστὶν ἁ ΚΗ τᾶς Ν καὶ τὸ Β τοῦ Ζ. ἐδείχθη δὲ τοῦ Ζ καὶ τὸ Α πολλαπλάσιον ἐόν· ὥστε τὸ Ζ τῶν Α, Β κοινόν ἐστι μέτρον. διαιρεθείσας οὖν τᾶς μὲν ΛΗ εἰς τὰς τᾷ Ν ἴσας, τοῦ δὲ Α εἰς τὰ τῷ Ζ ἴσα, τὰ ἐν τᾷ ΛΗ τμάματα ἰσομεγέθεα τᾷ Ν ἴσα ἐσσεῖται τῷ πλήθει τοῖς ἐν τῷ Α τμαμάτεσσιν ἴσοις ἐοῦσιν τῷ Ζ. ὥστε, ἂν ἐφ' ἕκαστον τῶν τμαμάτων τῶν ἐν τᾷ ΛΗ ἐπιτεθῇ μέγεθος ἴσον τῷ Ζ τὸ κέντρον τοῦ βάρεος ἔχον ἐπὶ μέσου τοῦ τμάματος, τά τε πάντα μεγέθεα ἴσα ἐντὶ τῷ Α, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον ἐσσεῖται τοῦ βάρεος τὸ Ε· ἄρτιά τε γάρ ἐστι τὰ πάντα τῷ πλήθει, καὶ τὰ ἐφ' ἑκάτερα τοῦ Ε ἴσα τῷ πλήθει διὰ τὸ ἴσαν εἶμεν τὰν ΛΕ τᾷ ΗΕ.

Ὁμοίως δὲ δειχθήσεται, ὅτι κἄν, εἴ κα ἐφ' ἕκαστον τῶν ἐν τᾷ ΚΗ τμαμάτων ἐπιτεθῇ μέγεθος ἴσον τῷ Ζ κέντρον τοῦ βάρεος ἔχον ἐπὶ τοῦ μέσου τοῦ τμάματος, τά τε πάντα μεγέθεα ἴσα ἐσσεῖται τῷ Β, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον τοῦ βάρεος ἐσσεῖται τὸ Δ· ἐσσεῖται οὖν τὸ μὲν Α ἐπικείμενον κατὰ τὸ Ε, τὸ δὲ Β κατὰ τὸ Δ. ἐσσεῖται δὴ μεγέθεα ἴσα ἀλλάλοις ἐπ' εὐθείας κείμενα, ὧν τὰ κέντρα τοῦ βάρεος ἴσα ἀπ' ἀλλάλων διέστακεν, [συγκείμενα][1] ἄρτια τῷ πλήθει· δῆλον οὖν, ὅτι τοῦ ἐκ πάντων συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεος ἁ διχοτομία τᾶς εὐθείας τᾶς ἐχούσας τὰ κέντρα τῶν μέσων μεγεθέων. ἐπεὶ δ' ἴσαι ἐντὶ

then $\Lambda H : N = A : Z$. [Eucl. v., Def. 5
And $KH : \Lambda H = B : A$; [Eucl. v. 7, coroll.
therefore, *ex aequo*,

$KH : N = B : Z$; [Eucl. v. 22

therefore Z is the same part of B as N is of KH. Now A was proved to be a multiple of Z; therefore Z is a common measure of A, B. Therefore, if ΛH is divided into segments equal to N and A into segments equal to Z, the segments in ΛH equal in magnitude to N will be equal in number to the segments of A equal to Z. It follows that, if there be placed on each of the segments in ΛH a magnitude equal to Z, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to A, and the centre of gravity of the figure compounded of them all will be E; for they are even in number, and the numbers on either side of E will be equal because ΛE = HE. [Prop. 5, coroll. 2.]

Similarly it may be proved that, if a magnitude equal to Z be placed on each of the segments [equal to N] in KH, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to B, and the centre of gravity of the figure compounded of them all will be Δ [Prop. 5, coroll. 2]. Therefore A may be regarded as placed at E, and B at Δ. But they will be a set of magnitudes lying on a straight line, equal one to another, with their centres of gravity at equal intervals, and even in number; it is therefore clear that the centre of gravity of the magnitude compounded of them all is the point of bisection of the line containing the centres [of gravity] of the middle magnitudes [from Prop. 5, coroll. 2].

[1] συγκείμενα om. Heiberg.

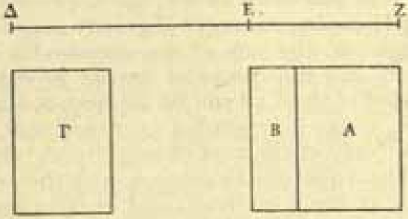
ἁ μὲν ΛΕ τᾷ ΓΔ, ἁ δὲ ΕΓ τᾷ ΔΚ, καὶ ὅλα ἄρα ἁ ΛΓ ἴσα τᾷ ΓΚ· ὥστε τοῦ ἐκ πάντων μεγέθεος κέντρον τοῦ βάρεος τὸ Γ σαμεῖον. τοῦ μὲν ἄρα Α κειμένου κατὰ τὸ Ε, τοῦ δὲ Β κατὰ τὸ Δ, ἰσορροπησοῦντι κατὰ τὸ Γ.

ζʹ

Καὶ τοίνυν, εἴ κα ἀσύμμετρα ἔωντι τὰ μεγέθεα, ὁμοίως ἰσορροπησοῦντι ἀπὸ μακέων ἀντιπεπονθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς μεγέθεσιν.

Ἔστω ἀσύμμετρα μεγέθεα τὰ ΑΒ, Γ, μάκεα δὲ τὰ ΔΕ, ΕΖ, ἐχέτω δὲ τὸ ΑΒ ποτὶ τὸ Γ τὸν αὐτὸν λόγον, ὃν καὶ τὸ ΕΔ ποτὶ τὸ ΕΖ μᾶκος· λέγω, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν ΑΒ, Γ κέντρον τοῦ βάρεός ἐστι τὸ Ε.

Εἰ γὰρ μὴ ἰσορροπήσει τὸ ΑΒ τεθὲν ἐπὶ τῷ Ζ τῷ Γ τεθέντι ἐπὶ τῷ Δ, ἤτοι μεῖζόν ἐστι τὸ ΑΒ



τοῦ Γ ἢ ὥστε ἰσορροπεῖν [τῷ Γ][1] ἢ οὔ. ἔστω μεῖζον, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΑΒ ἔλασσον τᾶς ὑπεροχᾶς, ᾇ μεῖζόν ἐστι τὸ ΑΒ τοῦ Γ ἢ ὥστε ἰσορροπεῖν, ὥστε [τὸ][2] λοιπὸν τὸ Α σύμμετρον

And since $\mathrm{AE} = \Gamma\Delta$ and $\mathrm{E}\Gamma = \Delta\mathrm{K}$, therefore $\mathrm{A}\Gamma = \Gamma\mathrm{K}$; so that the centre of gravity of the magnitude compounded of them all is the point Γ. Therefore if A is placed at E and B at Δ, they will balance about Γ.

Prop. 7

And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

Let (A + B), Γ be incommensurable magnitudes,[a] and let ΔE, EZ be distances, and let

$$(\mathrm{A} + \mathrm{B}) : \Gamma = \mathrm{E}\Delta : \mathrm{EZ};$$

I say that the centre of gravity of the magnitude composed of both (A + B), Γ is E.

For if (A + B) placed at Z do not balance Γ placed at Δ, either (A + B) is too much greater than Γ to balance or less. Let it [first] be too much greater, and let there be subtracted from (A + B) a magnitude less than the excess by which (A + B) is too much greater than Γ to balance, so that the remainder A is

[a] As becomes clear later in the proof, the first magnitude is regarded as made up of two parts—A, which is commensurate with Γ and B, which is not commensurate; if (A + B) is too big for equilibrium with Γ, then B is so chosen that, when it is taken away, the remainder A is still too big for equilibrium with Γ. Similarly if (A + B) is too small for equilibrium.

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[1] τῷ Γ om. Eutocius.
[2] τὸ om. Eutocius.

εἶμεν τῷ Γ. ἐπεὶ οὖν σύμμετρά ἐστι τὰ Α, Γ μεγέθεα, καὶ ἐλάσσονα λόγον ἔχει τὸ Α ποτὶ τὸ Γ ἢ ἁ ΔΕ ποτὶ ΕΖ, οὐκ ἰσορροπησοῦντι τὰ Α, Γ ἀπὸ τῶν ΔΕ, ΕΖ μακέων, τεθέντος τοῦ μὲν Α ἐπὶ τῷ Ζ, τοῦ δὲ Γ ἐπὶ τῷ Δ. διὰ ταὐτὰ δ', οὐδ' εἰ τὸ Γ μεῖζόν ἐστιν ἢ ὥστε ἰσορροπεῖν τῷ ΑΒ.

(iii.) *Centre of Gravity of a Parallelogram*

Ibid., Props. 9 et 10, Archim. ed. Heiberg ii. 140. 16–144. 4

θ'

Παντὸς παραλληλογράμμου τὸ κέντρον τοῦ βάρεός ἐστιν ἐπὶ τᾶς εὐθείας τᾶς ἐπιζευγνυούσας τὰς διχοτομίας τᾶν κατ' ἐναντίον τοῦ παραλληλογράμμου πλευρᾶν.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, ἐπὶ δὲ τὰν διχοτομίαν τᾶν ΑΒ, ΓΔ ἁ ΕΖ· φαμὶ δή, ὅτι τοῦ ΑΒΓΔ παραλληλογράμμου τὸ κέντρον τοῦ βάρεος ἐσσεῖται ἐπὶ τᾶς ΕΖ.

Μὴ γάρ, ἀλλ', εἰ δυνατόν, ἔστω τὸ Θ, καὶ ἄχθω παρὰ τὰν ΑΒ ἁ ΘΙ. τᾶς [δὲ][1] δὴ ΕΒ διχοτομουμένας αἰεὶ ἐσσεῖταί ποκα ἁ καταλειπομένα ἐλάσσων

[1] δὲ om. Heiberg.

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[a] The proof is incomplete and obscure; it may be thus completed.

Since $A : \Gamma < \Delta E : EZ$,

Δ will be depressed, which is impossible, since there has been taken away from (A + B) a magnitude less than the deduc-

commensurate with Γ. Then, since A, Γ are commensurable magnitudes, and

$$A : \Gamma < \Delta E : EZ,$$

A, Γ will not balance at the distances ΔE, EZ, A being placed at Z and Γ at Δ. By the same reasoning, they will not do so if Γ is greater than the magnitude necessary to balance (A + B).[a]

(iii.) *Centre of Gravity of a Parallelogram* [b]

Ibid., Props. 9 and 10, Archim. ed. Heiberg ii. 140. 16–144. 4

Prop. 9

The centre of gravity of any parallelogram is on the straight line joining the points of bisection of opposite sides of the parallelogram.

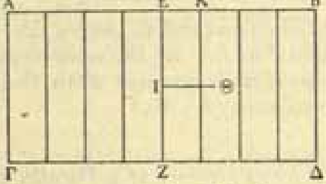
Let ABΓΔ be a parallelogram, and let EZ be the straight line joining the mid-points of AB, ΓΔ; then I say that the centre of gravity of the parallelogram ABΓΔ will be on EZ.

For if it be not, let it, if possible, be Θ, and let ΘI be drawn parallel to AB. Now if EB be bisected, and the half be bisected, and so on continually, there will be left some line less than IΘ; [let EK be less than

tion necessary to produce equilibrium, so that Z remains depressed. Therefore (A + B) is not greater than the magnitude necessary to produce equilibrium; in the same way it can be proved not to be less; therefore it is equal.

[b] The centres of gravity of a triangle and a trapezium are also found by Archimedes in the first book; the second book is wholly devoted to finding the centres of gravity of a parabolic segment and of a portion of it cut off by a parallel to the base.

τᾶς ΙΘ· καὶ διῃρήσθω ἑκατέρα τᾶν ΑΕ, ΕΒ εἰς τὰς τᾷ ΕΚ ἴσας, καὶ ἀπὸ τῶν κατὰ τὰς διαιρέσιας



σαμείων ἄχθωσαν παρὰ τὰν ΕΖ· διαιρεθήσεται δὴ τὸ ὅλον παραλληλόγραμμον εἰς παραλληλόγραμμα τὰ ἴσα καὶ ὁμοῖα τῷ ΚΖ. τῶν οὖν παραλληλογράμμων τῶν ἴσων καὶ ὁμοίων τῷ ΚΖ ἐφαρμοζομένων ἐπ' ἄλλαλα καὶ τὰ κέντρα τοῦ βάρεος αὐτῶν ἐπ' ἄλλαλα πεσοῦνται. ἐσσοῦνται δὴ μεγέθεά τινα, παραλληλόγραμμα ἴσα τῷ ΚΖ, ἄρτια τῷ πλήθει, καὶ τὰ κέντρα τοῦ βάρεος αὐτῶν ἐπ' εὐθείας κείμενα, καὶ τὰ μέσα ἴσα, καὶ πάντα τὰ ἐφ' ἑκάτερα τῶν μέσων αὐτά τε ἴσα ἐντὶ καὶ αἱ μεταξὺ τῶν κέντρων εὐθεῖαι ἴσαι· τοῦ ἐκ πάντων αὐτῶν ἄρα συγκειμένου μεγέθεος τὸ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τᾶς εὐθείας τᾶς ἐπιζευγνυούσας τὰ κέντρα τοῦ βάρεος τῶν μέσων χωρίων. οὐκ ἔστι δέ· τὸ γὰρ Θ ἐκτός ἐστι τῶν μέσων παραλληλογράμμων. φανερὸν οὖν, ὅτι ἐπὶ τᾶς ΕΖ εὐθείας τὸ κέντρον ἐστὶ τοῦ βάρεος τοῦ ΑΒΓΔ παραλληλογράμμου.

ι´

Παντὸς παραλληλογράμμου τὸ κέντρον τοῦ βάρεός ἐστι τὸ σαμεῖον, καθ' ὃ αἱ διαμέτροι συμπίπτοντι.

IΘ,] and let each of AE, EB be divided into parts equal to EK, and from the points of division let straight lines be drawn parallel to EZ; then the whole parallelogram will be divided into parallelograms equal and similar to KZ. Therefore, if these parallelograms equal and similar to KZ be applied to each other, their centres of gravity will also coincide [Post. 4]. Thus there will be a set of magnitudes, being parallelograms equal to KZ, which are even in number and whose centres of gravity lie on a straight line, and the middle magnitudes will be equal, and the magnitudes on either side of the middle magnitudes will also be equal, and the straight lines between their centres [of gravity] will be equal; therefore the centre of gravity of the magnitude compounded of them all will be on the straight line joining the centres of gravity of the middle areas [Prop. 5, coroll. 2]. But it is not; for Θ lies without the middle parallelograms. It is therefore manifest that the centre of gravity of the parallelogram ΑΒΓΔ will be on the straight line EZ.

Prop. 10

The centre of gravity of any parallelogram is the point in which the diagonals meet.

μαθήμασιν κατὰ τὸ ὑποπῖπτον θεωρίαν τετμηκότα ἐδοκίμασα γράψαι σοι καὶ εἰς τὸ αὐτὸ βιβλίον ἐξορίσαι τρόπου τινὸς ἰδιότητα, καθ' ὃν σοι παρεχόμενον ἔσται λαμβάνειν ἀφορμὰς εἰς τὸ δύνασθαί τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν. τοῦτο δὲ πέπεισμαι χρήσιμον εἶναι οὐδὲν ἧσσον καὶ εἰς τὴν ἀπόδειξιν αὐτῶν τῶν θεωρημάτων. καὶ γάρ τινα τῶν πρότερον μοι φανέντων μηχανικῶς ὕστερον γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρὶς ἀποδείξεως εἶναι τὴν διὰ τούτου τοῦ τρόπου θεωρίαν· ἑτοιμότερον γάρ ἐστι προλαβόντα διὰ τοῦ τρόπου γνῶσίν τινα τῶν ζητημάτων πορίσασθαι τὴν ἀπόδειξιν μᾶλλον ἢ μηδενὸς ἐγνωσμένου ζητεῖν. . . . γράφομεν οὖν πρῶτον τὸ καὶ πρῶτον φανὲν διὰ τῶν μηχανικῶν, ὅτι πᾶν τμῆμα ὀρθογωνίου κώνου τομῆς ἐπίτριτόν ἐστιν τριγώνου τοῦ βάσιν ἔχοντος τὴν αὐτὴν καὶ ὕψος ἴσον.

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

Ἔστω τμῆμα τὸ ΑΒΓ περιεχόμενον ὑπὸ εὐθείας τῆς ΑΓ καὶ ὀρθογωνίου κώνου τομῆς τῆς ΑΒΓ, καὶ τετμήσθω δίχα ἡ ΑΓ τῷ Δ, καὶ παρὰ τὴν διάμετρον ἤχθω ἡ ΔΒΕ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ.

Λέγω, ὅτι ἐπίτριτόν ἐστιν τὸ ΑΒΓ τμῆμα τοῦ ΑΒΓ τριγώνου.

Ἤχθωσαν ἀπὸ τῶν Α, Γ σημείων ἡ μὲν ΑΖ παρὰ τὴν ΔΒΕ, ἡ δὲ ΓΖ ἐπιψαύουσα τῆς τομῆς, καὶ ἐκβεβλήσθω ἡ ΓΒ ἐπὶ τὸ Κ, καὶ κείσθω τῇ ΓΚ ἴση ἡ ΚΘ. νοείσθω ζυγὸς ὁ ΓΘ καὶ μέσον

who gives due honour to mathematical inquiries when they arise, I have thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, with which furnished you will be able to make a beginning in the investigation by mechanics of some of the problems in mathematics. I am persuaded that this method is no less useful even for the proof of the theorems themselves. For some things first became clear to me by mechanics, though they had later to be proved geometrically owing to the fact that investigation by this method does not amount to actual proof; but it is, of course, easier to provide the proof when some knowledge of the things sought has been acquired by this method rather than to seek it with no prior knowledge. . . . At the outset therefore I will write out the very first theorem that became clear to me through mechanics, that *any segment of a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.*

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

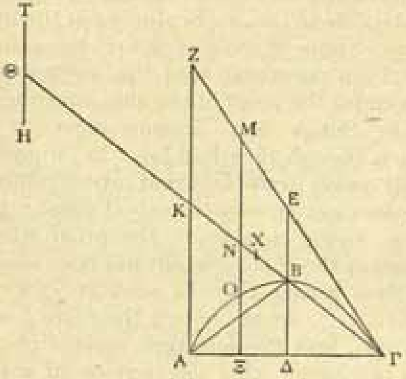
Let ΑΒΓ be a segment bounded by the straight line ΑΓ and the section ΑΒΓ of a right-angled cone, and let ΑΓ be bisected at Δ, and let ΔΒΕ be drawn parallel to the axis, and let ΑΒ, ΒΓ be joined.

I say that the segment ΑΒΓ is four-thirds of the triangle ΑΒΓ.

From the points Α, Γ let ΑΖ be drawn parallel to ΔΒΕ, and let ΓΖ be drawn to touch the section, and let ΓΒ be produced to Κ, and let ΚΘ be placed equal to ΓΚ. Let ΓΘ be imagined to be a balance

αὐτοῦ τὸ Κ καὶ τῇ ΕΔ παράλληλος τυχοῦσα ἡ ΜΞ.

Ἐπεὶ οὖν παραβολή ἐστιν ἡ ΓΒΑ, καὶ ἐφάπτεται



ἡ ΓΖ, καὶ τεταγμένως ἡ ΓΔ, ἴση ἐστὶν ἡ ΕΒ τῇ ΒΔ· τοῦτο γὰρ ἐν τοῖς στοιχείοις δείκνυται· διὰ δὴ τοῦτο, καὶ διότι παράλληλοί εἰσιν αἱ ΖΑ, ΜΞ τῇ ΕΔ, ἴση ἐστὶν καὶ ἡ μὲν ΜΝ τῇ ΝΞ, ἡ δὲ ΖΚ τῇ ΚΑ. καὶ ἐπεί ἐστιν, ὡς ἡ ΓΑ πρὸς ΑΞ, οὕτως ἡ ΜΞ πρὸς ΞΟ [τοῦτο γὰρ ἐν λήμματι δείκνυται],[1] ὡς δὲ ἡ ΓΑ πρὸς ΑΞ, οὕτως ἡ ΓΚ πρὸς ΚΝ, καὶ ἴση ἐστὶν ἡ ΓΚ τῇ ΚΘ, ὡς ἄρα ἡ ΘΚ πρὸς ΚΝ, οὕτως ἡ ΜΞ πρὸς ΞΟ. καὶ ἐπεὶ τὸ Ν σημεῖον κέντρον τοῦ βάρους τῆς ΜΞ εὐθείας ἐστίν, ἐπείπερ ἴση ἐστὶν ἡ ΜΝ τῇ ΝΞ, ἐὰν ἄρα τῇ ΞΟ ἴσην θῶμεν τὴν ΤΗ καὶ κέντρον τοῦ βάρους αὐτῆς τὸ Θ, ὅπως ἴση ᾖ ἡ ΤΘ τῇ ΘΗ, ἰσορροπήσει ἡ ΤΘΗ τῇ ΜΞ αὐτοῦ μενούσῃ διὰ τὸ ἀντιπεπονθότως τετμῆσθαι

with mid-point K, and let MΞ be drawn parallel to EΔ.

Then since ΓBA is a parabola,[a] and ΓZ touches it, and ΓΔ is a semi-ordinate, EB = BΔ—for this is proved in the elements [b]; for this reason, and because ZA, MΞ are parallel to EΔ, MN = NΞ and ZK = KA [Eucl. vi. 4, v. 9]. And since

$$\Gamma A : A\Xi = M\Xi : \Xi O, \qquad \text{[\textit{Quad. parab.} 5, Eucl. v. 18}$$

and $$\Gamma A : A\Xi = \Gamma K : KN, \qquad \text{[Eucl. vi. 2, v. 18}$$

while $$\Gamma K = K\Theta,$$

therefore $$\Theta K : KN = M\Xi : \Xi O.$$

And since the point N is the centre of gravity of the straight line MΞ, inasmuch as MN = NΞ [Lemma 4], if we place TH = ΞO, with Θ for its centre of gravity, so that TΘ = ΘH [Lemma 4], then TΘH will balance MΞ in its present position, because ΘN is cut

[a] Archimedes would have said "section of a right-angled cone"—ὀρθογωνίου κώνου τομά.

[b] The reference will be to the *Elements of Conics* by Euclid and Aristaeus for which *v.* vol. i. pp. 486-491 and *infra*, p. 280 n. *a*; *cf.* similar expressions in *On Conoids and Spheroids*, Prop. 3 and *Quadrature of a Parabola*, Prop. 3; the theorem is *Quadrature of a Parabola*, Prop. 2.

[1] τοῦτο . . . δείκνυται om. Heiberg. It is probably an interpolator's reference to a marginal lemma.

τὴν ΘΝ τοῖς ΤΗ, ΜΞ βάρεσιν, καὶ ὡς τὴν ΘΚ πρὸς ΚΝ, οὕτως τὴν ΜΞ πρὸς τὴν ΗΤ· ὥστε τοῦ ἐξ ἀμφοτέρων βάρους κέντρον ἐστὶν τοῦ βάρους τὸ Κ. ὁμοίως δὲ καί, ὅσαι ἂν ἀχθῶσιν ἐν τῷ ΖΑΓ τριγώνῳ παράλληλοι τῇ ΕΔ, ἰσορροπήσουσιν αὐτοῦ μένουσαι ταῖς ἀπολαμβανομέναις ἀπ' αὐτῶν ὑπὸ τῆς τομῆς μετενεχθείσαις ἐπὶ τὸ Θ, ὥστε εἶναι τοῦ ἐξ ἀμφοτέρων κέντρον τοῦ βάρους τὸ Κ. καὶ ἐπεὶ ἐκ μὲν τῶν ἐν τῷ ΓΖΑ τριγώνῳ τὸ ΓΖΑ τρίγωνον συνέστηκεν, ἐκ δὲ τῶν ἐν τῇ τομῇ ὁμοίως τῇ ΞΟ λαμβανομένων συνέστηκε τὸ ΑΒΓ τμῆμα, ἰσορροπήσει ἄρα τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ τμήματι τῆς τομῆς τεθέντι περὶ κέντρον τοῦ βάρους τὸ Θ κατὰ τὸ Κ σημεῖον, ὥστε τοῦ ἐξ ἀμφοτέρων κέντρον εἶναι τοῦ βάρους τὸ Κ. τετμήσθω δὴ ἡ ΓΚ τῷ Χ, ὥστε τριπλασίαν εἶναι τὴν ΓΚ τῆς ΚΧ· ἔσται ἄρα τὸ Χ σημεῖον κέντρον βάρους τοῦ ΑΖΓ τριγώνου· δέδεικται γὰρ ἐν τοῖς Ἰσορροπικοῖς. ἐπεὶ οὖν ἰσόρροπον τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ ΒΑΓ τμήματι κατὰ τὸ Κ τεθέντι περὶ τὸ Θ κέντρον τοῦ βάρους, καί ἐστιν τοῦ ΖΑΓ τριγώνου κέντρον βάρους τὸ Χ, ἔστιν ἄρα, ὡς τὸ ΑΖΓ τρίγωνον πρὸς τὸ ΑΒΓ τμῆμα κείμενον περὶ τὸ Θ κέντρον, οὕτως ἡ ΘΚ πρὸς ΧΚ. τριπλασία δέ ἐστιν ἡ ΘΚ τῆς ΚΧ· τριπλάσιον ἄρα καὶ τὸ ΑΖΓ τρίγωνον τοῦ ΑΒΓ τμήματος. ἔστι δὲ καὶ τὸ ΖΑΓ τρίγωνον τετραπλάσιον τοῦ ΑΒΓ τριγώνου διὰ τὸ ἴσην εἶναι τὴν μὲν ΖΚ τῇ ΚΑ, τὴν δὲ ΑΔ τῇ ΔΓ· ἐπίτριτον ἄρα ἐστὶν τὸ ΑΒΓ τμῆμα τοῦ ΑΒΓ τριγώνου. [τοῦτο οὖν φανερόν ἐστιν].[1]

[1] τοῦτο . . . ἐστιν om. Heiberg.

in the inverse proportion of the weights TH, MΞ,

and $\Theta K : KN = M\Xi : HT$;

therefore the centre of gravity of both [TH, MΞ] taken together is K. In the same way, as often as parallels to EΔ are drawn in the triangle ZAΓ, these parallels, remaining in the same position, will balance the parts cut off from them by the section and transferred to Θ, so that the centre of gravity of both together is K. And since the triangle ΓZA is composed of the [straight lines drawn] in ΓZA, and the segment ABΓ is composed of the lines in the section formed in the same way as ΞO, therefore the triangle ZAΓ in its present position will be balanced about K by the segment of the section placed with Θ for its centre of gravity, so that the centre of gravity of both combined is K. Now let ΓK be cut at X so that $\Gamma K = 3KX$; then the point X will be the centre of gravity of the triangle AZΓ; for this has been proved in the books *On Equilibriums*.[a] Then since the triangle ZAΓ in its present position is balanced about K by the segment BAΓ placed so as to have Θ for its centre of gravity, and since the centre of gravity of the triangle ZAΓ is X, therefore the ratio of the triangle AZΓ to the segment ABΓ placed about Θ as its centre [of gravity] is equal to $\Theta K : XK$. But $\Theta K = 3KX$; therefore

$$\text{triangle } AZ\Gamma = 3 \,.\, \text{segment } AB\Gamma.$$

And $\qquad \text{triangle } ZA\Gamma = 4 \,.\, \text{triangle } AB\Gamma,$

because $\quad ZK = KA$ and $A\Delta = \Delta\Gamma$;

therefore $\quad \text{segment } AB\Gamma = \frac{4}{3} \text{ triangle } AB\Gamma.$

[a] *Cf. De Plan. Equil.* i. 15.

Τοῦτο δὴ διὰ μὲν τῶν νῦν εἰρημένων οὐκ ἀποδέδεικται, ἔμφασιν δέ τινα πεποίηκε τὸ συμπέρασμα ἀληθὲς εἶναι· διόπερ ἡμεῖς ὁρῶντες μὲν οὐκ ἀποδεδειγμένον, ὑπονοοῦντες δὲ τὸ συμπέρασμα ἀληθὲς εἶναι, τάξομεν τὴν γεωμετρουμένην ἀπόδειξιν ἐξευρόντες αὐτοὶ τὴν ἐκδοθεῖσαν πρότερον.[1]

Archim. *Quadr. Parab.*, Praef., Archim. ed. Heiberg ii. 262. 2–266. 4

Ἀρχιμήδης Δοσιθέῳ εὖ πράττειν.

Ἀκούσας Κόνωνα μὲν τετελευτηκέναι, ὃς ἦν οὐδὲν ἐπιλείπων ἁμὶν ἐν φιλίᾳ, τὶν δὲ Κόνωνος γνώριμον γεγενῆσθαι καὶ γεωμετρίας οἰκεῖον εἶμεν τοῦ μὲν τετελευτηκότος εἵνεκεν ἐλυπήθημες ὡς καὶ φίλου τοῦ ἀνδρὸς γεναμένου καὶ ἐν τοῖς μαθημάτεσσι θαυμαστοῦ τινος, ἐπροχειριζάμεθα δὲ ἀποστεῖλαί τοι γράψαντες, ὡς Κόνωνι γράφειν ἐγνωκότες ἦμες, γεωμετρικῶν θεωρημάτων, ὃ πρότερον μὲν οὐκ ἦν τεθεωρημένον, νῦν δὲ ὑφ' ἁμῶν τεθεώρηται, πρότερον μὲν διὰ μηχανικῶν εὑρεθέν, ἔπειτα δὲ καὶ διὰ τῶν γεωμετρικῶν ἐπιδειχθέν. τῶν μὲν οὖν πρότερον περὶ γεωμετρίαν πραγματευθέντων ἐπεχείρησάν τινες γράφειν ὡς δυνατὸν ἐὸν κύκλῳ τῷ δοθέντι καὶ κύκλου τμάματι τῷ δοθέντι χωρίον εὑρεῖν εὐθύγραμμον ἴσον, καὶ μετὰ ταῦτα τὸ περιεχόμενον χωρίον ὑπό τε τᾶς

[1] τοῦτο . . . πρότερον. In the MS. the whole paragraph from τοῦτο to πρότερον comes at the beginning of Prop. 2; it is more appropriate at the end of Prop. 1.

This, indeed, has not been actually demonstrated by the arguments now used, but they have given some indication that the conclusion is true; seeing, therefore, that the theorem is not demonstrated, but suspecting that the conclusion is true, we shall have recourse [a] to the geometrical proof which I myself discovered and have already published.[b]

Archimedes, *Quadrature of a Parabola*, Preface, Archim. ed. Heiberg ii. 262. 2–266. 4

Archimedes to Dositheus greeting.

On hearing that Conon, who fulfilled in the highest degree the obligations of friendship, was dead, but that you were an acquaintance of Conon and also versed in geometry, while I grieved for the death of a friend and an excellent mathematician, I set myself the task of communicating to you, as I had determined to communicate to Conon, a certain geometrical theorem, which had not been investigated before, but has now been investigated by me, and which I first discovered by means of mechanics and later proved by means of geometry. Now some of those who in former times engaged in mathematics tried to find a rectilineal area equal to a given circle [c] and to a given segment of a circle, and afterwards they tried to square the area bounded by the section

[a] I have followed Heath's rendering of τάξομεν, which seems more probable than Heiberg's "suo loco proponemus," though it is a difficult meaning to extract from τάξομεν.

[b] Presumably *Quadr. Parab.* 24, the second of the proofs now to be given. The theorem has not been demonstrated, of course, because the triangle and the segment may not be supposed to be composed of straight lines.

[c] This seems to indicate that Archimedes had not at this time written his own book *On the Measurement of a Circle*. For attempts to square the circle, v. vol. i. pp. 308-347.

ὅλου τοῦ κώνου τομᾶς καὶ εὐθείας τετραγωνίζειν ἐπειρῶντο λαμβάνοντες οὐκ εὐπαραχώρητα λήμματα, διόπερ αὐτοῖς ὑπὸ τῶν πλείστων οὐκ εὑρισκόμενα ταῦτα κατεγνώσθεν. τὸ δὲ ὑπ' εὐθείας τε καὶ ὀρθογωνίου κώνου τομᾶς τμᾶμα περιεχόμενον οὐδένα τῶν προτέρων ἐγχειρήσαντα τετραγωνίζειν ἐπιστάμεθα, ὃ δὴ νῦν ὑφ' ἁμῶν εὕρηται· δείκνυται γάρ, ὅτι πᾶν τμᾶμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς ἐπίτριτόν ἐστι τοῦ τριγώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν καὶ ὕψος ἴσον τῷ τμάματι λαμβανομένου τοῦδε τοῦ λήμματος ἐς τὰν ἀπόδειξιν αὐτοῦ· τῶν ἀνίσων χωρίων τὰν ὑπεροχάν, ᾇ ὑπερέχει τὸ μεῖζον τοῦ ἐλάσσονος, δυνατὸν εἶμεν αὐτὰν ἑαυτᾷ συντιθεμέναν παντὸς ὑπερέχειν τοῦ προτεθέντος πεπερασμένου χωρίου. κέχρηνται δὲ καὶ οἱ πρότερον γεωμέτραι τῷδε τῷ λήμματι· τούς τε γὰρ κύκλους διπλασίονα λόγον ἔχειν ποτ' ἀλλάλους τᾶν διαμέτρων ἀποδεδείχασιν αὐτῷ τούτῳ τῷ λήμματι χρώμενοι, καὶ τὰς σφαίρας ὅτι τριπλασίονα λόγον ἔχοντι ποτ' ἀλλάλας τᾶν διαμέτρων, ἔτι δὲ καὶ ὅτι πᾶσα πυραμὶς τρίτον μέρος ἐστὶ τοῦ πρίσματος τοῦ τὰν αὐτὰν βάσιν ἔχοντος τᾷ πυραμίδι καὶ ὕψος ἴσον· καὶ διότι πᾶς κῶνος τρίτον μέρος ἐστὶ τοῦ κυλίνδρου τοῦ τὰν αὐτὰν βάσιν ἔχοντος τῷ κώνῳ καὶ ὕψος ἴσον, ὁμοῖον τῷ προειρημένῳ λῆμμά τι λαμβάνοντες ἔγραφον. συμβαίνει δὲ τῶν προειρημένων θεωρημάτων ἕκαστον μηδενὸς ἧσσον τῶν ἄνευ τούτου τοῦ λήμματος ἀποδεδειγμένων πεπιστευκέναι· ἀρκεῖ δὲ ἐς τὰν ὁμοίαν πίστιν τούτοις ἀναγμένων τῶν ὑφ' ἁμῶν ἐκδιδομένων. ἀναγράψαντες οὖν αὐτοῦ τὰς ἀποδείξιας ἀποστέλλομες

of the whole cone and a straight line,[a] assuming lemmas far from obvious, so that it was recognized by most people that the problem had not been solved. But I do not know that any of my predecessors has attempted to square the area bounded by a straight line and a section of a right-angled cone, the solution of which problem I have now discovered; for it is shown that *any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle which has the same base and height equal to the segment*, and for the proof this lemma is assumed: *given [two] unequal areas, the excess by which the greater exceeds the less can, by being added to itself, be made to exceed any given finite area.* Earlier geometers have also used this lemma: for, by using this same lemma, they proved that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and also that any pyramid is a third part of the prism having the same base as the pyramid and equal height; and, further, by assuming a lemma similar to that aforesaid, they proved that any cone is a third part of the cylinder having the same base as the cone and equal height.[b] In the event, each of the aforesaid theorems has been accepted, no less than those proved without this lemma; and it will satisfy me if the theorems now published by me obtain the same degree of acceptance. I have therefore written out the proofs, and now send them, first

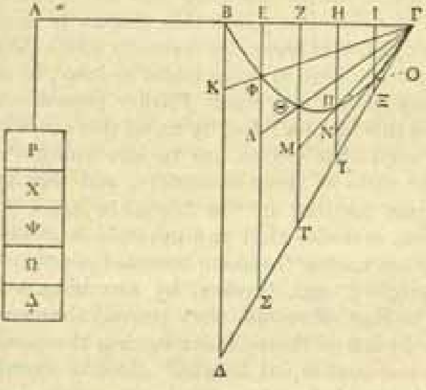
[a] A "section of the whole cone" is probably a section cutting right through it, *i.e.*, an ellipse, but the expression is odd.

[b] For this lemma, *v. supra*, p. 46 n. *a*.

πρῶτον μέν, ὡς διὰ τῶν μηχανικῶν ἐθεωρήθη, μετὰ ταῦτα δὲ καί, ὡς διὰ τῶν γεωμετρουμένων ἀποδείκνυται. προγράφεται δὲ καὶ στοιχεῖα κωνικὰ χρείαν ἔχοντα ἐς τὰς ἀπόδειξιν. ἔρρωσο.

Ibid., Prop. 14, Archim. ed. Heiberg ii. 284. 24–290. 17

Ἔστω τμᾶμα τὸ ΒΘΓ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς. ἔστω δὴ πρῶτον



ἁ ΒΓ ποτ' ὀρθὰς τᾷ διαμέτρῳ, καὶ ἄχθω ἀπὸ μὲν τοῦ Β σαμείου ἁ ΒΔ παρὰ τὰν διάμετρον, ἀπὸ δὲ τοῦ Γ ἁ ΓΔ ἐπιψαύουσα τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Γ· ἐσσεῖται δὴ τὸ ΒΓΔ τρίγωνον ὀρθογώνιον. διῃρήσθω δὴ ἁ ΒΓ ἐς ἴσα τμάματα ὁποσαοῦν τὰ ΒΕ, ΕΖ, ΖΗ, ΗΙ, ΙΓ, καὶ ἀπὸ τᾶν τομᾶν ἄχθωσαν παρὰ τὰν διάμετρον αἱ ΕΣ, ΖΤ, ΗΥ, ΙΞ, ἀπὸ δὲ τῶν σαμείων, καθ' ἃ τέμνοντι

as they were investigated by means of mechanics, and also as they may be proved by means of geometry. By way of preface are included the elements of conics which are needed in the demonstration. Farewell.

Ibid., Prop. 14, Archim. ed. Heiberg ii. 284. 24–290. 17

Let ΒΘΓ be a segment bounded by a straight line and a section of a right-angled cone. First let ΒΓ be at right angles to the axis, and from Β let ΒΔ be drawn parallel to the axis, and from Γ let ΓΔ be drawn touching the section of the cone at Γ; then the triangle ΒΓΔ will be right-angled [Eucl. i. 29]. Let ΒΓ be divided into any number of equal segments ΒΕ, ΕΖ, ΖΗ, ΗΙ, ΙΓ, and from the points of section let ΕΣ, ΖΤ, ΗΥ, ΙΞ be drawn parallel to the axis, and from the points in which these cut the

αὗται τὰν τοῦ κώνου τομάν, ἐπεζεύχθωσαν ἐπὶ τὸ Γ καὶ ἐκβεβλήσθωσαν. φαμὶ δὴ τὸ τρίγωνον τὸ ΒΔΓ τῶν μὲν τραπεζίων τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ καὶ τοῦ ΞΙΓ τριγώνου ἔλασσον εἶμεν ἢ τριπλάσιον, τῶν δὲ τραπεζίων τῶν ΖΦ, ΗΘ, ΙΠ καὶ τοῦ ΙΟΓ τριγώνου μεῖζόν [ἐστιν][1] ἢ τριπλάσιον.

Διάχθω γὰρ εὐθεῖα ἁ ΑΒΓ, καὶ ἀπολελάφθω ἁ ΑΒ ἴσα τᾷ ΒΓ, καὶ νοείσθω ζύγιον τὸ ΑΓ· μέσον δὲ αὐτοῦ ἐσσεῖται τὸ Β· καὶ κρεμάσθω ἐκ τοῦ Β, κρεμάσθω δὲ καὶ τὸ ΒΔΓ ἐκ τοῦ ζυγοῦ κατὰ τὰ Β, Γ, ἐκ δὲ τοῦ θατέρου μέρεος τοῦ ζυγοῦ κρεμάσθω τὰ Ρ, Χ, Ψ, Ω, Δ̣ χωρία κατὰ τὸ Α, καὶ ἰσορροπείτω τὸ μὲν Ρ χωρίον τῷ ΔΕ τραπεζίῳ οὕτως ἔχοντι, τὸ δὲ Χ τῷ ΖΣ τραπεζίῳ, τὸ δὲ Ψ τῷ ΤΗ, τὸ δὲ Ω τῷ ΥΙ, τὸ δὲ Δ̣ τῷ ΞΙΓ τριγώνῳ· ἰσορροπήσει δὴ καὶ τὸ ὅλον τῷ ὅλῳ· ὥστε τριπλάσιον ἂν εἴη τὸ ΒΔΓ τρίγωνον τοῦ ΡΧΨΩΔ̣ χωρίου. καὶ ἐπεί ἐστιν τμᾶμα τὸ ΒΓΘ, ὃ περιέχεται ὑπό τε εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς, καὶ ἀπὸ μὲν τοῦ Β παρὰ τὰν διάμετρον ἄκται ἁ ΒΔ, ἀπὸ δὲ τοῦ Γ ἁ ΓΔ ἐπιψαύουσα τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Γ, ἄκται δέ τις καὶ ἄλλα παρὰ τὰν διάμετρον ἁ ΣΕ, τὸν αὐτὸν ἔχει λόγον ἁ ΒΓ ποτὶ τὰν ΒΕ, ὃν ἁ ΣΕ ποτὶ τὰν ΕΦ· ὥστε καὶ ἁ ΒΑ ποτὶ τὰν ΒΕ τὸν αὐτὸν ἔχει λόγον, ὃν τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ. ὁμοίως δὲ δειχθήσεται ἁ ΑΒ ποτὶ τὰν ΒΖ τὸν αὐτὸν ἔχουσα λόγον, ὃν τὸ ΣΖ τραπέζιον ποτὶ τὸ ΛΖ, ποτὶ δὲ τὰν ΒΗ, ὃν τὸ ΤΗ ποτὶ τὸ ΜΗ, ποτὶ δὲ τὰν ΒΙ, ὃν τὸ ΥΙ ποτὶ τὸ ΝΙ. ἐπεὶ οὖν ἐστι τραπέζιον τὸ ΔΕ τὰς

[1] ἐστιν om. Heiberg.

section of the cone let straight lines be drawn to Γ and produced. Then I say that the triangle ΒΔΓ is less than three times the trapezia ΚΕ, ΛΖ, ΜΗ, ΝΙ and the triangle ΞΙΓ, but greater than three times the trapezia ΖΦ, ΗΘ, ΙΠ and the triangle ΙΟΓ.

For let the straight line ΑΒΓ be drawn, and let ΑΒ be cut off equal to ΒΓ, and let ΑΓ be imagined to be a balance; its middle point will be Β; let it be suspended from Β, and let the triangle ΒΔΓ be suspended from the balance at Β, Γ, and from the other part of the balance let the areas Ρ, Χ, Ψ, Ω, $\underset{\cdot}{\Delta}$ be suspended at Α, and let the area Ρ balance the trapezium ΔΕ in this position, let Χ balance the trapezium ΖΣ, let Ψ balance ΤΗ, let Ω balance ΥΙ, and let $\underset{\cdot}{\Delta}$ balance the triangle ΞΙΓ; then the whole will balance the whole; so that the triangle ΒΔΓ will be three times the area $Ρ + Χ + Ψ + Ω + \underset{\cdot}{\Delta}$ [Prop. 6]. And since ΒΓΘ is a segment bounded by a straight line and a section of a right-angled cone, and ΒΔ has been drawn from Β parallel to the axis, and ΓΔ has been drawn from Γ touching the section of a cone at Γ, and another straight line ΣΕ has been drawn parallel to the axis,

$$ΒΓ : ΒΕ = ΣΕ : ΕΦ; \qquad \text{[Prop. 5}$$

therefore $ΒΑ : ΒΕ =$ trapezium ΔΕ : trapezium ΚΕ.[a]

Similarly it may be proved that

$$ΑΒ : ΒΖ = ΣΖ : ΛΖ,$$

$$ΑΒ : ΒΗ = ΤΗ : ΜΗ,$$

$$ΑΒ : ΒΙ = ΥΙ : ΝΙ.$$

Therefore, since ΔΕ is a trapezium with right angles

[a] For ΒΑ = ΒΓ and ΔΕ : ΚΕ = ΣΕ : ΕΦ.

μὲν ποτὶ τοῖς Β, Ε σαμείοις γωνίας ὀρθὰς ἔχον, τὰς δὲ πλευρὰς ἐπὶ τὸ Γ νευούσας, ἰσορροπεῖ δέ τι χωρίον αὐτῷ τὸ Ρ κρεμάμενον ἐκ τοῦ ζυγοῦ κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς ἁ ΒΑ ποτὶ τὰν ΒΕ, οὕτως τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ, μεῖζον ἄρα ἐστὶν τὸ ΚΕ χωρίον τοῦ Ρ χωρίου· δέδεικται γὰρ τοῦτο. πάλιν δὲ καὶ τὸ ΖΣ τραπέζιον τὰς μὲν ποτὶ τοῖς Ζ, Ε γωνίας ὀρθὰς ἔχον, τὰν δὲ ΣΤ νεύουσαν ἐπὶ τὸ Γ, ἰσορροπεῖ δὲ αὐτῷ χωρίον τὸ Χ ἐκ τοῦ ζυγοῦ κρεμάμενον κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς μὲν ἁ ΑΒ ποτὶ τὰν ΒΕ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΖΦ, ὡς δὲ ἁ ΑΒ ποτὶ τὰν ΒΖ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΛΖ· εἴη οὖν κα τὸ Χ χωρίον τοῦ μὲν ΛΖ τραπεζίου ἔλασσον, τοῦ δὲ ΖΦ μεῖζον· δέδεικται γὰρ καὶ τοῦτο. διὰ τὰ αὐτὰ δὴ καὶ τὸ Ψ χωρίον τοῦ μὲν ΜΗ τραπεζίου ἔλασσον, τοῦ δὲ ΘΗ μεῖζον, καὶ τὸ Ω χωρίον τοῦ μὲν ΝΟΙΗ τραπεζίου ἔλασσον, τοῦ δὲ ΠΙ μεῖζον, ὁμοίως δὲ καὶ τὸ Ϙ χωρίον τοῦ μὲν ΞΙΓ τριγώνου ἔλασσον, τοῦ δὲ ΓΙΟ μεῖζον. ἐπεὶ οὖν τὸ μὲν ΚΕ τραπέζιον μεῖζόν ἐστι τοῦ Ρ χωρίου, τὸ δὲ ΛΖ τοῦ Χ, τὸ δὲ ΜΗ τοῦ Ψ, τὸ δὲ ΝΙ τοῦ Ω, τὸ δὲ ΞΙΓ τρίγωνον τοῦ Ϙ, φανερόν, ὅτι καὶ πάντα τὰ εἰρημένα χωρία μείζονά ἐστι τοῦ ΡΧΨΩϘ χωρίου. ἔστιν δὲ τὸ ΡΧΨΩϘ τρίτον μέρος τοῦ ΒΓΔ τριγώνου· δῆλον ἄρα, ὅτι τὸ ΒΓΔ τρίγωνον ἔλασσόν ἐστιν ἢ τριπλάσιον τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ τραπεζίων καὶ τοῦ ΞΙΓ τριγώνου. πάλιν, ἐπεὶ τὸ μὲν ΖΦ τραπέζιον ἔλασσόν ἐστι τοῦ Χ χωρίου, τὸ δὲ ΘΗ τοῦ Ψ, τὸ δὲ ΙΠ τοῦ Ω, τὸ δὲ ΙΟΓ τρίγωνον τοῦ Ϙ, φανερόν, ὅτι καὶ πάντα

at the points B, E and with sides converging on Γ, and it balances the area P suspended from the balance at A, if the trapezium be in its present position, while

$$\mathrm{BA} : \mathrm{BE} = \Delta\mathrm{E} : \mathrm{KE},$$

therefore $$\mathrm{KE} > \mathrm{P};$$

for this has been proved [Prop. 10]. Again, since ZΣ is a trapezium with right angles at the points Z, E and with ΣT converging on Γ, and it balances the area X suspended from the balance at A, if the trapezium be in its present position, while

$$\mathrm{AB} : \mathrm{BE} = \mathrm{Z}\Sigma : \mathrm{Z}\Phi,$$

$$\mathrm{AB} : \mathrm{BZ} = \mathrm{Z}\Sigma : \Lambda\mathrm{Z},$$

therefore $$\Lambda\mathrm{Z} > \mathrm{X} > \mathrm{Z}\Phi;$$

for this also has been proved [Prop. 12]. By the same reasoning

$$\mathrm{MH} > \Psi > \Theta\mathrm{H},$$

and $$\mathrm{NOIH} > \Omega > \Pi\mathrm{I},$$

and similarly $$\Xi\mathrm{I}\Gamma > \underset{s}{\Delta} > \Gamma\mathrm{IO}.$$

Then, since $\mathrm{KE} > \mathrm{P}$, $\Lambda\mathrm{Z} > \mathrm{X}$, $\mathrm{MH} > \Psi$, $\mathrm{NI} > \Omega$, $\Xi\mathrm{I}\Gamma > \underset{s}{\Delta}$, it is clear that the sum of the aforesaid areas is greater than the area $\mathrm{P} + \mathrm{X} + \Psi + \Omega + \underset{s}{\Delta}$. But

$$\mathrm{P} + \mathrm{X} + \Psi + \Omega + \underset{s}{\Delta} = \tfrac{1}{3}\,\mathrm{B}\Gamma\Delta; \qquad \text{[Prop. 6}$$

it is therefore plain that

$$\mathrm{B}\Gamma\Delta < 3(\mathrm{KE} + \Lambda\mathrm{Z} + \mathrm{MH} + \mathrm{NI} + \Xi\mathrm{I}\Gamma).$$

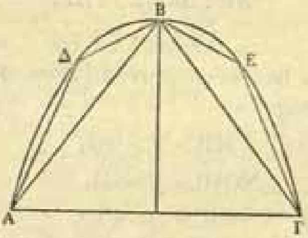
Again, since $\mathrm{Z}\Phi < \mathrm{X}$, $\Theta\mathrm{H} < \Psi$, $\mathrm{I}\Pi < \Omega$, $\mathrm{IO}\Gamma < \underset{s}{\Delta}$, it is

τὰ εἰρημένα ἐλάσσονά ἐστι τοῦ ΔΩΨΧ χωρίου· φανερὸν οὖν, ὅτι καὶ τὸ ΒΔΓ τρίγωνον μεῖζόν ἐστιν ἢ τριπλάσιον τῶν ΦΖ, ΘΗ, ΙΠ τραπεζίων καὶ τοῦ ΙΓΟ τριγώνου, ἔλασσον δὲ ἢ τριπλάσιον τῶν προγεγραμμένων.

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Πᾶν τμᾶμα τὸ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς ἐπίτριτόν ἐστι τριγώνου τοῦ τὰν αὐτὰν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον.

Ἔστω γὰρ τὸ ΑΔΒΕΓ τμᾶμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς, τὸ δὲ ΑΒΓ τρίγωνον ἔστω τὰν αὐτὰν βάσιν ἔχον τῷ τμάματι



καὶ ὕψος ἴσον, τοῦ δὲ ΑΒΓ τριγώνου ἔστω ἐπίτριτον τὸ Κ χωρίον. δεικτέον, ὅτι ἴσον ἐστὶ τῷ ΑΔΒΕΓ τμάματι.

Εἰ γὰρ μή ἐστιν ἴσον, ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω πρότερον, εἰ δυνατόν, μεῖζον τὸ ΑΔΒΕΓ τμᾶμα τοῦ Κ χωρίου. ἐνέγραψα δὴ τὰ ΑΔΒ, ΒΕΓ τρίγωνα, ὡς εἴρηται, ἐνέγραψα δὲ καὶ εἰς τὰ περιλειπόμενα τμάματα ἄλλα τρίγωνα τὰν αὐτὰν

clear that the sum of the aforesaid areas is greater than the area $\underset{s}{\Delta} + \Omega + \Psi + X$;

it is therefore manifest that

$$B\Delta\Gamma > 3(\Phi Z + \Theta H + I\Pi + I\Gamma O),^{a}$$

but is less than thrice the aforementioned areas.[b]

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

For let AΔBEΓ be a segment bounded by a straight line and a section of a right-angled cone, and let ABΓ be a triangle having the same base as the segment and equal height, and let the area K be four-thirds of the triangle ABΓ. It is required to prove that it is equal to the segment AΔBEΓ.

For if it is not equal, it is either greater or less. Let the segment AΔBEΓ first be, if possible, greater than the area K. Now I have inscribed the triangles AΔB, BEΓ, as aforesaid,[c] and I have inscribed in the remaining segments other triangles having the same

[a] For $B\Delta\Gamma = 3(P + X + \Psi + \Omega + \underset{s}{\Delta}) > 3(\underset{s}{\Delta} + \Omega + \Psi + X)$.

[b] In Prop. 15 Archimedes shows that the same theorem holds good even if BΓ is not at right angles to the axis. It is then proved in Prop. 16, by the method of exhaustion, that the segment is equal to one-third of the triangle BΓΔ. This is done by showing, on the basis of the "Axiom of Archimedes," that by taking enough parts the difference between the circumscribed and the inscribed figures can be made as small as we please. It is equivalent to integration. From this it is easily proved that the segment is equal to four-thirds of a triangle with the same base and equal height (Prop. 17).

[c] In earlier propositions Archimedes has used the same procedure as he now describes. Δ, E are the points in which the diameter through the mid-points of AB, BΓ meet the curve.

βάσιν ἔχοντα τοῖς τμαμάτεσσιν καὶ ὕψος τὸ αὐτό, καὶ ἀεὶ εἰς τὰ ὕστερον γινόμενα τμάματα ἐγγράφω δύο τρίγωνα τὰν αὐτὰν βάσιν ἔχοντα τοῖς τμαμάτεσσιν καὶ ὕψος τὸ αὐτό· ἐσσοῦνται δὴ τὰ καταλειπόμενα τμάματα ἐλάσσονα τᾶς ὑπεροχᾶς, ᾇ ὑπερέχει τὸ ΑΔΒΕΓ τμᾶμα τοῦ Κ χωρίου. ὥστε τὸ ἐγγραφόμενον πολύγωνον μεῖζον ἐσσεῖται τοῦ Κ· ὅπερ ἀδύνατον. ἐπεὶ γάρ ἐστιν ἑξῆς κείμενα χωρία ἐν τῷ τετραπλασίονι λόγῳ, πρῶτον μὲν τὸ ΑΒΓ τρίγωνον τετραπλάσιον τῶν ΑΔΒ, ΒΕΓ τριγώνων, ἔπειτα δὲ αὐτὰ ταῦτα τετραπλάσια τῶν εἰς τὰ ἑπόμενα τμάματα ἐγγραφέντων καὶ ἀεὶ οὕτω, δῆλον, ὡς σύμπαντα τὰ χωρία ἐλάσσονά ἐστιν ἢ ἐπίτριτά τοῦ μεγίστου, τὸ δὲ Κ ἐπίτριτόν ἐστι τοῦ μεγίστου χωρίου. οὐκ ἄρα ἐστὶν μεῖζον τὸ ΑΔΒΕΓ τμᾶμα τοῦ Κ χωρίου.

Ἔστω δέ, εἰ δυνατόν, ἔλασσον. κείσθω δὴ τὸ μὲν ΑΒΓ τρίγωνον ἴσον τῷ Ζ, τοῦ δὲ Ζ τέταρτον τὸ Η, καὶ ὁμοίως τοῦ Η τὸ Θ, καὶ ἀεὶ ἑξῆς τιθέσθω, ἕως κα γένηται τὸ ἔσχατον ἔλασσον τᾶς

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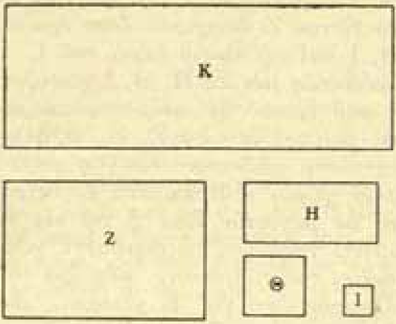
[a] This was proved geometrically in Prop. 23, and is proved generally in Eucl. ix. 35. It is equivalent to the summation

$$1+\left(\tfrac{1}{4}\right)+\left(\tfrac{1}{4}\right)^2+\ldots\left(\tfrac{1}{4}\right)^{n-1}=\tfrac{4}{3}-\tfrac{1}{3}\left(\tfrac{1}{4}\right)^{n-1}$$
$$=\frac{1-\left(\tfrac{1}{4}\right)^n}{1-\tfrac{1}{4}}.$$

base as the segments and equal height, and so on continually I inscribe in the resulting segments two



triangles having the same base as the segments and equal height; then there will be left [at some time] segments less than the excess by which the segment ΑΔΒΕΓ exceeds the area K [Prop. 20, coroll.]. Therefore the inscribed polygon will be greater than K; which is impossible. For since the areas successively formed are each four times as great as the next, the triangle ΑΒΓ being four times the triangles ΑΔΒ, ΒΕΓ [Prop. 21], then these last triangles four times the triangles inscribed in the succeeding segments, and so on continually, it is clear that the sum of all the areas is less than four-thirds of the greatest [Prop. 23],[a] and K is equal to four-thirds of the greatest area. Therefore the segment ΑΔΒΕΓ is not greater than the area K.

Now let it be, if possible, less. Then let

$$Z = AB\Gamma,\ H = \tfrac{1}{4}Z,\ \Theta = \tfrac{1}{4}H,$$

and so on continually, until the last [area] is less than

ὑπεροχᾶς, ᾇ ὑπερέχει τὸ Κ χωρίον τοῦ τμάματος, καὶ ἔστω ἔλασσον τὸ Ι. ἔστιν δὴ τὰ Ζ, Η, Θ, Ι χωρία καὶ τὸ τρίτον τοῦ Ι ἐπίτριτα τοῦ Ζ. ἔστιν δὲ καὶ τὸ Κ τοῦ Ζ ἐπίτριτον· ἴσον ἄρα τὸ Κ τοῖς Ζ, Η, Θ, Ι καὶ τῷ τρίτῳ μέρει τοῦ Ι. ἐπεὶ οὖν τὸ Κ χωρίον τῶν μὲν Ζ, Η, Θ, Ι χωρίων ὑπερέχει ἐλάσσονι τοῦ Ι, τοῦ δὲ τμάματος μείζονι τοῦ Ι, δῆλον, ὡς μείζονά ἐντι τὰ Ζ, Η, Θ, Ι χωρία τοῦ τμάματος· ὅπερ ἀδύνατον· ἐδείχθη γάρ, ὅτι, ἐὰν ᾖ ὁποσαοῦν χωρία ἑξῆς κείμενα ἐν τετραπλασίονι λόγῳ, τὸ δὲ μέγιστον ἴσον ᾖ τῷ εἰς τὸ τμᾶμα ἐγγραφομένῳ τριγώνῳ, τὰ σύμπαντα χωρία ἐλάσσονα ἐσσεῖται τοῦ τμάματος. οὐκ ἄρα τὸ ΑΔΒΕΓ τμᾶμα ἔλασσόν ἐστι τοῦ Κ χωρίου. ἐδείχθη δέ, ὅτι οὐδὲ μεῖζον· ἴσον ἄρα ἐστὶν τῷ Κ. τὸ δὲ Κ χωρίον ἐπίτριτόν ἐστι τοῦ τριγώνου τοῦ ΑΒΓ· καὶ τὸ ΑΔΒΕΓ ἄρα τμᾶμα ἐπίτριτόν ἐστι τοῦ ΑΒΓ τριγώνου.

(*k*) Hydrostatics

(i.) *Postulates*

Archim. *De Corpor. Fluit.* i., Archim. ed. Heiberg ii. 318. 2-8

Ὑποκείσθω τὸ ὑγρὸν φύσιν ἔχον τοιαύταν, ὥστε τῶν μερέων αὐτοῦ τῶν ἐξ ἴσου κειμένων καὶ συν-

[a] The Greek text of the book *On Floating Bodies*, the earliest extant treatise on hydrostatics, first became available in 1906 when Heiberg discovered at Constantinople the ms. which he terms C. Unfortunately many of the readings are doubtful, and those who are interested in the text should consult the Teubner edition. Still more unfortunately, it is incomplete; but, as the whole treatise was translated into Latin in 1269 by William of Moerbeke from a Greek ms.

the excess by which the area K exceeds the segment [Eucl. x. 1], and let I be [the area] less [than this excess]. Now

$$Z + H + \Theta + I + \tfrac{1}{3}I = \tfrac{4}{3}Z. \qquad \text{[Prop. 23}$$

But $$K = \tfrac{4}{3}Z;$$

therefore $$K = Z + H + \Theta + I + \tfrac{1}{3}I.$$

Therefore since the area K exceeds the areas Z, H, Θ, I by an excess less than I, and exceeds the segment by an excess greater than I, it is clear that the areas Z, H, Θ, I are greater than the segment; which is impossible; for it was proved that, if there be any number of areas in succession such that each is four times the next, and the greatest be equal to the triangle inscribed in the segment, then the sum of the areas will be less than the segment [Prop. 22]. Therefore the segment AΔBEΓ is not less than the area K. And it was proved not to be greater; therefore it is equal to K. But the area K is four-thirds of the triangle ABΓ; and therefore the segment AΔBEΓ is four-thirds of the triangle ABΓ.

(*k*) Hydrostatics

(i.) *Postulates*

Archimedes, *On Floating Bodies* [a] i., Archim. ed. Heiberg ii. 318. 2-8

Let the nature of a fluid be assumed to be such that, of its parts which lie evenly and are continuous,

since lost, it is possible to supply the missing parts in Latin, as is done for part of Prop. 2. From a comparison with the Greek, where it survives, William's translation is seen to be so literal as to be virtually equivalent to the original. In each case Heiberg's figures are taken from William's translation, as they are almost unrecognizable in C; for convenience in reading the Greek, the figures are given the appropriate Greek letters in this edition.

εχέων ἐόντων ἐξωθεῖσθαι τὸ ἧσσον θλιβόμενον ὑπὸ τοῦ μᾶλλον θλιβομένου, καὶ ἕκαστον δὲ τῶν μερέων αὐτοῦ θλίβεσθαι τῷ ὑπεράνω αὐτοῦ ὑγρῷ κατὰ κάθετον ἐόντι, εἴ κα μὴ τὸ ὑγρὸν ᾖ καθειργμένον ἔν τινι καὶ ὑπὸ ἄλλου τινὸς θλιβόμενον.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

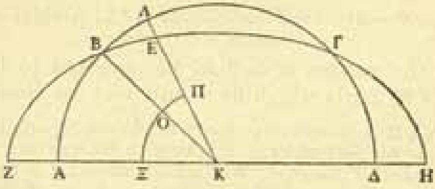
Ὑποκείσθω, τῶν ἐν τῷ ὑγρῷ ἄνω φερομένων ἕκαστον ἀναφέρεσθαι κατὰ τὰν κάθετον τὰν διὰ τοῦ κέντρου τοῦ βάρεος αὐτοῦ ἀγμέναν.

(ii.) *Surface of Fluid at Rest*

Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7–320. 30

Omnis humidi consistentis ita, ut maneat inmotum, superficies habebit figuram sperae habentis centrum idem cum terra.

Intelligatur enim humidum consistens ita, ut maneat non motum, et secetur ipsius superficies plano per centrum terrae, sit autem terrae centrum K, superficiei autem sectio linea ABGD. Dico itaque,



lineam ABGD circuli esse periferiam, centrum autem ipsius K.

Si enim non est, rectae a K ad lineam ABGD

that which is under the lesser pressure is driven along by that under the greater pressure, and each of its parts is under pressure from the fluid which is perpendicularly above it, except when the fluid is enclosed in something and is under pressure from something else.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

Let it be assumed that, of bodies which are borne upwards in a fluid, each is borne upwards along the perpendicular drawn through its centre of gravity.[a]

(ii.) *Surface of Fluid at Rest*

Ibid. i., Prop. 2,[b] Archim. ed. Heiberg ii. 319. 7–320. 30

The surface of any fluid at rest is the surface of a sphere having the same centre as the earth.

For let there be conceived a fluid at rest, and let its surface be cut by a plane through the centre of the earth, and let the centre of the earth be K, and let the section of the surface be the curve ΑΒΓΔ. Then I say that the curve ΑΒΓΔ is an arc of a circle whose centre is K.

For if it is not, straight lines drawn from K to the

[a] These are the only assumptions, other than the assumptions of Euclidean geometry, made in this book by Archimedes; if the object of mathematics be to base the conclusions on the fewest and most "self-evident" axioms, Archimedes' treatise *On Floating Bodies* must indeed be ranked highly.

[b] The earlier part of this proposition has to be given from William of Moerbeke's translation. The diagram is here given with the appropriate Greek letters.

occurrentes non erunt aequales. Sumatur itaque aliqua recta, quae est quarundam quidem a K occurrentium ad lineam ABGD maior, quarundam autem minor, et centro quidem K, distantia autem sumptae lineae circulus describatur; cadet igitur periferia circuli habens hoc quidem extra lineam ABGD, hoc autem intra, quoniam quae ex centro quarundam quidem a K occurrentium ad lineam ABGD est maior, quarundam autem minor. Sit igitur descripti circuli periferia quae ZBH, et a B ad K recta ducatur, et copulentur quae ZK, KEL aequales facientes angulos, describatur autem et centro K periferia quaedam quae XOP in plano et in humido; partes itaque humidi quae secundum XOP periferiam ex aequo sunt positae et continuae inuicem. Et premuntur quae quidem secundum XO periferiam humido quod secundum ZB locum, quae autem secundum periferiam OP humido quod secundum BE locum; inaequaliter igitur premuntur partes humidi quae secundum periferiam XO ei quae [ᾇ][1] κατὰ τὰν ΟΠ· ὥστε ἐξωθήσονται τὰ ἧσσον θλιβόμενα ὑπὸ τῶν μᾶλλον θλιβομένων· οὐ μένει ἄρα τὸ ὑγρόν. ὑπέκειτο δὲ καθεστακὸς εἶμεν ὥστε μένειν ἀκίνητον· ἀναγκαῖον ἄρα τὰν ΑΒΓΔ γραμμὰν κύκλου περιφέρειαν εἶμεν καὶ κέντρον αὐτᾶς τὸ Κ. ὁμοίως δὴ δειχθήσεται καί, ὅπως κα ἄλλως ἁ ἐπιφάνεια τοῦ ὑγροῦ ἐπιπέδῳ τμαθῇ διὰ τοῦ κέντρου τᾶς γᾶς, ὅτι ἁ τομὰ ἐσσεῖται κύκλου περιφέρεια, καὶ κέντρον αὐτᾶς ἐσσεῖται, ὃ καὶ τᾶς γᾶς ἐστι κέντρον. δῆλον οὖν, ὅτι ἁ ἐπιφάνεια τοῦ ὑγροῦ καθεστακότος ἀκινήτου σφαίρας ἔχει τὸ σχῆμα τὸ αὐτὸ κέντρον ἐχούσας τᾷ γᾷ, ἐπειδὴ

[1] ᾇ om. Heiberg.

curve ΑΒΓΔ will not be equal. Let there be taken, therefore, any straight line which is greater than some of the straight lines drawn from K to the curve ΑΒΓΔ, but less than others, and with centre K and radius equal to the straight line so taken let a circle be described; the circumference of the circle will fall partly outside the curve ΑΒΓΔ, partly inside, inasmuch as its radii are greater than some of the straight lines drawn from K to the curve ΑΒΓΔ, but less than others. Let the arc of the circle so described be ΖΒΗ, and from B let a straight line be drawn to K, and let ZK, ΚΕΛ be drawn making equal angles [with KB], and with centre K let there be described, in the plane and in the fluid, an arc ΞΟΠ; then the parts of the fluid along ΞΟΠ lie evenly and are continuous [*v. supra*, p. 243]. And the parts along the arc ΞΟ are under pressure from the portion of the fluid between it and ZB, while the parts along the arc ΟΠ are under pressure from the portion of the fluid between it and BE; therefore the parts of the fluid along ΞΟ and the parts of the fluid along ΟΠ are under unequal pressures; so that the parts under the lesser pressure are thrust along by the parts under the greater pressure [*v. supra*, p. 245]; therefore the fluid will not remain at rest. But it was postulated that the fluid would remain unmoved; therefore the curve ΑΒΓΔ must be an arc of a circle with centre K. Similarly it may be shown that, in whatever other manner the surface be cut by a plane through the centre of the earth, the section is an arc of a circle and its centre will also be the centre of the earth. It is therefore clear that the surface of the fluid remaining at rest has the form of a sphere with the same centre as the earth, since it is such

τοιαύτα ἐστίν, ὥστε διὰ τοῦ αὐτοῦ σαμείου τμαθεῖσαν τὰν τομὰν ποιεῖν περιφέρειαν κύκλου κέντρον ἔχοντος τὸ σαμεῖον, δι' οὗ τέμνεται τῷ ἐπιπέδῳ.

(iii.) *Solid immersed in a Fluid*

Ibid. i., Prop. 7, Archim. ed. Heiberg ii. 332. 21–336. 13

Τὰ βαρύτερα τοῦ ὑγροῦ ἀφεθέντα εἰς τὸ ὑγρὸν οἰσεῖται κάτω, ἔστ' ἂν καταβᾶντι, καὶ ἐσσοῦνται κουφότερα ἐν τῷ ὑγρῷ τοσοῦτον, ὅσον ἔχει τὸ βάρος τοῦ ὑγροῦ τοῦ ταλικοῦτον ὄγκον ἔχοντος, ἁλίκος ἐστὶν ὁ τοῦ στερεοῦ μεγέθεος ὄγκος.

Ὅτι μὲν οὖν οἰσεῖται ἐς τὸ κάτω, ἔστ' ἂν καταβᾶντι, δῆλον· τὰ γὰρ ὑποκάτω αὐτοῦ μέρεα τοῦ ὑγροῦ θλιβησοῦνται μᾶλλον τῶν ἐξ ἴσου αὐτοῖς κειμένων μερέων, ἐπειδὴ βαρύτερον ὑπόκειται τὸ στερεὸν μέγεθος τοῦ ὑγροῦ· ὅτι δὲ κουφότερα ἐσσοῦνται, ὡς εἴρηται, δειχθήσεται.

Ἔστω τι μέγεθος τὸ Α, ὃ ἐστι βαρύτερον τοῦ ὑγροῦ, βάρος δὲ ἔστω τοῦ μὲν ἐν ᾧ Α μεγέθεος τὸ ΒΓ, τοῦ δὲ ὑγροῦ τοῦ ἴσον ὄγκον ἔχοντος τῷ Α τὸ Β. δεικτέον, ὅτι τὸ Α μέγεθος ἐν τῷ ὑγρῷ ἐὸν βάρος ἕξει ἴσον τῷ Γ.

Λελάφθω γάρ τι μέγεθος τὸ ἐν ᾧ τὸ Δ κουφότερον τοῦ ὑγροῦ τοῦ ἴσον ὄγκον ἔχοντος αὐτῷ, ἔστω δὲ τοῦ μὲν ἐν ᾧ τὸ Δ μεγέθεος βάρος ἴσον τῷ Β βάρει, τοῦ δὲ ὑγροῦ τοῦ ἴσον ὄγκον ἔχοντος τῷ Δ μεγέθει τὸ βάρος ἔστω ἴσον τῷ ΒΓ βάρει.

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[a] Or, as we should say, "lighter by the weight of fluid displaced."

that, when it is cut [by a plane] always passing through the same point, the section is an arc of a circle having for centre the point through which it is cut by the plane [Prop. 1].

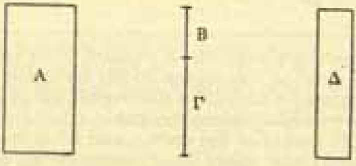
(iii.) *Solid immersed in a Fluid*

Ibid. i., Prop. 7, Archim. ed. Heiberg ii. 332. 21–336. 13

Solids heavier than a fluid will, if placed in the fluid, sink to the bottom, and they will be lighter [if weighed] in the fluid by the weight of a volume of the fluid equal to the volume of the solid.[a]

That they will sink to the bottom is manifest; for the parts of the fluid under them are under greater pressure than the parts lying evenly with them, since it is postulated that the solid is heavier than water; that they will be lighter, as aforesaid will be [thus] proved.

Let A be any magnitude heavier than the fluid, let the weight of the magnitude A be B + Γ, and let the weight of fluid having the same volume as A be B. It is required to prove that in the fluid the magnitude A will have a weight equal to Γ.



For let there be taken any magnitude Δ lighter than the same volume of the fluid such that the weight of the magnitude Δ is equal to the weight B, while the weight of the fluid having the same volume as the magnitude Δ is equal to the weight B + Γ.

συντεθέντων δὴ ἐς τὸ αὐτὸ τῶν μεγεθέων, ἐν οἷς τὰ Α, Δ, τὸ τῶν συναμφοτέρων μέγεθος ἰσοβαρὲς ἐσσεῖται τῷ ὑγρῷ· ἔστι γὰρ τῶν μεγεθέων συναμφοτέρων τὸ βάρος ἴσον συναμφοτέροις τοῖς βάρεσιν τῷ τε ΒΓ καὶ τῷ Β, τοῦ δὲ ὑγροῦ τοῦ ἴσον ὄγκον ἔχοντος ἀμφοτέροις τοῖς μεγέθεσι τὸ βάρος ἴσον ἐστὶ τοῖς αὐτοῖς βάρεσιν. ἀφεθέντων οὖν τῶν μεγεθέων ἐς τὸ ὑγρὸν ἰσορροπησοῦνται τῷ ὑγρῷ καὶ οὔτε εἰς τὸ ἄνω οἰσοῦνται οὔτε εἰς τὸ κάτω· διὰ τὸ μὲν ἐν ᾧ Α μέγεθος οἰσεῖται ἐς τὸ κάτω καὶ τοσαύτᾳ βίᾳ ὑπὸ τοῦ ἐν ᾧ Δ μεγέθεος ἀνέλκεται ἐς τὸ ἄνω, τὸ δὲ ἐν ᾧ Δ μέγεθος, ἐπεὶ κουφότερόν ἐστι τοῦ ὑγροῦ, ἀνοισεῖται εἰς τὸ ἄνω τοσαύτα βίᾳ, ὅσον ἐστὶ τὸ Γ βάρος· δέδεικται γάρ, ὅτι τὰ κουφότερα τοῦ ὑγροῦ μεγέθεα στερεὰ βιασθέντα ἐς τὸ ὑγρὸν ἀναφέρονται τοσαύτᾳ βίᾳ ἐς τὸ ἄνω, ὅσον ἐστὶ τὸ βάρος, ᾧ βαρύτερόν ἐστι τοῦ μεγέθεος τὸ ὑγρὸν τὸ ἴσογκον τῷ μεγέθει. ἔστι δὲ τῷ Γ βάρει βαρύτερον τοῦ Δ μεγέθεος τὸ ὑγρὸν τὸ ἴσον ὄγκον ἔχον τῷ Δ· δῆλον οὖν, ὅτι καὶ τὸ ἐν ᾧ Α μέγεθος ἐς τὸ κάτω οἰσεῖται τοσούτῳ βάρει, ὅσον ἐστὶ τὸ Γ.

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[a] This proposition suggests a method, alternative to that given by Vitruvius (v. *supra*, pp. 36-39, especially p. 38 n. *a*), whereby Archimedes may have discovered the proportions of gold and silver in King Hiero's crown.

Let w be the weight of the crown, and let w_1 and w_2 be the weights of gold and silver in it respectively, so that $w = w_1 + w_2$.

Take a weight w of gold and weigh it in a fluid, and let the loss of weight be P_1. Then the loss of weight when a weight w_1 of gold is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{w_1}{w} \cdot P_1$.

Then if we combine the magnitudes A, Δ, the combined magnitude will be equal to the weight of the same volume of the fluid; for the weight of the combined magnitudes is equal to the weight (B + Γ) + B, while the weight of the fluid having the same volume as both the magnitudes is equal to the same weight. Therefore, if the [combined] magnitudes are placed in the fluid, they will balance the fluid, and will move neither upwards nor downwards [Prop. 3]; for this reason the magnitude A will move downwards, and will be subject to the same force as that by which the magnitude Δ is thrust upwards, and since Δ is lighter than the fluid it will be thrust upwards by a force equal to the weight Γ; for it has been proved that when solid magnitudes lighter than the fluid are forcibly immersed in the fluid, they will be thrust upwards by a force equal to the difference in weight between the magnitude and an equal volume of the fluid [Prop. 6]. But the fluid having the same volume as Δ is heavier than the magnitude Δ by the weight Γ; it is therefore plain that the magnitude A will be borne upwards by a force equal to Γ.[a]

Now take a weight w of silver and weigh it in the fluid, and let the loss of weight be P_2. Then the loss of weight when a weight w_2 of silver is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{w_2}{w} \cdot P_2$.

Finally, weigh the crown itself in the fluid, and let the loss of weight, and consequently the weight of fluid displaced, be P.

It follows that $$\frac{w_1}{w} \cdot P_1 + \frac{w_2}{w} \cdot P_2 = P,$$

whence $$\frac{w_1}{w_2} = \frac{P_2 - P}{P - P_1}.$$

(iv.) *Stability of a Paraboloid of Revolution*

Ibid. ii., Prop. 2, Archim. ed. Heiberg ii. 348. 10–352. 19

Τὸ ὀρθὸν τμᾶμα τοῦ ὀρθογωνίου κωνοειδέος, ὅταν τὸν ἄξονα ἔχῃ μὴ μείζονα ἢ ἡμιόλιον τᾶς μέχρι τοῦ ἄξονος, πάντα λόγον ἔχον ποτὶ τὸ ὑγρὸν τῷ βάρει, ἀφεθὲν εἰς τὸ ὑγρὸν οὕτως, ὥστε τὰν βάσιν αὐτοῦ μὴ ἅπτεσθαι τοῦ ὑγροῦ, τεθὲν κεκλιμένον οὐ μενεῖ κεκλιμένον, ἀλλὰ ἀποκαταστασεῖται ὀρθόν. ὀρθὸν δὲ λέγω καθεστακέναι τὸ τοιοῦτο τμᾶμα, ὁπόταν τὸ ἀποτετμακὸς αὐτὸ ἐπίπεδον παρὰ τὰν ἐπιφάνειαν ᾖ τοῦ ὑγροῦ.

Ἔστω τμᾶμα ὀρθογωνίου κωνοειδέος, οἷον εἴρηται, καὶ κείσθω κεκλιμένον. δεικτέον, ὅτι οὐ μενεῖ, ἀλλ' ἀποκαταστασεῖται ὀρθόν.

Τμαθέντος δὴ αὐτοῦ ἐπιπέδῳ διὰ τοῦ ἄξονος ὀρθῷ ποτὶ τὸ ἐπίπεδον τὸ ἐπὶ τᾶς ἐπιφανείας τοῦ ὑγροῦ τμάματος ἔστω τομὰ ἁ ΑΠΟΛ ὀρθογωνίου κώνου τομά, ἄξων δὲ τοῦ τμάματος καὶ διάμετρος τᾶς τομᾶς ἁ ΝΟ, τᾶς δὲ τοῦ ὑγροῦ ἐπιφανείας τομὰ ἁ ΙΣ. ἐπεὶ οὖν τὸ τμᾶμα οὐκ ἔστιν ὀρθόν, οὐκ ἂν εἴη παράλληλος ἁ ΑΛ τᾷ ΙΣ· ὥστε οὐ ποιήσει ὀρθὰν γωνίαν ἁ ΝΟ ποτὶ τὰν ΙΣ. ἄχθω

[a] Writing of the treatise *On Floating Bodies*, Heath (*H.G.M.* ii. 94-95) justly says: "Book ii., which investigates fully the conditions of stability of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and the principal parameter of the generating parabola, is a veritable *tour de force* which must be read in full to be appreciated."

[b] In this technical term the "axis" is the axis of the

(iv.) *Stability of a Paraboloid of Revolution* [a]

Ibid. ii., Prop. 2, Archim. ed. Heiberg ii. 348. 10–352. 19

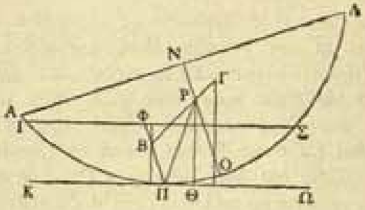
If there be a right segment of a right-angled conoid, whose axis is not greater than one-and-a-half times the line drawn as far as the axis,[b] *and whose weight relative to the fluid may have any ratio, and if it be placed in the fluid in an inclined position in such a manner that its base do not touch the fluid, it will not remain inclined but will return to the upright position.* I mean by returning to the upright position the figure formed when the plane cutting off the segment is parallel to the surface of the fluid.

Let there be a segment of a right-angled conoid, such as has been stated, and let it be placed in an inclined position. It is required to prove that it will not remain there but will return to the upright position.

Let the segment be cut by a plane through the axis perpendicular to the plane which forms the surface of the fluid, and let ΑΠΟΛ be the section of the segment, being a section of a right-angled cone [*De Con. et Sphaer.* 11], and let ΝΟ be the axis of the segment and the axis of the section, and let ΙΣ be the section of the surface of the liquid. Then since the segment is not upright, ΑΛ will not be parallel to ΙΣ; and therefore ΝΟ will not make a right angle

right-angled cone from which the generating parabola is derived. The *latus rectum* is "the line which is double of the line drawn as far as the axis" (ἁ διπλασία τᾶς μέχρι τοῦ ἄξονος): and so the condition laid down by Archimedes is that the axis of the segment of the paraboloid of revolution shall not be greater than three-quarters of the *latus rectum* or *principal parameter* of the generating parabola.

οὖν παράλληλος ἁ ἐφαπτομένα ἁ ΚΩ τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Π, καὶ ἀπὸ τοῦ Π παρὰ τὰν



ΝΟ ἄχθω ἁ ΠΦ· τέμνει δὴ ἁ ΠΦ δίχα τὰν ΙΣ· δέδεικται γὰρ ἐν τοῖς κωνικοῖς. τετμάσθω ἁ ΠΦ, ὥστε εἶμεν διπλασίαν τὰν ΠΒ τᾶς ΒΦ, καὶ ἁ ΝΟ κατὰ τὸ Ρ τετμάσθω, ὥστε καὶ τὰν ΟΡ τᾶς ΡΝ διπλασίαν εἶμεν· ἐσσεῖται δὴ τοῦ μείζονος ἀποτμάματος τοῦ στερεοῦ κέντρον τοῦ βάρεος τὸ Ρ, τοῦ δὲ κατὰ τὰν ΙΠΟΣ τὸ Β· δέδεικται γὰρ ἐν ταῖς Ἰσορροπίαις, ὅτι παντὸς ὀρθογωνίου κωνοειδέος τμάματος τὸ κέντρον τοῦ βάρεός ἐστιν ἐπὶ τοῦ ἄξονος διῃρημένου οὕτως, ὥστε τὸ ποτὶ τᾷ κορυφᾷ τοῦ ἄξονος τμᾶμα διπλάσιον εἶμεν τοῦ λοιποῦ. ἀφαιρεθέντος δὴ τοῦ κατὰ τὰν ΙΠΟΣ τμάματος στερεοῦ ἀπὸ τοῦ ὅλου τοῦ λοιποῦ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τᾶς ΒΓ εὐθείας· δέδεικται γὰρ τοῦτο ἐν τοῖς Στοιχείοις τῶν μηχανικῶν, ὅτι, εἴ κα μέγεθος ἀφαιρεθῇ μὴ τὸ αὐτὸ κέντρον ἔχον τοῦ βάρεος τῷ ὅλῳ μεγέθει, τοῦ λοιποῦ τὸ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τᾶς εὐθείας τᾶς ἐπιζευγνυούσας τὰ κέντρα τοῦ τε ὅλου μεγέθεος καὶ τοῦ

with IΣ. Therefore let KΩ be drawn parallel [to IΣ] and touching the section of the cone at Π, and from Π let ΠΦ be drawn parallel to NO; then ΠΦ bisects IΣ—for this is proved in the [*Elements of*] *Conics*.[a] Let ΠΦ be cut so that ΠB = 2BΦ, and let NO be cut at P so that OP = 2PN; then P will be the centre of gravity of the greater segment of the solid, and B that of IΠOΣ; for it is proved in the books *On Equilibriums* that the centre of gravity of any segment of a right-angled conoid is at the point dividing the axis in such a manner that the segment towards the vertex of the axis is double of the remainder.[b] Now if the solid segment IΠOΣ be taken away from the whole, the centre of gravity of the remainder will lie upon the straight line BΓ; for it has been proved in the *Elements of Mechanics* that if any magnitude be taken away not having the same centre of gravity as the whole magnitude, the centre of gravity of the remainder will be on the straight line joining the centres [of gravity] of the whole magnitude and of the part

[a] Presumably in the works of Aristaeus or Euclid, but it is also *Quad. Parab.* 1.

[b] The proof is not in any extant work by Archimedes.

ἀφῃρημένου ἐκβεβλημένας ἐπὶ τὰ αὐτά, ἐφ' ἃ τὸ κέντρον τοῦ ὅλου μεγέθεός ἐστιν. ἐκβεβλήσθω δὴ ἁ ΒΡ ἐπὶ τὸ Γ, καὶ ἔστω τὸ Γ τὸ κέντρον τοῦ βάρεος τοῦ λοιποῦ μεγέθεος. ἐπεὶ οὖν ἁ ΝΟ τᾶς μὲν ΟΡ ἡμιολία, τᾶς δὲ μέχρι τοῦ ἄξονος οὐ μείζων ἢ ἡμιολία, δῆλον, ὅτι ἁ ΡΟ τᾶς μέχρι τοῦ ἄξονος οὐκ ἐστὶ μείζων· ἡ ΠΡ ἄρα ποτὶ τὰν ΚΩ γωνίας ἀνίσους ποιεῖ, καὶ ἁ ὑπὸ τῶν ΡΠΩ γίνεται ὀξεῖα· ἁ ἀπὸ τοῦ Ρ ἄρα κάθετος ἐπὶ τὰν ΠΩ ἀγομένα μεταξὺ πεσεῖται τῶν Π, Ω. πιπτέτω ὡς ἁ ΡΘ· ἁ ΡΘ ἄρα ὀρθά ἐστιν ποτὶ τὸ ⟨ἀποτετμακὸς⟩[1] ἐπίπεδον, ἐν ᾧ ἐστιν ἁ ΣΙ, ὅ ἐστιν ἐπὶ τᾶς ἐπιφανείας τοῦ ὑγροῦ. ἄχθωσαν δή τινες ἀπὸ τῶν Β, Γ παρὰ τὰν ΡΘ· ἐνεχθήσεται δὴ τὸ μὲν ἐκτὸς τοῦ ὑγροῦ τοῦ μεγέθεος εἰς τὸ κάτω κατὰ τὰν διὰ τοῦ Γ ἀγομέναν κάθετον· ὑπόκειται γὰρ ἕκαστον τῶν βαρέων εἰς τὸ κάτω φέρεσθαι κατὰ τὰν κάθετον τὰν διὰ τοῦ κέντρου ἀγομέναν· τὸ δὲ ἐν τῷ ὑγρῷ μέγεθος, ἐπεὶ κουφότερον γίνεται τοῦ ὑγροῦ, ἐνεχθήσεται εἰς τὸ ἄνω κατὰ τὰν κάθετον τὰν διὰ τοῦ Β ἀγομέναν. ἐπεὶ δὲ οὐ κατὰ τὰν αὐτὰν κάθετον ἀλλάλοις ἀντιθλίβονται, οὐ μενεῖ τὸ σχῆμα, ἀλλὰ τὰ μὲν κατὰ τὸ Α εἰς τὸ ἄνω ἐνεχθήσεται, τὰ δὲ κατὰ τὸ Λ εἰς τὸ κάτω, καὶ τοῦτο ἀεὶ ἐσσεῖται, ἕως ἂν ὀρθὸν ἀποκατασταθῇ.

[1] ἀποτετμακὸς, *cf. supra*, p. 252 line 8; Heiberg prints . . . . κ . . ος.

[a] If the normal at Π meets the axis in M, then OM is greater than "the line drawn as far as the axis" except in the case where Π coincides with the vertex, which case is excluded by the conditions of this proposition. Hence OM is always greater than OP; and because the angle ΩΠΜ is right, the angle ΩΠΡ must be acute.

taken away, produced from the extremity which is the centre of gravity of the whole magnitude [*De Plan. Aequil.* i. 8]. Let BP then be produced to Γ, and let Γ be the centre of gravity of the remaining magnitude. Then, since $NO = \frac{2}{3} \cdot OP$, and $NO >$ $\frac{2}{3} \cdot$ (the line drawn as far as the axis), it is clear that $PO >$ (the line drawn as far as the axis); therefore ΠP makes unequal angles with KΩ, and the angle PΠΩ is acute [a]; therefore the perpendicular drawn from P to ΠΩ will fall between Π, Ω. Let it fall as PΘ; then PΘ is perpendicular to the cutting plane containing ΣI, which is on the surface of the fluid. Now let lines be drawn from B, Γ parallel to PΘ; then the portion of the magnitude outside the fluid will be subject to a downward force along the line drawn through Γ—for it is postulated that each weight is subject to a downward force along the perpendicular drawn through its centre of gravity [b]; and since the magnitude in the fluid is lighter than the fluid,[c] it will be subject to an upward force along the perpendicular drawn through B.[d] But, since they are not subject to contrary forces along the same perpendicular, the figure will not remain at rest but the portion on the side of A will move upwards and the portion on the side of Λ will move downwards, and this will go on continually until it is restored to the upright position.

[b] *Cf. supra*, p. 245; possibly a similar assumption to this effect has fallen out of the text.

[c] A tacit assumption, which limits the generality of the opening statement of the proposition that the segment may have any weight relative to the fluid.

[d] *v. supra*, p. 251.

XVIII. ERATOSTHENES

XVIII. ERATOSTHENES

(a) General

Suidas, *s.v.* Ἐρατοσθένης

Ἐρατοσθένης, Ἀγλαοῦ, οἱ δὲ Ἀμβροσίου· Κυρηναῖος, μαθητὴς φιλοσόφου Ἀρίστωνος Χίου, γραμματικοῦ δὲ Λυσανίου τοῦ Κυρηναίου καὶ Καλλιμάχου τοῦ ποιητοῦ. μετεπέμφθη δὲ ἐξ Ἀθηνῶν ὑπὸ τοῦ τρίτου Πτολεμαίου καὶ διέτριψε μέχρι τοῦ πέμπτου. διὰ δὲ τὸ δευτερεύειν ἐν παντὶ εἴδει παιδείας τοῖς ἄκροις ἐγγίσαντα[1] Βῆτα[2] ἐπεκλήθη. οἱ δὲ καὶ δεύτερον ἢ νέον Πλάτωνα, ἄλλοι Πένταθλον ἐκάλεσαν. ἐτέχθη δὲ ρκϛ´ Ὀλυμ-

[1] ἐγγίσαντα Meursius, ἐγγίσασι Adler.
[2] Βῆτα Gloss. in Psalmos, Hesych. Mil., τὰ βήματα codd.

[a] Several of Eratosthenes' achievements have already been described—his solution of the Delian problem (vol. i. pp. 290-297), and his *sieve* for finding successive odd numbers (vol. i. pp. 100-103). Archimedes, as we have seen, dedicated the *Method* to him, and the *Cattle Problem*, as we have also seen, is said to have been sent through him to the Alexandrian mathematicians. It is generally supposed that Ptolemy credits him with having calculated the distance between the tropics (or twice the obliquity of the ecliptic) at 11/83rds. of a complete circle or 47° 29′ 39″, but Ptolemy's meaning is not clear. Eratosthenes also calculated the distances of the sun and moon from the earth and the size of the sun. Fragments of an astronomical poem which he wrote under the title

XVIII. ERATOSTHENES [a]

(a) General

Suidas, *s.v. Eratosthenes*

Eratosthenes, son of Aglaus, others say of Ambrosius; a Cyrenean, a pupil of the philosopher Ariston of Chios, of the grammarian Lysanias of Cyrene and of the poet Callimachus [b]; he was sent for from Athens by the third Ptolemy [c] and stayed till the fifth.[d] Owing to taking second place in all branches of learning, though approaching the highest excellence, he was called *Beta*. Others called him a *Second* or *New Plato*, and yet others *Pentathlon*. He was born in the 126th Olympiad [e] and died at the age

Hermes have survived. He was the first person to attempt a scientific chronology from the siege of Troy in two separate works, and he wrote a geographical work in three books. His writings are critically discussed in Bernhardy's *Eratosthenica* (Berlin, 1822).

[b] Callimachus, the famous poet and grammarian, was also a Cyrenean. He opened a school in the suburbs of Alexandria and was appointed by Ptolemy Philadelphus chief librarian of the Alexandrian library, a post which he held till his death *c.* 240 B.C. Eratosthenes later held the same post.

[c] Euergetes I (reigned 246–221 B.C.), who sent for him to be tutor to his son and successor Philopator (*v.* vol. i. pp. 256, 296).

[d] Epiphanes (reigned 204–181 B.C.).

[e] 276–273 B.C.

πιάδι καὶ ἐτελεύτησεν π̄ ἐτῶν γεγονώς, ἀποσχόμενος τροφῆς διὰ τὸ ἀμβλυώττειν, μαθητὴν ἐπίσημον καταλιπὼν Ἀριστοφάνην τὸν Βυζάντιον· οὗ πάλιν Ἀρίσταρχος μαθητής. μαθηταὶ δὲ αὐτοῦ Μνασέας καὶ Μένανδρος καὶ Ἄριστις. ἔγραψε δὲ φιλόσοφα καὶ ποιήματα καὶ ἱστορίας, Ἀστρονομίαν ἢ Καταστερισμούς,[1] Περὶ τῶν κατὰ φιλοσοφίαν αἱρέσεων, Περὶ ἀλυπίας, διαλόγους πολλοὺς καὶ γραμματικὰ συχνά.

(b) On Means

Papp. *Coll.* vii. 3, ed. Hultsch 636. 18-25

Τῶν δὲ προειρημένων τοῦ Ἀναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη . . . Ἐρατοσθένους περὶ μεσοτήτων δύο.

Papp. *Coll.* vii. 21, ed. Hultsch 660. 18–662. 18

Τῶν τόπων καθόλου οἱ μέν εἰσιν ἐφεκτικοί, ὡς καὶ Ἀπολλώνιος πρὸ τῶν ἰδίων στοιχείων λέγει σημείου μὲν τόπον σημεῖον, γραμμῆς δὲ τόπον γραμμήν, ἐπιφανείας δὲ ἐπιφάνειαν, στερεοῦ δὲ στερεόν, οἱ δὲ διεξοδικοί, ὡς σημείου μὲν γραμμήν, γραμμῆς δ' ἐπιφάνειαν, ἐπιφανείας δὲ στερεόν,

[1] Καταστερισμοὺς coni. Portus, Καταστηρίγμους codd.

[a] Not, of course, Aristarchus of Samos, the mathematician, but the celebrated Samothracian grammarian.

[b] Mnaseas was the author of a work entitled Περίπλους, whose three sections dealt with Europe, Asia and Africa, and a collection of oracles given at Delphi.

[c] This work is extant, but is not thought to be genuine in

of eighty of voluntary starvation, having lost his sight; he left a distinguished pupil, Aristophanes of Byzantium; of whom in turn Aristarchus[a] was a pupil. Among his pupils were Mnaseas,[b] Menander and Aristis. He wrote philosophical works, poems and histories, *Astronomy* or *Placings Among the Stars*,[c] *On Philosophical Divisions*, *On Freedom from Pain*, many dialogues and numerous grammatical works.

(b) On Means

Pappus, *Collection* vii. 3, ed. Hultsch 636. 18–25

The order of the aforesaid books in the *Treasury of Analysis* is as follows . . . the two books of Eratosthenes *On Means*.[d]

Pappus, *Collection* vii. 21, ed. Hultsch 660. 18–662. 18

Loci in general are termed *fixed*, as when Apollonius at the beginning of his own *Elements* says the locus of a point is a point, the locus of a line is a line, the locus of a surface is a surface and the locus of a solid is a solid; or *progressive*, as when it is said that the locus of a point is a line, the locus of a line is a surface and the locus of a surface is a solid; or *circumambient* as

its extant form: it contains a mythology and description of the constellations under forty-four heads. The general title Ἀστρονομία may be a mistake for Ἀστροθεσία; elsewhere it is alluded to under the title Κατάλογοι.

[d] The inclusion of this work in the *Treasury of Analysis*, along with such works as those of Euclid, Aristaeus and Apollonius, shows that it was a standard treatise. It is not otherwise mentioned, but the *loci with references to means* referred to in the passage from Pappus next cited were presumably discussed in it.

οἱ δὲ ἀναστροφικοί, ὡς σημείου μὲν ἐπιφάνειαν, γραμμῆς δὲ στερεόν. [. . . οἱ δὲ ὑπὸ Ἐρατοσθένους ἐπιγραφέντες τόποι πρὸς μεσότητας ἐκ τῶν προειρημένων εἰσὶν τῷ γένει, ἀπὸ δὲ τῆς ἰδιότητος τῶν ὑποθέσεων . . . ἐκείνοις.][1]

(c) The "Platonicus"

Theon Smyr., ed. Hiller 81. 17–82. 5

Ἐρατοσθένης δὲ ἐν τῷ Πλατωνικῷ φησι, μὴ ταὐτὸν εἶναι διάστημα καὶ λόγον. ἐπειδὴ λόγος μέν ἐστι δύο μεγεθῶν ἡ πρὸς ἄλληλα ποιὰ σχέσις, γίνεται δ' αὕτη καὶ ἐν διαφόροις ⟨καὶ ἐν ἀδιαφόροις⟩.[2] οἷον ἐν ᾧ λόγῳ ἐστὶ τὸ αἰσθητὸν πρὸς τὸ νοητόν, ἐν τούτῳ δόξα πρὸς ἐπιστήμην, καὶ διαφέρει καὶ τὸ νοητὸν τοῦ ἐπιστητοῦ ᾧ καὶ ἡ δόξα τοῦ αἰσθητοῦ. διάστημα δὲ ἐν διαφέρουσι μόνον, ἢ κατὰ τὸ μέγεθος ἢ κατὰ ποιότητα ἢ κατὰ θέσιν ἢ ἄλλως ὁπωσοῦν. δῆλον δὲ καὶ ἐντεῦθεν,

[1] The passage of which this forms the concluding sentence is attributed by Hultsch to an interpolator. To fill the lacuna before ἐκείνοις he suggests ἀνόμοιοι ἐκείνοις, following Halley's rendering, "diversa sunt ab illis."

[2] καὶ ἐν ἀδιαφόροις add. Hiller.

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[a] Tannery conjectured that these were the loci of points such that their distances from three fixed lines provided a "médiété," *i.e.*, loci (straight lines and conics) which can be represented in trilinear co-ordinates by such equations as

$$2y=x+z,\quad y^2=xz,\quad y(x+z)=2xz,\quad x(x-y)=z(y-z),$$
$$x(x-y)=y(y-z);$$

these represent respectively the arithmetic, geometric and harmonic means, and the means subcontrary to the harmonic and geometric means (*v.* vol. i. pp. 122-125). Zeuthen has

when it is said that the locus of a point is a surface and the locus of a line is a solid. [. . . the loci described by Eratosthenes as having reference to means belong to one of the aforesaid classes, but from a peculiarity in the assumptions are unlike them.][a]

(c) The "Platonicus"

Theon of Smyrna, ed. Hiller 81. 17–82. 5

Eratosthenes in the *Platonicus*[b] says that *interval* and *ratio* are not the same. Inasmuch as a ratio is a sort of relationship of two magnitudes one towards the other,[c] there exists a ratio both between terms that are different and also between terms that are not different. For example, the ratio of the perceptible to the intelligible is the same as the ratio of opinion to knowledge, and the difference between the intelligible and the known is the same as the difference of opinion from the perceptible.[d] But there can be an interval only between terms that are different, according to magnitude or quality or position or in some other way. It is thence clear that ratio is

an alternative conjecture on similar lines (*Die Lehre von den Kegelschnitten im Altertum*, pp. 320-321).

[b] Theon cites this work in one other passage (ed. Hiller 2. 3-12) telling how Plato was consulted about the doubling of the cube; it has already been cited (vol. i. p. 256). Eratosthenes' own solution of the problem has already been given in vol. i. pp. 290-297, and a letter purporting to be from Eratosthenes to Ptolemy Euergetes is given in vol. i. pp. 256-261. Whether the *Platonicus* was a commentary on Plato or a dialogue in which Plato was an interlocutor cannot be decided.

[c] *Cf.* Eucl. v. Def. 3, cited in vol. i. p. 444.

[d] A reference to Plato, *Rep.* vi. 509 D—511 E, vii. 517 A—518 B.

ὅτι λόγος διαστήματος ἕτερον· τὸ γὰρ ἥμισυ πρὸς τὸ διπλάσιον ⟨καὶ τὸ διπλάσιον πρὸς τὸ ἥμισυ⟩[1] λόγον μὲν οὐ τὸν αὐτὸν ἔχει, διάστημα δὲ τὸ αὐτό.

(*d*) Measurement of the Earth

Cleom. *De motu circ.* i. 10. 52, ed. Ziegler 94. 23–100. 23

Καὶ ἡ μὲν τοῦ Ποσειδωνίου ἔφοδος περὶ τοῦ κατὰ τὴν γῆν μεγέθους τοιαύτη, ἡ δὲ τοῦ Ἐρατοσθένους γεωμετρικῆς ἐφόδου ἐχομένη, καὶ δοκοῦσά τι ἀσαφέστερον ἔχειν. ποιήσει δὲ σαφῆ τὰ λεγόμενα ὑπ' αὐτοῦ τάδε προϋποτιθεμένων ἡμῶν. ὑποκείσθω ἡμῖν πρῶτον μὲν κἀνταῦθα, ὑπὸ τῷ αὐτῷ μεσημβρινῷ κεῖσθαι Συήνην καὶ Ἀλεξάνδρειαν, καὶ δεύτερον, τὸ διάστημα τὸ μεταξὺ τῶν πόλεων πεντακισχιλίων σταδίων εἶναι, καὶ τρίτον, τὰς καταπεμπομένας ἀκτῖνας ἀπὸ διαφόρων μερῶν τοῦ ἡλίου ἐπὶ διάφορα τῆς γῆς μέρη παραλλήλους εἶναι· οὕτως γὰρ ἔχειν αὐτὰς οἱ γεωμέτραι ὑποτίθενται. τέταρτον ἐκεῖνο ὑποκείσθω, δεικνύμενον παρὰ τοῖς γεωμέτραις, τὰς εἰς παραλλήλους ἐμπιπτούσας εὐθείας τὰς ἐναλλὰξ γωνίας ἴσας ποιεῖν, πέμπτον, τὰς ἐπὶ ἴσων γωνιῶν βεβηκυίας περιφερείας ὁμοίας εἶναι, τουτέστι τὴν αὐτὴν ἀναλογίαν καὶ τὸν αὐτὸν λόγον ἔχειν πρὸς τοὺς οἰκείους κύκλους, δεικνυμένου καὶ τούτου παρὰ τοῖς γεωμέτραις. ὁπόταν γὰρ περιφέρειαι ἐπὶ ἴσων γωνιῶν ὦσι βεβηκυῖαι, ἂν μία ἡτισοῦν

[1] καὶ . . . ἥμισυ add. Hiller.

[a] The difference between ratio and interval is explained a little more neatly by Theon himself (ed. Hiller 81. 6-9):

different from interval; for the relationship of the half to the double and of the double to the half does not furnish the same ratio, but it does furnish the same interval.[a]

(d) Measurement of the Earth

Cleomedes,[b] *On the Circular Motion of the Heavenly Bodies* i. 10. 52, ed. Ziegler 94. 23–100. 23

Such then is Posidonius's method of investigating the size of the earth, but Eratosthenes' method depends on a geometrical argument, and gives the impression of being more obscure. What he says will, however, become clear if the following assumptions are made. Let us suppose, in this case also, first that Syene and Alexandria lie under the same meridian circle; secondly, that the distance between the two cities is 5000 stades; and thirdly, that the rays sent down from different parts of the sun upon different parts of the earth are parallel; for the geometers proceed on this assumption. Fourthly, let us assume that, as is proved by the geometers, straight lines falling on parallel straight lines make the alternate angles equal, and fifthly, that the arcs subtended by equal angles are similar, that is, have the same proportion and the same ratio to their proper circles—this also being proved by the geometers. For whenever arcs of circles are subtended by equal angles, if any one of these is (say) one-tenth

διαφέρει δὲ διάστημα καὶ λόγος, ἐπειδὴ διάστημα μέν ἐστι τὸ μεταξὺ τῶν ὁμογενῶν τε καὶ ἀνίσων ὅρων, λόγος δὲ ἁπλῶς ἡ τῶν ὁμογενῶν ὅρων πρὸς ἀλλήλους σχέσις.

[b] Cleomedes probably wrote about the middle of the first century B.C. His handbook *De motu circulari corporum caelestium* is largely based on Posidonius.

αὐτῶν δέκατον ᾖ μέρος τοῦ οἰκείου κύκλου, καὶ αἱ λοιπαὶ πᾶσαι δέκατα μέρη γενήσονται τῶν οἰκείων κύκλων.

Τούτων ὁ κατακρατήσας οὐκ ἂν χαλεπῶς τὴν ἔφοδον τοῦ Ἐρατοσθένους καταμάθοι ἔχουσαν οὕτως. ὑπὸ τῷ αὐτῷ κεῖσθαι μεσημβρινῷ φησι Συήνην καὶ Ἀλεξάνδρειαν. ἐπεὶ οὖν μέγιστοι τῶν ἐν τῷ κόσμῳ οἱ μεσημβρινοί, δεῖ καὶ τοὺς ὑποκειμένους τούτοις τῆς γῆς κύκλους μεγίστους εἶναι ἀναγκαίως. ὥστε ἡλίκον ἂν τὸν διὰ Συήνης καὶ Ἀλεξανδρείας ἥκοντα κύκλον τῆς γῆς ἡ ἔφοδος ἀποδείξει αὕτη, τηλικοῦτος καὶ ὁ μέγιστος ἔσται τῆς γῆς κύκλος. φησὶ τοίνυν, καὶ ἔχει οὕτως, τὴν Συήνην ὑπὸ τῷ θερινῷ τροπικῷ κεῖσθαι κύκλῳ. ὁπόταν οὖν ἐν καρκίνῳ γενόμενος ὁ ἥλιος καὶ θερινὰς ποιῶν τροπὰς ἀκριβῶς μεσουρανήσῃ, ἄσκιοι γίνονται οἱ τῶν ὡρολογίων γνώμονες ἀναγκαίως, κατὰ κάθετον ἀκριβῆ τοῦ ἡλίου ὑπερκειμένου· καὶ τοῦτο γίνεσθαι λόγος ἐπὶ σταδίους τριακοσίους τὴν διάμετρον. ἐν Ἀλεξανδρείᾳ δὲ τῇ αὐτῇ ὥρᾳ ἀποβάλλουσιν οἱ τῶν ὡρολογίων γνώμονες σκιάν, ἅτε πρὸς ἄρκτῳ μᾶλλον τῆς Συήνης ταύτης τῆς πόλεως κειμένης. ὑπὸ τῷ αὐτῷ μεσημβρινῷ τοίνυν καὶ μεγίστῳ κύκλῳ τῶν πόλεων κειμένων, ἂν περιαγάγωμεν περιφέρειαν ἀπὸ τοῦ ἄκρου τῆς τοῦ γνώμονος σκιᾶς ἐπὶ τὴν βάσιν αὐτὴν τοῦ γνώμονος τοῦ ἐν Ἀλεξανδρείᾳ ὡρολογίου, αὕτη ἡ περιφέρεια τμῆμα γενήσεται τοῦ μεγίστου τῶν ἐν τῇ σκάφῃ κύκλων, ἐπεὶ μεγίστῳ κύκλῳ ὑπόκειται ἡ τοῦ ὡρολογίου σκάφη. εἰ οὖν ἑξῆς νοήσαιμεν εὐθείας διὰ τῆς γῆς ἐκβαλλομένας ἀφ' ἑκατέρου τῶν γνωμόνων, πρὸς τῷ κέντρῳ τῆς γῆς

of its proper circle, all the remaining arcs will be tenth parts of their proper circles.

Anyone who has mastered these facts will have no difficulty in understanding the method of Eratosthenes, which is as follows. Syene and Alexandria, he asserts, are under the same meridian. Since meridian circles are great circles in the universe, the circles on the earth which lie under them are necessarily great circles also. Therefore, of whatever size this method shows the circle on the earth through Syene and Alexandria to be, this will be the size of the great circle on the earth. He then asserts, as is indeed the case, that Syene lies under the summer tropic. Therefore, whenever the sun, being in the Crab at the summer solstice, is exactly in the middle of the heavens, the pointers of the sundials necessarily throw no shadows, the sun being in the exact vertical line above them; and this is said to be true over a space 300 stades in diameter. But in Alexandria at the same hour the pointers of the sundials throw shadows, because this city lies farther to the north than Syene. As the two cities lie under the same meridian great circle, if we draw an arc from the extremity of the shadow of the pointer to the base of the pointer of the sundial in Alexandria, the arc will be a segment of a great circle in the bowl of the sundial, since the bowl lies under the great circle. If then we conceive straight lines produced in order from each of the pointers through the earth, they

συμπεσοῦνται. ἐπεὶ οὖν τὸ ἐν Συήνῃ ὡρολόγιον κατὰ κάθετον ὑπόκειται τῷ ἡλίῳ, ἂν ἐπινοήσωμεν εὐθεῖαν ἀπὸ τοῦ ἡλίου ἥκουσαν ἐπ' ἄκρον τὸν τοῦ ὡρολογίου γνώμονα, μία γενήσεται εὐθεῖα ἡ ἀπὸ τοῦ ἡλίου μέχρι τοῦ κέντρου τῆς γῆς ἥκουσα. ἐὰν οὖν ἑτέραν εὐθεῖαν νοήσωμεν ἀπὸ τοῦ ἄκρου τῆς σκιᾶς τοῦ γνώμονος δι' ἄκρου τοῦ γνώμονος ἐπὶ τὸν ἥλιον ἀναγομένην ἀπὸ τῆς ἐν Ἀλεξανδρείᾳ σκάφης, αὕτη καὶ ἡ προειρημένη εὐθεῖα παράλληλοι γενήσονται ἀπὸ διαφόρων γε τοῦ ἡλίου μερῶν ἐπὶ διάφορα μέρη τῆς γῆς διήκουσαι. εἰς ταύτας τοίνυν παραλλήλους οὔσας ἐμπίπτει εὐθεῖα ἡ ἀπὸ τοῦ κέντρου τῆς γῆς ἐπὶ τὸν ἐν Ἀλεξανδρείᾳ γνώμονα ἥκουσα, ὥστε τὰς ἐναλλὰξ γωνίας ἴσας ποιεῖν· ὧν ἡ μέν ἐστι πρὸς τῷ κέντρῳ τῆς γῆς κατὰ σύμπτωσιν τῶν εὐθειῶν, αἳ ἀπὸ τῶν ὡρολογίων ἤχθησαν ἐπὶ τὸ κέντρον τῆς γῆς, γινομένη, ἡ δὲ κατὰ σύμπτωσιν ἄκρου τοῦ ἐν Ἀλεξανδρείᾳ γνώμονος καὶ τῆς ἀπ' ἄκρου τῆς σκιᾶς αὐτοῦ ἐπὶ τὸν ἥλιον διὰ τῆς πρὸς αὐτὸν ψαύσεως ἀναχθείσης γεγενημένη. καὶ ἐπὶ μὲν ταύτης βέβηκε περιφέρεια ἡ ἀπ' ἄκρου τῆς σκιᾶς τοῦ γνώμονος ἐπὶ τὴν βάσιν αὐτοῦ περιαχθεῖσα, ἐπὶ δὲ τῆς πρὸς τῷ κέντρῳ τῆς γῆς ἡ ἀπὸ Συήνης διήκουσα εἰς Ἀλεξάνδρειαν. ὅμοιαι τοίνυν αἱ περιφέρειαί εἰσιν ἀλλήλαις ἐπ' ἴσων γε γωνιῶν βεβηκυῖαι. ὃν ἄρα λόγον ἔχει ἡ ἐν τῇ σκάφῃ πρὸς τὸν οἰκεῖον κύκλον, τοῦτον ἔχει τὸν λόγον καὶ ἡ ἀπὸ Συήνης εἰς Ἀλεξάνδρειαν ἥκουσα. ἡ δέ γε ἐν τῇ σκάφῃ πεντηκοστὸν μέρος εὑρίσκεται τοῦ οἰκείου κύκλου. δεῖ οὖν ἀναγκαίως καὶ τὸ ἀπὸ Συήνης εἰς Ἀλεξάνδρειαν διάστημα πεντηκοστὸν εἶναι μέρος τοῦ

will meet at the centre of the earth. Now since the sundial at Syene is vertically under the sun, if we conceive a straight line drawn from the sun to the top of the pointer of the sundial, the line stretching from the sun to the centre of the earth will be one straight line. If now we conceive another straight line drawn upwards from the extremity of the shadow of the pointer of the sundial in Alexandria, through the top of the pointer to the sun, this straight line and the aforesaid straight line will be parallel, being straight lines drawn through from different parts of the sun to different parts of the earth. Now on these parallel straight lines there falls the straight line drawn from the centre of the earth to the pointer at Alexandria, so that it makes the alternate angles equal; one of these is formed at the centre of the earth by the intersection of the straight lines drawn from the sundials to the centre of the earth; the other is at the intersection of the top of the pointer in Alexandria and the straight line drawn from the extremity of its shadow to the sun through the point where it meets the pointer. Now this latter angle subtends the arc carried round from the extremity of the shadow of the pointer to its base, while the angle at the centre of the earth subtends the arc stretching from Syene to Alexandria. But the arcs are similar since they are subtended by equal angles. Whatever ratio, therefore, the arc in the bowl of the sundial has to its proper circle, the arc reaching from Syene to Alexandria has the same ratio. But the arc in the bowl is found to be the fiftieth part of its proper circle. Therefore the distance from Syene to Alexandria must necessarily be a fiftieth part of the great

μεγίστου τῆς γῆς κύκλου· καὶ ἔστι τοῦτο σταδίων πεντακισχιλίων. ὁ ἄρα σύμπας κύκλος γίνεται μυριάδων εἴκοσι πέντε. καὶ ἡ μὲν Ἐρατοσθένους ἔφοδος τοιαύτη.

Heron, *Dioptra* 36, ed. H. Schöne 302. 10-17

Δέον δὲ ἔστω, εἰ τύχοι, τὴν μεταξὺ Ἀλεξανδρείας καὶ Ῥώμης ὁδὸν ἐκμετρῆσαι τὴν ἐπ' εὐθείας, τήν γε ἐπὶ κύκλου περιφερείας μεγίστου τοῦ ἐν τῇ γῇ, προσομολογουμένου τοῦ ὅτι περίμετρος τῆς γῆς σταδίων ἐστὶ $\overset{\beta}{\mu}$ καὶ ἔτι ,β, ὡς ὁ μάλιστα τῶν ἄλλων ἀκριβέστερον πεπραγματευμένος Ἐρατοσθένης δείκνυσιν ἐν ⟨τῷ⟩[1] ἐπιγραφομένῳ Περὶ τῆς ἀναμετρήσεως τῆς γῆς.

[1] τῷ add. H. Schöne.

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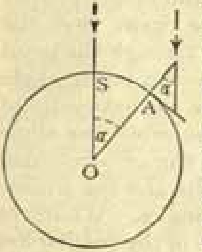
[a] The attached figure will help to elucidate Cleomedes.



S is Syene and A Alexandria; the centre of the earth is O. The sun's rays at the two places are represented by the broken straight lines. If α be the angle made by the sun's rays with the pointer of the sundial at Alexandria (OA produced), the angle SOA is also equal to α, or one-fiftieth of four right angles. The arc SA is known to be 5000 stades and it follows that the whole circumference of the earth must be 250000 stades.

circle of the earth. And this distance is 5000 stades. Therefore the whole great circle is 250000 stades. Such is the method of Eratosthenes.[a]

Heron, *Dioptra* 36, ed. H. Schöne 302. 10-17

Let it be required, perchance, to measure the distance between Alexandria and Rome along the arc of a great circle,[b] on the assumption that the perimeter of the earth is 252000 stades, as Eratosthenes, who investigated this question more accurately than others, shows in the book which he wrote *On the Measurement of the Earth*.[c]

[b] Lit. "along the circumference of the greatest circle on the earth."

[c] Strabo (ii. 5. 7) and Theon of Smyrna (ed. Hiller 124. 10-12) also give Eratosthenes' measurement as 252000 stades against the 250000 of Cleomedes. "The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250000 to 252000 for some reason, perhaps in order to get a figure devisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny (*N.H.* xii. 13. 53) is right in saying that Eratosthenes made 40 stades equal to the Egyptian σχοῖνος, then, taking the σχοῖνος at 12000 Royal cubits of 0·525 metres, we get 300 such cubits, or 157·5 metres, *i.e.*, 516·73 feet, as the length of the stade. On this basis 252000 stades works out to 24662 miles, and the diameter of the earth to about 7850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation" (Heath, *H.G.M.* ii. 107).

XIX. APOLLONIUS OF PERGA

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(a) The Conic Sections

(i.) *Relation to Previous Works*

Eutoc. *Comm. in Con.*, Apoll. Perg. ed. Heiberg ii. 168. 5–170. 26

Ἀπολλώνιος ὁ γεωμέτρης, ὦ φίλε ἑταῖρε Ἀνθέμιε, γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίᾳ ἐν χρόνοις τοῦ Εὐεργέτου Πτολεμαίου, ὡς ἱστορεῖ Ἡράκλειος ὁ τὸν βίον Ἀρχιμήδους γράφων, ὃς καί φησι τὰ κωνικὰ θεωρήματα ἐπινοῆσαι μὲν πρῶτον τὸν Ἀρχιμήδη, τὸν δὲ Ἀπολλώνιον αὐτὰ εὑρόντα ὑπὸ Ἀρχιμήδους μὴ ἐκδοθέντα ἰδιοποιήσασθαι, οὐκ ἀληθεύων κατά γε τὴν ἐμήν. ὅ τε γὰρ Ἀρχιμήδης ἐν πολλοῖς φαίνεται ὡς παλαιοτέρας τῆς στοιχειώσεως τῶν κωνικῶν μεμνημένος, καὶ ὁ Ἀπολλώνιος οὐχ ὡς ἰδίας ἐπινοίας γράφει· οὐ γὰρ ἂν ἔφη " ἐπὶ πλέον καὶ καθόλου μᾶλλον

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[a] Scarcely anything more is known of the life of one of the greatest geometers of all time than is stated in this brief reference. From Pappus, *Coll.* vii., ed. Hultsch 67 (quoted in vol. i. p. 488), it is known that he spent much time at Alexandria with Euclid's successors. Ptolemy Euergetes reigned 246–221 B.C., and as Ptolemaeus Chennus (apud Photii *Bibl.*, cod. cxc., ed. Bekker 151 b 18) mentions an astro-

XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) *Relation to Previous Works*

Eutocius, *Commentary on Apollonius's Conics*, Apoll. Perg. ed. Heiberg ii. 168. 5–170. 26

Apollonius the geometer, my dear Anthemius, flourished at Perga in Pamphylia during the time of Ptolemy Euergetes,[a] as is related in the life of Archimedes written by Heraclius,[b] who also says that Archimedes first conceived the theorems in conics and that Apollonius, finding they had been discovered by Archimedes but not published, appropriated them for himself, but in my opinion he errs. For in many places Archimedes appears to refer to the elements of conics as an older work, and moreover Apollonius does not claim to be giving his own discoveries; otherwise he would not have described his purpose as "to investigate these properties more fully and more

nomer named Apollonius who flourished in the time of Ptolemy Philopator (221–204 B.C.), the great geometer is probably meant. This fits in with Apollonius's dedication of Books iv.-viii. of his *Conics* to King Attalus I (247–197 B.C.). From the preface to Book i., quoted *infra* (p. 281), we gather that Apollonius visited Eudemus at Pergamum, and to Eudemus he dedicated the first two books of the second edition of his work.

[b] More probably Heraclides, *v. supra*, p. 18 n. *a*.

ἐξειργάσθαι ταῦτα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα." ἀλλ' ὅπερ φησὶν ὁ Γέμινος ἀληθές ἐστιν, ὅτι οἱ παλαιοὶ κῶνον ὁριζόμενοι τὴν τοῦ ὀρθογωνίου τριγώνου περιφορὰν μενούσης μιᾶς τῶν περὶ τὴν ὀρθὴν εἰκότως καὶ τοὺς κώνους πάντας ὀρθοὺς ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἑκάστῳ, ἐν μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλουμένην παραβολήν, ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολήν, ἐν δὲ τῷ ὀξυγωνίῳ τὴν ἔλλειψιν· καὶ ἔστι παρ' αὐτοῖς εὑρεῖν οὕτως ὀνομαζομένας τὰς τομάς. ὥσπερ οὖν τῶν ἀρχαίων ἐπὶ ἑνὸς ἑκάστου εἴδους τριγώνου θεωρησάντων τὰς δύο ὀρθὰς πρότερον ἐν τῷ ἰσοπλεύρῳ καὶ πάλιν ἐν τῷ ἰσοσκελεῖ καὶ ὕστερον ἐν τῷ σκαληνῷ οἱ μεταγενέστεροι καθολικὸν θεώρημα ἀπέδειξαν τοιοῦτο· παντὸς τριγώνου αἱ ἐντὸς τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· οὕτως καὶ ἐπὶ τῶν τοῦ κώνου τομῶν· τὴν μὲν γὰρ λεγομένην ὀρθογωνίου κώνου τομὴν ἐν ὀρθογωνίῳ μόνον κώνῳ ἐθεώρουν τεμνομένῳ ἐπιπέδῳ ὀρθῷ πρὸς μίαν πλευρὰν τοῦ κώνου, τὴν δὲ τοῦ ἀμβλυγωνίου κώνου τομὴν ἐν ἀμβλυγωνίῳ γινομένην κώνῳ ἀπεδείκνυσαν, τὴν δὲ τοῦ ὀξυγωνίου ἐν ὀξυγωνίῳ, ὁμοίως ἐπὶ πάντων τῶν κώνων ἄγοντες τὰ ἐπίπεδα ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κώνου· δηλοῖ δὲ καὶ αὐτὰ τὰ ἀρχαῖα ὀνόματα τῶν γραμμῶν. ὕστερον δὲ Ἀπολλώνιος ὁ Περγαῖος καθόλου τι ἐθεώρησεν, ὅτι ἐν παντὶ κώνῳ καὶ ὀρθῷ καὶ σκαληνῷ πᾶσαι αἱ τομαί εἰσι κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κῶνον προσβολήν· ὃν καὶ θαυμάσαντες οἱ κατ' αὐτὸν γενόμενοι διὰ τὸ θαυμάσιον τῶν ὑπ' αὐτοῦ δεδειγμένων κωνικῶν θεωρημάτων μέγαν γεωμέτρην ἐκάλουν. ταῦτα

generally than is done in the works of others."[a] But what Geminus says is correct: defining a cone as the figure formed by the revolution of a right-angled triangle about one of the sides containing the right angle, the ancients naturally took all cones to be right with one section in each—in the right-angled cone the section now called the parabola, in the obtuse-angled the hyperbola, and in the acute-angled the ellipse; and in this may be found the reason for the names they gave to the sections. Just as the ancients, investigating each species of triangle separately, proved that there were two right angles first in the equilateral triangle, then in the isosceles, and finally in the scalene, whereas the more recent geometers have proved the general theorem, that *in any triangle the three internal angles are equal to two right angles*, so it has been with the sections of the cone; for the ancients investigated the so-called *section of a right-angled cone* in a right-angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the *section of an obtuse-angled cone* in an obtuse-angled cone and the *section of an acute-angled cone* in the acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone. In admiration for this, and on account of the remarkable nature of the theorems in conics proved by him, his contemporaries called him the "Great

[a] This comes from the preface to Book i., v. *infra*, p. 283.

μὲν οὖν ὁ Γέμινος ἐν τῷ ἕκτῳ φησὶ τῆς Τῶν μαθημάτων θεωρίας.

(ii.) *Scope of the Work*

Apoll. *Conic.* i., Praef., Apoll. Perg. ed. Heiberg i. 2. 2–4. 28

Ἀπολλώνιος Εὐδήμῳ χαίρειν.

Εἰ τῷ τε σώματι εὖ ἐπανάγεις καὶ τὰ ἄλλα κατὰ γνώμην ἐστί σοι, καλῶς ἂν ἔχοι, μετρίως δὲ ἔχομεν καὶ αὐτοί. καθ᾽ ὃν δὲ καιρὸν ἤμην μετά σου ἐν Περγάμῳ, ἐθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κωνικῶν· πέπομφα οὖν σοι τὸ πρῶτον βιβλίον διορθωσάμενος, τὰ δὲ λοιπά, ὅταν εὐαρεστήσωμεν, ἐξαποστελοῦμεν· οὐκ ἀμνημονεῖν γὰρ οἴομαί σε παρ᾽ ἐμοῦ ἀκηκοότα, διότι τὴν περὶ ταῦτα ἔφοδον ἐποιησάμην ἀξιωθεὶς ὑπὸ Ναυκράτους τοῦ γεωμέτρου, καθ᾽ ὃν καιρὸν ἐσχόλαζε

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[a] Menaechmus, as shown in vol. i. pp. 278-283, and more particularly p. 283 n. *a*, solved the problem of the doubling of the cube by means of the intersection of a parabola with a hyperbola, and also by means of the intersection of two parabolas. This is the earliest mention of the conic sections in Greek literature, and therefore Menaechmus (*fl.* 360–350 B.C.) is generally credited with their discovery; and as Eratosthenes' epigram (vol. i. p. 296) speaks of "cutting the cone in the triads of Menaechmus," he is given credit for discovering the ellipse as well. He may have obtained them all by the method suggested by Geminus, but Heath (*H.G.M.* ii. 111-116) gives cogent reasons for thinking that he may have obtained his rectangular hyperbola by a section of a right-angled cone parallel to the axis.

A passage already quoted (vol. i. pp. 486-489) from Pappus (ed. Hultsch 672. 18–678. 24) informs us that treatises on the conic sections were written by Aristaeus and Euclid. Aristaeus' work, in five books, was entitled *Solid Loci*; Euclid's

Geometer." Geminus relates these details in the sixth book of his *Theory of Mathematics.*[a]

(ii.) *Scope of the Work*

Apollonius, *Conics* i., Preface, Apoll. Perg. ed. Heiberg i. 2. 2–4. 28

Apollonius to Eudemus[b] greeting.

If you are in good health and matters are in other respects as you wish, it is well; I am pretty well too. During the time I spent with you at Pergamum, I noticed how eager you were to make acquaintance with my work in conics; I have therefore sent to you the first book, which I have revised, and I will send the remaining books when I am satisfied with them. I suppose you have not forgotten hearing me say that I took up this study at the request of Naucrates the geometer, at the time when he came

Conics was in four books. The work of Aristaeus was obviously more original and more specialized; that of Euclid was admittedly a compilation largely based on Aristaeus. Euclid flourished about 300 B.C. As noted in vol. i. p. 495 n. *a*, the focus-directrix property must have been known to Euclid, and probably to Aristaeus; curiously, it does not appear in Apollonius's treatise.

Many properties of conics are assumed in the works of Archimedes without proof and several have been encountered in this work; they were no doubt taken from the works of Aristaeus or Euclid. As the reader will notice, Archimedes' terminology differs in several respects from that of Apollonius, apart from the fundamental difference on which Geminus laid stress.

The history of the conic sections in antiquity is admirably treated by Zeuthen, *Die Lehre von den Kegelschnitten im Altertum* (1886) and Heath, *Apollonius of Perga*, xvii–clvi.

[b] Not, of course, the pupil of Aristotle who wrote the famous *History of Geometry*, unhappily lost.

παρ' ἡμῖν παραγενηθεὶς εἰς Ἀλεξάνδρειαν, καὶ διότι πραγματεύσαντες αὐτὰ ἐν ὀκτὼ βιβλίοις ἐξ αὐτῆς μεταδεδώκαμεν αὐτὰ εἰς τὸ σπουδαιότερον διὰ τὸ πρὸς ἔκπλῳ αὐτὸν εἶναι οὐ διακαθάραντες, ἀλλὰ πάντα τὰ ὑποπίπτοντα ἡμῖν θέντες ὡς ἔσχατον ἐπελευσόμενοι. ὅθεν καιρὸν νῦν λαβόντες ἀεὶ τὸ τυγχάνον διορθώσεως ἐκδίδομεν. καὶ ἐπεὶ συμβέβηκε καὶ ἄλλους τινὰς τῶν συμμεμιχότων ἡμῖν μετειληφέναι τὸ πρῶτον καὶ τὸ δεύτερον βιβλίον πρὶν ἢ διορθωθῆναι, μὴ θαυμάσῃς, ἐὰν περιπίπτῃς αὐτοῖς ἑτέρως ἔχουσιν.

Ἀπὸ δὲ τῶν ὀκτὼ βιβλίων τὰ πρῶτα τέσσαρα πέπτωκεν εἰς ἀγωγὴν στοιχειώδη, περιέχει δὲ τὸ μὲν πρῶτον τὰς γενέσεις τῶν τριῶν τομῶν καὶ τῶν ἀντικειμένων καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργασμένα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα, τὸ δὲ δεύτερον τὰ περὶ τὰς διαμέτρους καὶ τοὺς ἄξονας τῶν τομῶν συμβαίνοντα καὶ τὰς ἀσυμπτώτους καὶ ἄλλα γενικὴν καὶ ἀναγκαίαν χρείαν παρεχόμενα πρὸς τοὺς διορισμούς· τίνας δὲ διαμέτρους καὶ τίνας ἄξονας καλῶ, εἰδήσεις ἐκ τούτου τοῦ βιβλίου. τὸ δὲ τρίτον πολλὰ καὶ παράδοξα θεωρήματα χρήσιμα πρός τε τὰς συνθέσεις τῶν στερεῶν τόπων καὶ τοὺς διορισμούς, ὧν τὰ πλεῖστα καὶ κάλλιστα ξένα, ἃ καὶ κατανοήσαντες συνείδομεν μὴ συντιθέμενον ὑπὸ Εὐκλείδου τὸν ἐπὶ τρεῖς καὶ τέσσαρας γραμμὰς τόπον, ἀλλὰ μόριον τὸ τυχὸν αὐτοῦ καὶ τοῦτο οὐκ εὐτυχῶς· οὐ γὰρ ἦν δυνατὸν ἄνευ τῶν προσευρημένων ἡμῖν τελειωθῆναι τὴν

[a] A necessary observation, because Archimedes had used the terms in a different sense.

to Alexandria and stayed with me, and that, when I had completed the investigation in eight books, I gave them to him at once, a little too hastily, because he was on the point of sailing, and so I was not able to correct them, but put down everything as it occurred to me, intending to make a revision at the end. Accordingly, as opportunity permits, I now publish on each occasion as much of the work as I have been able to correct. As certain other persons whom I have met have happened to get hold of the first and second books before they were corrected, do not be surprised if you come across them in a different form.

Of the eight books the first four form an elementary introduction. The first includes the methods of producing the three sections and the opposite branches [of the hyperbola] and their fundamental properties, which are investigated more fully and more generally than in the works of others. The second book includes the properties of the diameters and the axes of the sections as well as the asymptotes, with other things generally and necessarily used in determining limits of possibility; and what I call diameters and axes you will learn from this book.[a] The third book includes many remarkable theorems useful for the syntheses of solid loci and for determining limits of possibility; most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me.[b]

[b] For this locus, and Pappus's comments on Apollonius's claims, v. vol. i. pp. 486–489.

σύνθεσιν. τὸ δὲ τέταρτον, ποσαχῶς αἱ τῶν κώνων τομαὶ ἀλλήλαις τε καὶ τῇ τοῦ κύκλου περιφερείᾳ συμβάλλουσι, καὶ ἄλλα ἐκ περισσοῦ, ὧν οὐδέτερον ὑπὸ τῶν πρὸ ἡμῶν γέγραπται, κώνου τομὴ ἢ κύκλου περιφέρεια κατὰ πόσα σημεῖα συμβάλλουσι.

Τὰ δὲ λοιπά ἐστι περιουσιαστικώτερα· ἔστι γὰρ τὸ μὲν περὶ ἐλαχίστων καὶ μεγίστων ἐπὶ πλέον, τὸ δὲ περὶ ἴσων καὶ ὁμοίων κώνου τομῶν, τὸ δὲ περὶ διοριστικῶν θεωρημάτων, τὸ δὲ προβλημάτων κωνικῶν διωρισμένων. οὐ μὴν ἀλλὰ καὶ πάντων ἐκδοθέντων ἔξεστι τοῖς περιτυγχάνουσι κρίνειν αὐτά, ὡς ἂν αὐτῶν ἕκαστος αἱρῆται. εὐτύχει.

(iii.) *Definitions*

Ibid., Deff., Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

Ἐὰν ἀπό τινος σημείου πρὸς κύκλου περιφέρειαν, ὃς οὐκ ἔστιν ἐν τῷ αὐτῷ ἐπιπέδῳ τῷ σημείῳ, εὐθεῖα ἐπιζευχθεῖσα ἐφ᾿ ἑκάτερα προσεκβληθῇ, καὶ μένοντος τοῦ σημείου ἡ εὐθεῖα περιενεχθεῖσα περὶ τὴν τοῦ κύκλου περιφέρειαν εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὴν γραφεῖσαν ὑπὸ τῆς εὐθείας ἐπιφάνειαν, ἣ σύγκειται ἐκ δύο ἐπιφανειῶν κατὰ κορυφὴν ἀλλήλαις κειμένων, ὧν ἑκατέρα εἰς ἄπειρον αὔξεται τῆς

[a] Only the first four books survive in Greek. Books v.–vii. have survived in Arabic, but Book viii. is wholly lost. Halley (Oxford, 1710) edited the first seven books, and his edition is still the only source for Books vi. and vii. The first four books have since been edited by Heiberg (Leipzig, 1891–1893) and Book v. (up to Prop. 7) by L. Nix (Leipzig, 1889). The

The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas].

The remaining books are thrown in by way of addition: one of them discusses fully *minima* and *maxima*, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.[a]

(iii.) *Definitions*

Ibid., Definitions, Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line *a conical surface*; it is composed of two surfaces lying vertically opposite to each other, of which each

surviving books have been put into mathematical notation by T. L. Heath, *Apollonius of Perga* (Cambridge, 1896) and translated into French by Paul Ver Eecke, *Les Coniques d' Apollonius de Perga* (Bruges, 1923).

In ancient times Eutocius edited the first four books with a commentary which still survives and is published in Heiberg's edition. Serenus and Hypatia also wrote commentaries, and Pappus a number of lemmas.

γραφούσης εὐθείας εἰς ἄπειρον προσεκβαλλομένης, καλῶ κωνικὴν ἐπιφάνειαν, κορυφὴν δὲ αὐτῆς τὸ μεμενηκὸς σημεῖον, ἄξονα δὲ τὴν διὰ τοῦ σημείου καὶ τοῦ κέντρου τοῦ κύκλου ἀγομένην εὐθεῖαν.

Κῶνον δὲ τὸ περιεχόμενον σχῆμα ὑπό τε τοῦ κύκλου καὶ τῆς μεταξὺ τῆς τε κορυφῆς καὶ τῆς τοῦ κύκλου περιφερείας κωνικῆς ἐπιφανείας, κορυφὴν δὲ τοῦ κώνου τὸ σημεῖον, ὃ καὶ τῆς ἐπιφανείας ἐστὶ κορυφή, ἄξονα δὲ τὴν ἀπὸ τῆς κορυφῆς ἐπὶ τὸ κέντρον τοῦ κύκλου ἀγομένην εὐθεῖαν, βάσιν δὲ τὸν κύκλον.

Τῶν δὲ κώνων ὀρθοὺς μὲν καλῶ τοὺς πρὸς ὀρθὰς ἔχοντας ταῖς βάσεσι τοὺς ἄξονας, σκαληνοὺς δὲ τοὺς μὴ πρὸς ὀρθὰς ἔχοντας ταῖς βάσεσι τοὺς ἄξονας.

Πάσης καμπύλης γραμμῆς, ἥτις ἐστὶν ἐν ἑνὶ ἐπιπέδῳ, διάμετρον μὲν καλῶ εὐθεῖαν, ἥτις ἠγμένη ἀπὸ τῆς καμπύλης γραμμῆς πάσας τὰς ἀγομένας ἐν τῇ γραμμῇ εὐθείας εὐθείᾳ τινὶ παραλλήλους δίχα διαιρεῖ, κορυφὴν δὲ τῆς γραμμῆς τὸ πέρας τῆς εὐθείας τὸ πρὸς τῇ γραμμῇ, τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατῆχθαι ἑκάστην τῶν παραλλήλων.

Ὁμοίως δὲ καὶ δύο καμπύλων γραμμῶν ἐν ἑνὶ ἐπιπέδῳ κειμένων διάμετρον καλῶ πλαγίαν μέν, ἥτις εὐθεῖα τέμνουσα τὰς δύο γραμμὰς πάσας τὰς ἀγομένας ἐν ἑκατέρᾳ τῶν γραμμῶν παρά τινα εὐθεῖαν δίχα τέμνει, κορυφὰς δὲ τῶν γραμμῶν τὰ πρὸς ταῖς γραμμαῖς πέρατα τῆς διαμέτρου, ὀρθίαν δέ, ἥτις κειμένη μεταξὺ τῶν δύο γραμμῶν πάσας τὰς ἀγομένας παραλλήλους εὐθείας εὐθείᾳ τινὶ καὶ ἀπολαμβανομένας μεταξὺ τῶν γραμμῶν δίχα

extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the *vertex*, and the straight line drawn through this point and the centre of the circle I call the *axis*.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a *cone*, and by the *vertex of the cone* I mean the point which is the vertex of the surface, and by the *axis* I mean the straight line drawn from the vertex to the centre of the circle, and by the *base* I mean the circle.

Of cones, I term those *right* which have their axes at right angles to their bases, and *scalene* those which have their axes not at right angles to their bases.

In any plane curve I mean by a *diameter* a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the *vertex of the curve* I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn *ordinate-wise* to the diameter.

Similarly, in a pair of plane curves I mean by a *transverse diameter* a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the *vertices of the curves* I mean the extremities of the diameter on the curves; and by an *erect diameter* I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given

τέμνει, τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατῆχθαι ἑκάστην τῶν παραλλήλων.

Συζυγεῖς καλῶ διαμέτρους [δύο][1] καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθείας, ὧν ἑκατέρα διάμετρος οὖσα τὰς τῇ ἑτέρᾳ παραλλήλους δίχα διαιρεῖ.

Ἄξονα δὲ καλῶ καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθεῖαν, ἥτις διάμετρος οὖσα τῆς γραμμῆς ἢ τῶν γραμμῶν πρὸς ὀρθὰς τέμνει τὰς παραλλήλους.

Συζυγεῖς καλῶ ἄξονας καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθείας, αἵτινες διάμετροι οὖσαι συζυγεῖς πρὸς ὀρθὰς τέμνουσι τὰς ἀλλήλων παραλλήλους.

(iv.) *Construction of the Sections*

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

ζ'

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὸ ἐπίπεδον, ἐν ᾧ ἐστιν ἡ βάσις τοῦ κώνου, κατ' εὐθεῖαν πρὸς ὀρθὰς οὖσαν ἤτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἢ τῇ ἐπ' εὐθείας αὐτῇ, αἱ ἀγόμεναι εὐθεῖαι ἀπὸ τῆς γενηθείσης τομῆς ἐν τῇ τοῦ κώνου ἐπιφανείᾳ, ἣν ἐποίησε τὸ τέμνον ἐπίπεδον, παράλληλοι τῇ πρὸς ὀρθὰς τῇ βάσει τοῦ τριγώνου εὐθείᾳ ἐπὶ τὴν κοινὴν τομὴν πεσοῦνται τοῦ τέμ-

[1] δύο om. Heiberg.

[a] This proposition defines a conic section in the most general way with reference to any diameter. It is only much

straight line; and I describe each of the parallels as drawn *ordinate-wise to the diameter.*

By *conjugate diameters* in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an *axis* of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By *conjugate axes* in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(iv.) *Construction of the Sections*

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

Prop. 7 [a]

If a cone be cut by a plane through the axis, and if it be also cut by another plane cutting the plane containing the base of the cone in a straight line perpendicular to the base of the axial triangle,[b] *or to the base produced, a section will be made on the surface of the cone by the cutting plane, and straight lines drawn in it parallel to the straight line perpendicular to the base of the axial triangle will meet the common section of the cutting plane and the axial*

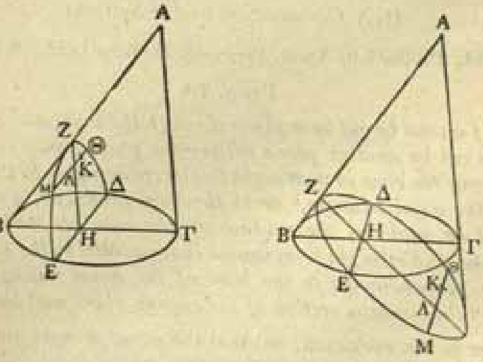
later in the work (i. 52-58) that the principal axes are introduced as diameters at right angles to their ordinates. The proposition is an excellent example of the generality of Apollonius's methods.

Apollonius followed rigorously the Euclidean form of proof. In consequence his general enunciations are extremely long and often can be made tolerable in an English rendering only by splitting them up; but, though Apollonius seems to have taken a malicious pleasure in their length, they are formed on a perfect logical pattern without a superfluous word.

[b] Lit. "the triangle through the axis."

νοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου καὶ προσεκβαλλόμεναι ἕως τοῦ ἑτέρου μέρους τῆς τομῆς δίχα τμηθήσονται ὑπ' αὐτῆς, καὶ ἐὰν μὲν ὀρθὸς ᾖ ὁ κῶνος, ἡ ἐν τῇ βάσει εὐθεῖα πρὸς ὀρθὰς ἔσται τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου, ἐὰν δὲ σκαληνός, οὐκ αἰεὶ πρὸς ὀρθὰς ἔσται, ἀλλ' ὅταν τὸ διὰ τοῦ ἄξονος ἐπίπεδον πρὸς ὀρθὰς ᾖ τῇ βάσει τοῦ κώνου.

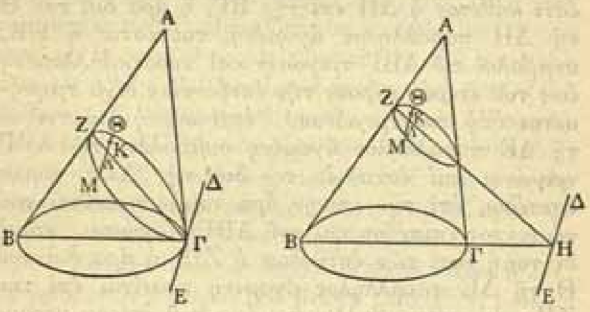
Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ



τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρίγωνον. τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὸ ἐπίπεδον, ἐν ᾧ ἐστιν ὁ ΒΓ κύκλος, κατ' εὐθεῖαν τὴν ΔΕ ἤτοι πρὸς ὀρθὰς οὖσαν τῇ ΒΓ ἢ τῇ ἐπ' εὐθείας αὐτῇ, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ· κοινὴ δὴ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου ἡ ΖΗ. καὶ

triangle and, if produced to the other part of the section, will be bisected by it; if the cone be right, the straight line in the base will be perpendicular to the common section of the cutting plane and the axial triangle; but if it be scalene, it will not in general be perpendicular, but only when the plane through the axis is perpendicular to the base of the cone.

Let there be a cone whose vertex is the point A and whose base is the circle BΓ, and let it be cut by a



plane through the axis, and let the section so made be the triangle ABΓ. Now let it be cut by another plane cutting the plane containing the circle BΓ in a straight line ΔE which is either perpendicular to BΓ or to BΓ produced, and let the section made on the surface of the cone be ΔZE [a]; then the common section of the cutting plane and of the triangle ABΓ

[a] This applies only to the first two of the figures given in the mss.

εἰλήφθω τι σημεῖον ἐπὶ τῆς ΔΖΕ τομῆς τὸ Θ, καὶ ἤχθω διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἡ ΘΚ. λέγω, ὅτι ἡ ΘΚ συμβαλεῖ τῇ ΖΗ καὶ ἐκβαλλομένη ἕως τοῦ ἑτέρου μέρους τῆς ΔΖΕ τομῆς δίχα τμηθήσεται ὑπὸ τῆς ΖΗ εὐθείας.

Ἐπεὶ γὰρ κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, τέτμηται ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖ τομὴν τὸ ΑΒΓ τρίγωνον, εἴληπται δέ τι σημεῖον ἐπὶ τῆς ἐπιφανείας, ὃ μή ἐστιν ἐπὶ πλευρᾶς τοῦ ΑΒΓ τριγώνου, τὸ Θ, καί ἐστι κάθετος ἡ ΔΗ ἐπὶ τὴν ΒΓ, ἡ ἄρα διὰ τοῦ Θ τῇ ΔΗ παράλληλος ἀγομένη, τουτέστιν ἡ ΘΚ, συμβαλεῖ τῷ ΑΒΓ τριγώνῳ καὶ προσεκβαλλομένη ἕως τοῦ ἑτέρου μέρους τῆς ἐπιφανείας δίχα τμηθήσεται ὑπὸ τοῦ τριγώνου. ἐπεὶ οὖν ἡ διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἀγομένη συμβάλλει τῷ ΑΒΓ τριγώνῳ καί ἐστιν ἐν τῷ διὰ τῆς ΔΖΕ τομῆς ἐπιπέδῳ, ἐπὶ τὴν κοινὴν ἄρα τομὴν πεσεῖται τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου. κοινὴ δὲ τομή ἐστι τῶν ἐπιπέδων ἡ ΖΗ· ἡ ἄρα διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἀγομένη πεσεῖται ἐπὶ τὴν ΖΗ· καὶ προσεκβαλλομένη ἕως τοῦ ἑτέρου μέρους τῆς ΔΖΕ τομῆς δίχα τμηθήσεται ὑπὸ τῆς ΖΗ εὐθείας.

Ἤτοι δὴ ὁ κῶνος ὀρθός ἐστιν, ἢ τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ ὀρθόν ἐστι πρὸς τὸν ΒΓ κύκλον, ἢ οὐδέτερον.

Ἔστω πρότερον ὁ κῶνος ὀρθός· εἴη ἂν οὖν καὶ τὸ ΑΒΓ τρίγωνον ὀρθὸν πρὸς τὸν ΒΓ κύκλον. ἐπεὶ οὖν ἐπίπεδον τὸ ΑΒΓ πρὸς ἐπίπεδον τὸ ΒΓ ὀρθόν ἐστι, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΒΓ ἐν ἑνὶ τῶν ἐπιπέδων τῷ ΒΓ πρὸς ὀρθὰς ἦκται ἡ ΔΕ,

is ZH. Let any point Θ be taken on ΔZE, and through Θ let ΘK be drawn parallel to ΔE. I say that ΘK intersects ZH and, if produced to the other part of the section ΔZE, it will be bisected by the straight line ZH.

For since the cone, whose vertex is the point A and base the circle ΒΓ, is cut by a plane through the axis and the section so made is the triangle ΑΒΓ, and there has been taken any point Θ on the surface, not being on a side of the triangle ΑΒΓ, and ΔH is perpendicular to ΒΓ, therefore the straight line drawn through Θ parallel to ΔH, that is ΘK, will meet the triangle ΑΒΓ and, if produced to the other part of the surface, will be bisected by the triangle [Prop. 6]. Therefore, since the straight line drawn through Θ parallel to ΔE meets the triangle ΑΒΓ and is in the plane containing the section ΔZE, it will fall upon the common section of the cutting plane and the triangle ΑΒΓ. But the common section of those planes is ZH; therefore the straight line drawn through Θ parallel to ΔE will meet ZH; and if it be produced to the other part of the section ΔZE it will be bisected by the straight line ZH.

Now the cone is right, or the axial triangle ΑΒΓ is perpendicular to the circle ΒΓ, or neither.

First, let the cone be right; then the triangle ΑΒΓ will be perpendicular to the circle ΒΓ [Def. 3; Eucl. xi. 18]. Then since the plane ΑΒΓ is perpendicular to the plane ΒΓ, and ΔE is drawn in one of the planes perpendicular to their common section ΒΓ, therefore

ἡ ΔΕ ἄρα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθάς· καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ΑΒΓ τριγώνῳ ὀρθή ἐστιν. ὥστε καὶ πρὸς τὴν ΖΗ ἐστι πρὸς ὀρθάς.

Μὴ ἔστω δὴ ὁ κῶνος ὀρθός. εἰ μὲν οὖν τὸ διὰ τοῦ ἄξονος τρίγωνον ὀρθόν ἐστι πρὸς τὸν ΒΓ κύκλον, ὁμοίως δείξομεν, ὅτι καὶ ἡ ΔΕ τῇ ΖΗ ἐστι πρὸς ὀρθάς. μὴ ἔστω δὴ τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ ὀρθὸν πρὸς τὸν ΒΓ κύκλον. λέγω, ὅτι οὐδὲ ἡ ΔΕ τῇ ΖΗ ἐστι πρὸς ὀρθάς. εἰ γὰρ δυνατόν, ἔστω· ἔστι δὲ καὶ τῇ ΒΓ πρὸς ὀρθάς· ἡ ἄρα ΔΕ ἑκατέρᾳ τῶν ΒΓ, ΖΗ ἐστι πρὸς ὀρθάς. καὶ τῷ διὰ τῶν ΒΓ, ΖΗ ἐπιπέδῳ ἄρα πρὸς ὀρθάς ἐστι. τὸ δὲ διὰ τῶν ΒΓ, ΗΖ ἐπίπεδόν ἐστι τὸ ΑΒΓ· καὶ ἡ ΔΕ ἄρα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθάς. καὶ πάντα ἄρα τὰ δι' αὐτῆς ἐπίπεδα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθάς. ἕν δέ τι τῶν διὰ τῆς ΔΕ ἐπιπέδων ἐστὶν ὁ ΒΓ κύκλος· ὁ ΒΓ ἄρα κύκλος πρὸς ὀρθάς ἐστι τῷ ΑΒΓ τριγώνῳ. ὥστε καὶ τὸ ΑΒΓ τρίγωνον ὀρθὸν ἔσται πρὸς τὸν ΒΓ κύκλον· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ἡ ΔΕ τῇ ΖΗ ἐστι πρὸς ὀρθάς.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς ΔΖΕ τομῆς διάμετρός ἐστιν ἡ ΖΗ, ἐπείπερ τὰς ἀγομένας παραλλήλους εὐθείᾳ τινὶ τῇ ΔΕ δίχα τέμνει, καὶ ὅτι δυνατόν ἐστιν ὑπὸ τῆς διαμέτρου τῆς ΖΗ παραλλήλους τινὰς δίχα τέμνεσθαι καὶ μὴ πρὸς ὀρθάς.

η´

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος,

ΔΕ is perpendicular to the triangle ΑΒΓ [Eucl. xi. Def. 4]; and therefore it is perpendicular to all the straight lines in the triangle ΑΒΓ which meet it [Eucl. xi. Def. 3]. Therefore it is perpendicular to ΖΗ.

Now let the cone be not right. Then, if the axial triangle is perpendicular to the circle ΒΓ, we may similarly show that ΔΕ is perpendicular to ΖΗ. Now let the axial triangle ΑΒΓ be not perpendicular to the circle ΒΓ. I say that neither is ΔΕ perpendicular to ΖΗ. For if it is possible, let it be; now it is also perpendicular to ΒΓ; therefore ΔΕ is perpendicular to both ΒΓ, ΖΗ. And therefore it is perpendicular to the plane through ΒΓ, ΖΗ [Eucl. xi. 4]. But the plane through ΒΓ, ΗΖ is ΑΒΓ; and therefore ΔΕ is perpendicular to the triangle ΑΒΓ. Therefore all the planes through it are perpendicular to the triangle ΑΒΓ [Eucl. xi. 18]. But one of the planes through ΔΕ is the circle ΒΓ; therefore the circle ΒΓ is perpendicular to the triangle ΑΒΓ. Therefore the triangle ΑΒΓ is perpendicular to the circle ΒΓ; which is contrary to hypothesis. Therefore ΔΕ is not perpendicular to ΖΗ.

Corollary

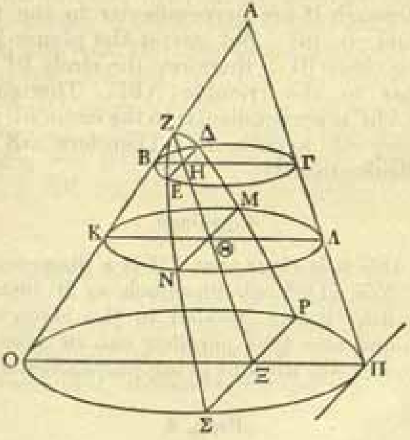
From this it is clear that ΖΗ is a diameter of the section ΔΖΕ [Def. 4], inasmuch as it bisects the straight lines drawn parallel to the given straight line ΔΕ, and also that parallels can be bisected by the diameter ΖΗ without being perpendicular to it.

Prop. 8

If a cone be cut by a plane through the axis, and it be

τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ' εὐθεῖαν πρὸς ὀρθὰς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, ἡ δὲ διάμετρος τῆς γινομένης ἐν τῇ ἐπιφανείᾳ τομῆς ἤτοι παρὰ μίαν ᾖ τῶν τοῦ τριγώνου πλευρῶν ἢ συμπίπτῃ αὐτῇ ἐκτὸς τῆς κορυφῆς τοῦ κώνου, προσεκβάλληται δὲ ἥ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον εἰς ἄπειρον, καὶ ἡ τομὴ εἰς ἄπειρον αὐξηθήσεται, καὶ ἀπὸ τῆς διαμέτρου τῆς τομῆς πρὸς τῇ κορυφῇ πάσῃ τῇ δοθείσῃ εὐθείᾳ ἴσην ἀπολήψεταί τις εὐθεῖα ἀγομένη ἀπὸ τῆς τοῦ κώνου τομῆς παρὰ τὴν ἐν τῇ βάσει τοῦ κώνου εὐθεῖαν.

Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ



διὰ τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρί-

also cut by another plane cutting the base of the cone in a line perpendicular to the base of the axial triangle, and if the diameter of the section made on the surface be either parallel to one of the sides of the triangle or meet it beyond the vertex of the cone, and if the surface of the cone and the cutting plane be produced to infinity, the section will also increase to infinity, and a straight line can be drawn from the section of the cone parallel to the straight line in the base of the cone so as to cut off from the diameter of the section towards the vertex an intercept equal to any given straight line.

Let there be a cone whose vertex is the point A and base the circle BΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle

γωνον· τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὸν ΒΓ κύκλον κατ' εὐθεῖαν τὴν ΔΕ πρὸς ὀρθὰς οὖσαν τῇ ΒΓ, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τὴν ΔΖΕ γραμμήν· ἡ δὲ διάμετρος τῆς ΔΖΕ τομῆς ἡ ΖΗ ἤτοι παράλληλος ἔστω τῇ ΑΓ ἢ ἐκβαλλομένη συμπιπτέτω αὐτῇ ἐκτὸς τοῦ Α σημείου. λέγω, ὅτι καί, ἐὰν ἥ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον ἐκβάλληται εἰς ἄπειρον, καὶ ἡ ΔΖΕ τομὴ εἰς ἄπειρον αὐξηθήσεται.

Ἐκβεβλήσθω γὰρ ἥ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον· φανερὸν δή, ὅτι καὶ αἱ ΑΒ, ΑΓ, ΖΗ συνεκβληθήσονται. ἐπεὶ ἡ ΖΗ τῇ ΑΓ ἤτοι παράλληλός ἐστιν ἢ ἐκβαλλομένη συμπίπτει αὐτῇ ἐκτὸς τοῦ Α σημείου, αἱ ΖΗ, ΑΓ ἄρα ἐκβαλλόμεναι ὡς ἐπὶ τὰ Γ, Η μέρη οὐδέποτε συμπεσοῦνται. ἐκβεβλήσθωσαν οὖν, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς ΖΗ τυχὸν τὸ Θ, καὶ διὰ τοῦ Θ σημείου τῇ μὲν ΒΓ παράλληλος ἤχθω ἡ ΚΘΛ, τῇ δὲ ΔΕ παράλληλος ἡ ΜΘΝ· τὸ ἄρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΒΓ, ΔΕ. κύκλος ἄρα ἐστὶ τὸ ΚΛΜΝ ἐπίπεδον. καὶ ἐπεὶ τὰ Δ, Ε, Μ, Ν σημεῖα ἐν τῷ τέμνοντί ἐστιν ἐπιπέδῳ, ἔστι δὲ καὶ ἐν τῇ ἐπιφανείᾳ τοῦ κώνου, ἐπὶ τῆς κοινῆς ἄρα τομῆς ἐστιν· ηὔξηται ἄρα ἡ ΔΖΕ μέχρι τῶν Μ, Ν σημείων. αὐξηθείσης ἄρα τῆς ἐπιφανείας τοῦ κώνου καὶ τοῦ τέμνοντος ἐπιπέδου μέχρι τοῦ ΚΛΜΝ κύκλου ηὔξηται καὶ ἡ ΔΖΕ τομὴ μέχρι τῶν Μ, Ν σημείων. ὁμοίως δὴ δείξομεν, ὅτι καί, ἐὰν εἰς ἄπειρον ἐκβάλληται ἥ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον, καὶ ἡ ΜΔΖΕΝ τομὴ εἰς ἄπειρον αὐξηθήσεται.

Καὶ φανερόν, ὅτι πάσῃ τῇ δοθείσῃ εὐθείᾳ ἴσην

ΑΒΓ ; now let it be cut by another plane cutting the circle ΒΓ in the straight line ΔΕ perpendicular to ΒΓ, and let the section made on the surface be the curve ΔΖΕ ; let ΖΗ, the diameter of the section ΔΖΕ, be either parallel to ΑΓ or let it, when produced, meet ΑΓ beyond the point Α. I say that if the surface of the cone and the cutting plane be produced to infinity, the section ΔΖΕ will also increase to infinity.

For let the surface of the cone and the cutting plane be produced ; it is clear that the straight lines, ΑΒ, ΑΓ, ΖΗ are simultaneously produced. Since ΖΗ is either parallel to ΑΓ or meets it, when produced, beyond the point Α, therefore ΖΗ, ΑΓ when produced in the directions Η, Γ, will never meet. Let them be produced accordingly, and let there be taken any point Θ at random upon ΖΗ, and through the point Θ let ΚΘΛ be drawn parallel to ΒΓ, and let ΜΘΝ be drawn parallel to ΔΕ ; the plane through ΚΛ, ΜΝ is therefore parallel to the plane through ΒΓ, ΔΕ [Eucl. xi. 15]. Therefore the plane ΚΛΜΝ is a circle [Prop. 4]. And since the points Δ, Ε, Μ, Ν are in the cutting plane, and are also on the surface of the cone, they are therefore upon the common section ; therefore ΔΖΕ has increased to Μ, Ν. Therefore, when the surface of the cone and the cutting plane increase up to the circle ΚΛΜΝ, the section ΔΖΕ increases up to the points Μ, Ν. Similarly we may prove that, if the surface of the cone and the cutting plane be produced to infinity, the section ΜΔΖΕΝ will increase to infinity.

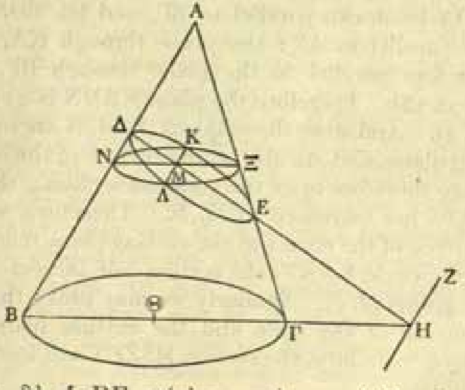
And it is clear that there can be cut off from the

ἀπολήψεταί τις ἀπὸ τῆς ΖΘ εὐθείας πρὸς τῷ Ζ σημείῳ. ἐὰν γὰρ τῇ δοθείσῃ ἴσην θῶμεν τὴν ΖΞ καὶ διὰ τοῦ Ξ τῇ ΔΕ παράλληλον ἀγάγωμεν, συμπεσεῖται τῇ τομῇ, ὥσπερ καὶ ἡ διὰ τοῦ Θ ἀπεδείχθη συμπίπτουσα τῇ τομῇ κατὰ τὰ Μ, Ν σημεῖα· ὥστε ἄγεταί τις εὐθεῖα συμπίπτουσα τῇ τομῇ παράλληλος οὖσα τῇ ΔΕ ἀπολαμβάνουσα ἀπὸ τῆς ΖΗ εὐθεῖαν ἴσην τῇ δοθείσῃ πρὸς τῷ Ζ σημείῳ.

θ′

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ συμπίπτοντι μὲν ἑκατέρᾳ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν ἠγμένῳ μήτε ὑπεναντίως, ἡ τομὴ οὐκ ἔσται κύκλος.

Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον,



βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ τινὶ μήτε παραλλήλῳ ὄντι τῇ βάσει μήτε ὑπ-

straight line ZΘ in the direction of the point Z an intercept equal to any given straight line. For if we place ZΞ equal to the given straight line and through Ξ draw a parallel to ΔE, it will meet the section, just as the parallel through Θ was shown to meet the section at the points M, N; therefore a straight line parallel to ΔE has been drawn to meet the section so as to cut off from ZH in the direction of the point Z an intercept equal to the given straight line.

Prop. 9

If a cone be cut by a plane meeting either side of the axial triangle, but neither parallel to the base nor subcontrary,[a] *the section will not be a circle.*

Let there be a cone whose vertex is the point A and base the circle BΓ, and let it be cut by a plane neither parallel to the base nor subcontrary, and let

[a] In the figure of this theorem, the section of the cone by the plane ΔE would be a *subcontrary section* (ὑπεναντία τομή) if the triangle AΔE were similar to the triangle ABΓ, but in a contrary sense, *i.e.*, if angle AΔE = angle AΓB. Apollonius proves in i. 5 that subcontrary sections of the cone are circles; it was proved in i. 4 that all sections parallel to the base are circles.

εναντίως, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τὴν ΔΚΕ γραμμήν. λέγω, ὅτι ἡ ΔΚΕ γραμμὴ οὐκ ἔσται κύκλος.

Εἰ γὰρ δυνατόν, ἔστω, καὶ συμπιπτέτω τὸ τέμνον ἐπίπεδον τῇ βάσει, καὶ ἔστω τῶν ἐπιπέδων κοινὴ τομὴ ἡ ΖΗ, τὸ δὲ κέντρον τοῦ ΒΓ κύκλου ἔστω τὸ Θ, καὶ ἀπ' αὐτοῦ κάθετος ἤχθω ἐπὶ τὴν ΖΗ ἡ ΘΗ, καὶ ἐκβεβλήσθω διὰ τῆς ΗΘ καὶ τοῦ ἄξονος ἐπίπεδον καὶ ποιείτω τομὰς ἐν τῇ κωνικῇ ἐπιφανείᾳ τὰς ΒΑ, ΑΓ εὐθείας. ἐπεὶ οὖν τὰ Δ, Ε, Η σημεῖα ἔν τε τῷ διὰ τῆς ΔΚΕ ἐπιπέδῳ ἐστίν, ἔστι δὲ καὶ ἐν τῷ διὰ τῶν Α, Β, Γ, τὰ ἄρα Δ, Ε, Η σημεῖα ἐπὶ τῆς κοινῆς τομῆς τῶν ἐπιπέδων ἐστίν· εὐθεῖα ἄρα ἐστὶν ἡ ΗΕΔ. εἰλήφθω δή τι ἐπὶ τῆς ΔΚΕ γραμμῆς σημεῖον τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΖΗ παράλληλος ἤχθω ἡ ΚΛ· ἔσται δὴ ἴση ἡ ΚΜ τῇ ΜΛ. ἡ ἄρα ΔΕ διάμετρός ἐστι τοῦ ΔΚΛΕ κύκλου. ἤχθω δὴ διὰ τοῦ Μ τῇ ΒΓ παράλληλος ἡ ΝΜΞ· ἔστι δὲ καὶ ἡ ΚΛ τῇ ΖΗ παράλληλος· ὥστε τὸ διὰ τῶν ΝΞ, ΚΜ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΒΓ, ΖΗ, τουτέστι τῇ βάσει, καὶ ἔσται ἡ τομὴ κύκλος. ἔστω ὁ ΝΚΞ. καὶ ἐπεὶ ἡ ΖΗ τῇ ΒΗ πρὸς ὀρθάς ἐστι, καὶ ἡ ΚΜ τῇ ΝΞ πρὸς ὀρθάς ἐστιν· ὥστε τὸ ὑπὸ τῶν ΝΜΞ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΚΜ. ἔστι δὲ τὸ ὑπὸ τῶν ΔΜΕ ἴσον τῷ ἀπὸ τῆς ΚΜ· κύκλος γὰρ ὑπόκειται ἡ ΔΚΕΛ γραμμή, καὶ διάμετρος αὐτοῦ ἡ ΔΕ. τὸ ἄρα ὑπὸ τῶν ΝΜΞ ἴσον ἐστὶ τῷ ὑπὸ ΔΜΕ. ἔστιν ἄρα ὡς ἡ ΜΝ πρὸς ΜΔ, οὕτως ἡ ΕΜ πρὸς ΜΞ. ὅμοιον ἄρα ἐστὶ τὸ ΔΜΝ τρίγωνον τῷ ΞΜΕ τριγώνῳ, καὶ ἡ ὑπὸ ΔΝΜ γωνία ἴση ἐστὶ τῇ ὑπὸ ΜΕΞ. ἀλλὰ

the section so made on the surface be the curve ΔΚΕ. I say that the curve ΔΚΕ will not be a circle.

For, if possible, let it be, and let the cutting plane meet the base, and let the common section of the planes be ΖΗ, and let the centre of the circle ΒΓ be Θ, and from it let ΘΗ be drawn perpendicular to ΖΗ, and let the plane through ΗΘ and the axis be produced, and let the sections made on the conical surface be the straight lines ΒΑ, ΑΓ. Then since the points Δ, Ε, Η are in the plane through ΔΚΕ, and are also in the plane through Α, Β, Γ, therefore the points Δ, Ε, Η are on the common section of the planes; therefore ΗΕΔ is a straight line [Eucl. xi. 3]. Now let there be taken any point Κ on the curve ΔΚΕ, and through Κ let ΚΛ be drawn parallel to ΖΗ; then ΚΜ will be equal to ΜΛ [Prop. 7]. Therefore ΔΕ is a diameter of the circle ΔΚΕΛ [Prop. 7, coroll.]. Now let ΝΜΞ be drawn through Μ parallel to ΒΓ; but ΚΛ is parallel to ΖΗ; therefore the plane through ΝΞ, ΚΜ is parallel to the plane through ΒΓ, ΖΗ [Eucl. xi. 15], that is to the base, and the section will be a circle [Prop. 4]. Let it be ΝΚΞ. And since ΖΗ is perpendicular to ΒΗ, ΚΜ is also perpendicular to ΝΞ [Eucl. xi. 10]; therefore $ΝΜ . ΜΞ = ΚΜ^2$. But $ΔΜ . ΜΕ = ΚΜ^2$; for the curve ΔΚΕΛ is by hypothesis a circle, and ΔΕ is a diameter in it. Therefore $ΝΜ . ΜΞ = ΔΜ . ΜΕ$. Therefore $ΜΝ : ΜΔ = ΕΜ : ΜΞ$. Therefore the triangle ΔΜΝ is similar to the triangle ΞΜΕ, and the angle ΔΝΜ is equal to the angle ΜΕΞ.

ἡ ὑπὸ ΔΝΜ γωνία τῇ ὑπὸ ΑΒΓ ἐστιν ἴση· παράλληλος γὰρ ἡ ΝΞ τῇ ΒΓ· καὶ ἡ ὑπὸ ΑΒΓ ἄρα ἴση ἐστὶ τῇ ὑπὸ ΜΕΞ. ὑπεναντία ἄρα ἐστὶν ἡ τομή· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα κύκλος ἐστὶν ἡ ΔΚΕ γραμμή.

(v.) *Fundamental Properties*

Ibid., Props. 11-14, Apoll. Perg. ed. Heiberg i. 36. 26–58. 7

ια΄

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ᾽ εὐθεῖαν πρὸς ὀρθὰς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, ἔτι δὲ ἡ διάμετρος τῆς τομῆς παράλληλος ᾖ μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, ἥτις ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου μέχρι τῆς διαμέτρου τῆς τομῆς, δυνήσεται τὸ περιεχόμενον ὑπό τε τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς καὶ ἄλλης τινὸς εὐθείας, ἣ λόγον ἔχει πρὸς τὴν μεταξὺ τῆς τοῦ κώνου γωνίας καὶ τῆς κορυφῆς τῆς τομῆς, ὃν τὸ τετράγωνον τὸ ἀπὸ τῆς βάσεως τοῦ διὰ τοῦ ἄξονος τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν· καλείσθω δὲ ἡ τοιαύτη τομὴ παραβολή.

Ἔστω κῶνος, οὗ τὸ Α σημεῖον κορυφή, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ᾽ εὐθεῖαν τὴν ΔΕ πρὸς ὀρθὰς

But the angle ΔNM is equal to the angle ABΓ; for NΞ is parallel to BΓ; and therefore the angle ABΓ is equal to the angle MEΞ. Therefore the section is subcontrary [Prop. 5]; which is contrary to hypothesis. Therefore the curve ΔKE is not a circle.

(v.) *Fundamental Properties*

Ibid., Props. 11-14, Apoll. Perg. ed. Heiberg i. 36. 26–58. 7

Prop. 11

Let a cone be cut by a plane through the axis, and let it be also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and further let the diameter of the section be parallel to one side of the axial triangle; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the rectangle bounded by the intercept made by it on the diameter in the direction of the vertex of the section and a certain other straight line; this straight line will bear the same ratio to the intercept between the angle of the cone and the vertex of the segment as the square on the base of the axial triangle bears to the rectangle bounded by the remaining two sides of the triangle; and let such a section be called a parabola.

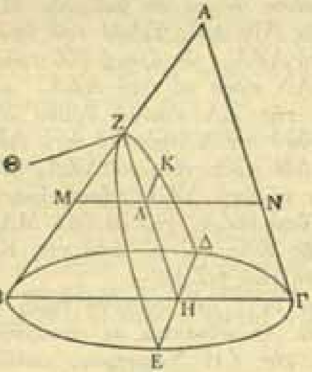
For let there be a cone whose vertex is the point A and whose base is the circle BΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle ABΓ, and let it be cut by another plane cutting the base of the cone in the straight line

οὖσαν τῇ ΒΓ, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ, ἡ δὲ διάμετρος τῆς τομῆς ἡ ΖΗ παράλληλος ἔστω μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου τῇ ΑΓ, καὶ ἀπὸ τοῦ Ζ σημείου τῇ ΖΗ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΖΘ, καὶ πεποιήσθω, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ, οὕτως ἡ ΖΘ πρὸς ΖΑ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τυχὸν τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΔΕ παράλληλος ἡ ΚΛ. λέγω, ὅτι τὸ ἀπὸ τῆς ΚΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΘΖΛ.

Ἤχθω γὰρ διὰ τοῦ Λ τῇ ΒΓ παράλληλος ἡ ΜΝ· ἔστι δὲ καὶ ἡ ΚΛ τῇ ΔΕ παράλληλος· τὸ ἄρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΒΓ, ΔΕ ἐπιπέδῳ, τουτέστι τῇ βάσει τοῦ κώνου. τὸ ἄρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον κύκλος ἐστίν, οὗ διάμετρος ἡ ΜΝ. καὶ ἔστι κάθετος ἐπὶ τὴν ΜΝ ἡ ΚΛ, ἐπεὶ καὶ ἡ ΔΕ ἐπὶ τὴν ΒΓ· τὸ ἄρα ὑπὸ τῶν ΜΛΝ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ τῶν ΒΑΓ, οὕτως ἡ ΘΖ πρὸς ΖΑ, τὸ δὲ

ΔE perpendicular to BΓ, and let the section so made on the surface of the cone be ΔZE, and let ZH, the diameter of the section, be parallel to AΓ, one side of the axial triangle, and from the point Z let ZΘ be drawn perpendicular to ZH, and let $B\Gamma^2 : BA \cdot A\Gamma = Z\Theta : ZA$, and let any point K be taken at random on the section, and through K let KΛ be drawn parallel to ΔE. I say that $K\Lambda^2 = \Theta Z \cdot Z\Lambda$.

For let MN be drawn through Λ parallel to BΓ; but KΛ is parallel to ΔE; therefore the plane through



KΛ, MN is parallel to the plane through BΓ, ΔE [Eucl. xi. 15], that is to the base of the cone. Therefore the plane through KΛ, MN is a circle, whose diameter is MN [Prop. 4]. And KΛ is perpendicular to MN, since ΔE is perpendicular to BΓ [Eucl. xi. 10]; therefore $M\Lambda \cdot \Lambda N = K\Lambda^2$.

And since $B\Gamma^2 : BA \cdot A\Gamma = \Theta Z : ZA$,

ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ τῶν ΒΑΓ λόγον ἔχει τὸν συγκείμενον ἔκ τε τοῦ, ὃν ἔχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ, ὁ ἄρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΒΓ πρὸς ΓΑ καὶ τοῦ τῆς ΓΒ πρὸς ΒΑ. ἀλλ' ὡς μὲν ἡ ΒΓ πρὸς ΓΑ, οὕτως ἡ ΜΝ πρὸς ΝΑ, τουτέστιν ἡ ΜΛ πρὸς ΛΖ, ὡς δὲ ἡ ΒΓ πρὸς ΒΑ, οὕτως ἡ ΜΝ πρὸς ΜΑ, τουτέστιν ἡ ΛΜ πρὸς ΜΖ, καὶ λοιπὴ ἡ ΝΛ πρὸς ΖΑ. ὁ ἄρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΜΛ πρὸς ΛΖ καὶ τοῦ τῆς ΝΛ πρὸς ΖΑ. ὁ δὲ συγκείμενος λόγος ἐκ τοῦ τῆς ΜΛ πρὸς ΛΖ καὶ τοῦ τῆς ΛΝ πρὸς ΖΑ ὁ τοῦ ὑπὸ ΜΛΝ ἐστι πρὸς τὸ ὑπὸ ΛΖΑ. ὡς ἄρα ἡ ΘΖ πρὸς ΖΑ, οὕτως τὸ ὑπὸ ΜΛΝ πρὸς τὸ ὑπὸ ΛΖΑ. ὡς δὲ ἡ ΘΖ πρὸς ΖΑ, τῆς ΖΛ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ ΘΖΛ πρὸς τὸ ὑπὸ ΛΖΑ· ὡς ἄρα τὸ ὑπὸ ΜΛΝ πρὸς τὸ ὑπὸ ΛΖΑ, οὕτως τὸ ὑπὸ ΘΖΛ πρὸς τὸ ὑπὸ ΛΖΑ. ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΜΛΝ τῷ ὑπὸ ΘΖΛ. τὸ δὲ ὑπὸ ΜΛΝ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ· καὶ τὸ ἀπὸ τῆς ΚΛ ἄρα ἴσον ἐστὶ τῷ ὑπὸ τῶν ΘΖΛ.

Καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ παραβολή, ἡ δὲ ΘΖ παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν ΖΗ διάμετρον, καλείσθω δὲ καὶ ὀρθία.

ιβ'

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν

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[a] A *parabola* (παραβολή) because the square on the ordinate ΚΛ is *applied* (παραβαλεῖν) to the parameter ΘΖ in the form

while $\quad B\Gamma^2 : BA \,.\, A\Gamma = (B\Gamma : \Gamma A)(B\Gamma : BA)$,

therefore $\quad \Theta Z : Z A = (B\Gamma : \Gamma A)(\Gamma B : BA)$.

But $\quad B\Gamma : \Gamma A = MN : N A$

$\quad = M\Lambda : \Lambda Z$, [Eucl. vi. 4

and $\quad B\Gamma : BA = MN : MA$

$\quad = \Lambda M : MZ$ [*ibid.*

$\quad = N\Lambda : Z\Lambda$. [Eucl. vi. 2

Therefore $\quad \Theta Z : Z A = (M\Lambda : \Lambda Z)(N\Lambda : Z\Lambda)$.

But $\quad (M\Lambda : \Lambda Z)(\Lambda N : Z\Lambda) = M\Lambda \,.\, \Lambda N : \Lambda Z \,.\, Z\Lambda$.

Therefore $\quad \Theta Z : Z A = M\Lambda \,.\, \Lambda N : \Lambda Z \,.\, Z\Lambda$.

But $\quad \Theta Z : Z A = \Theta Z \,.\, Z\Lambda : \Lambda Z \,.\, Z\Lambda$,

by taking a common height $Z\Lambda$;

therefore $\quad M\Lambda \,.\, \Lambda N : \Lambda Z \,.\, Z\Lambda = \Theta Z \,.\, Z\Lambda : \Lambda Z \,.\, Z\Lambda$.

Therefore $\quad M\Lambda \,.\, \Lambda N = \Theta Z \,.\, Z\Lambda$. [Eucl. v. 9

But $\quad M\Lambda \,.\, \Lambda N = K\Lambda^2$;

and therefore $\quad K\Lambda^2 = \Theta Z \,.\, Z\Lambda$.

Let such a section be called a *parabola*, and let ΘZ be called the *parameter of the ordinates* to the diameter ZH, and let it also be called the *erect side* (*latus rectum*).[a]

Prop. 12

Let a cone be cut by a plane through the axis, and let it be cut by another plane cutting the base of the cone in

of the rectangle ΘZ . ZΛ, and is exactly equal to this rectangle. It was Apollonius's most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the *application of areas* (παραβολὴ τῶν χωρίων), for which v. vol. i. pp. 186-215. The explanation of the term *latus rectum* will become more obvious in the cases of the hyperbola and the ellipse; v. *infra*, p. 317 n. *a*.

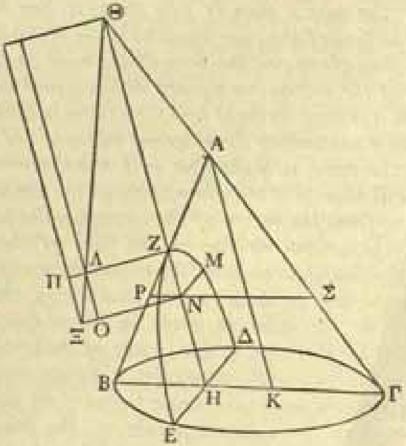
τοῦ κώνου κατ' εὐθεῖαν πρὸς ὀρθὰς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, καὶ ἡ διάμετρος τῆς τομῆς ἐκβαλλομένη συμπίπτῃ μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς τοῦ κώνου κορυφῆς, ἥτις ἂν ἀπὸ τῆς τομῆς ἀχθῇ παράλληλος τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου, ἕως τῆς διαμέτρου τῆς τομῆς δυνήσεταί τι χωρίον παρακείμενον παρά τινα εὐθεῖαν, πρὸς ἣν λόγον ἔχει ἡ ἐπ' εὐθείας μὲν οὖσα τῇ διαμέτρῳ τῆς τομῆς, ὑποτείνουσα δὲ τὴν ἐκτὸς τοῦ τριγώνου γωνίαν, ὃν τὸ τετράγωνον τὸ ἀπὸ τῆς ἠγμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διάμετρον τῆς τομῆς ἕως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν τῆς βάσεως τμημάτων, ὧν ποιεῖ ἡ ἀχθεῖσα, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ὑπερβάλλον εἴδει ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ περιεχομένῳ ὑπό τε τῆς ὑποτεινούσης τὴν ἐκτὸς γωνίαν τοῦ τριγώνου καὶ τῆς παρ' ἣν δύνανται αἱ καταγόμεναι· καλείσθω δὲ ἡ τοιαύτη τομὴ ὑπερβολή.

Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ' εὐθεῖαν τὴν ΔΕ πρὸς ὀρθὰς οὖσαν τῇ ΒΓ βάσει τοῦ ΑΒΓ τριγώνου, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ γραμμήν, ἡ δὲ διάμετρος τῆς τομῆς ἡ ΖΗ ἐκβαλλομένη συμπιπτέτω μιᾷ πλευρᾷ τοῦ ΑΒΓ τριγώνου τῇ ΑΓ ἐκτὸς τῆς τοῦ κώνου κορυφῆς κατὰ τὸ Θ, καὶ διὰ τοῦ Α τῇ διαμέτρῳ τῆς τομῆς

a straight line perpendicular to the base of the axial triangle, and let the diameter of the section, when produced, meet one side of the axial triangle beyond the vertex of the cone; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the area applied to a certain straight line; this line is such that the straight line subtending the external angle of the triangle, lying in the same straight line with the diameter of the section, will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle bounded by the segments of the base made by the line so drawn; the breadth of the applied figure will be the intercept made by the ordinate on the diameter in the direction of the vertex of the section; and the applied figure will exceed by a figure similar and similarly situated to the rectangle bounded by the straight line subtending the external angle of the triangle and the parameter of the ordinates; and let such a section be called a hyperbola.

Let there be a cone whose vertex is the point A and whose base is the circle BΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle ABΓ, and let it be cut by another plane cutting the base of the cone in the straight line ΔE perpendicular to BΓ, the base of the triangle ABΓ, and let the section so made on the surface of the cone be the curve ΔZE, and let ZH, the diameter of the section, when produced, meet AΓ, one side of the triangle ABΓ, beyond the vertex of the cone at Θ, and through A let AK be drawn parallel to ZH, the

τῇ ΖΗ παράλληλος ἤχθω ἡ ΑΚ, καὶ τεμνέτω τὴν ΒΓ, καὶ ἀπὸ τοῦ Ζ τῇ ΖΗ πρὸς ὀρθὰς ἤχθω ἡ



ΖΛ, καὶ πεποιήσθω, ὡς τὸ ἀπὸ ΚΑ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΖΘ πρὸς ΖΛ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τυχὸν τὸ Μ, καὶ διὰ τοῦ Μ τῇ ΔΕ παράλληλος ἤχθω ἡ ΜΝ, διὰ δὲ τοῦ Ν τῇ ΖΛ παράλληλος ἡ ΝΟΞ, καὶ ἐπιζευχθεῖσα ἡ ΘΛ ἐκβεβλήσθω ἐπὶ τὸ Ξ, καὶ διὰ τῶν Λ, Ξ τῇ ΖΝ παράλληλοι ἤχθωσαν αἱ ΛΟ, ΞΠ. λέγω, ὅτι ἡ ΜΝ δύναται τὸ ΖΞ, ὃ παράκειται παρὰ τὴν ΖΛ, πλάτος ἔχον τὴν ΖΝ, ὑπερβάλλον εἴδει τῷ ΛΞ ὁμοίῳ ὄντι τῷ ὑπὸ τῶν ΘΖΛ.

Ἤχθω γὰρ διὰ τοῦ Ν τῇ ΒΓ παράλληλος ἡ ΡΝΣ· ἔστι δὲ καὶ ἡ ΝΜ τῇ ΔΕ παράλληλος· τὸ

diameter of the section, and let it cut $B\Gamma$, and from Z let $Z\Lambda$ be drawn perpendicular to ZH, and let $K A^2 : BK . K\Gamma = Z\Theta : Z\Lambda$, and let there be taken at random any point M on the section, and through M let MN be drawn parallel to ΔE, and through N let $NO\Xi$ be drawn parallel to $Z\Lambda$, and let $\Theta\Lambda$ be joined and produced to Ξ, and through Λ, Ξ, let ΛO, $\Xi\Pi$ be drawn parallel to ZN. I say that the square on MN is equal to $Z\Xi$, which is applied to the straight line $Z\Lambda$, having ZN for its breadth, and exceeding by the figure $\Lambda\Xi$ which is similar to the rectangle contained by ΘZ, $Z\Lambda$.

For let $PN\Sigma$ be drawn through N parallel to $B\Gamma$; but NM is parallel to ΔE; therefore the plane through

ἄρα διὰ τῶν ΜΝ, ΡΣ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΒΓ, ΔΕ, τουτέστι τῇ βάσει τοῦ κώνου. ἐὰν ἄρα ἐκβληθῇ τὸ διὰ τῶν ΜΝ, ΡΣ ἐπίπεδον, ἡ τομὴ κύκλος ἔσται, οὗ διάμετρος ἡ ΡΝΣ. καὶ ἔστιν ἐπ' αὐτὴν κάθετος ἡ ΜΝ· τὸ ἄρα ὑπὸ τῶν ΡΝΣ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΖΘ πρὸς ΖΛ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ἡ ΑΚ πρὸς ΚΓ καὶ ἡ ΑΚ πρὸς ΚΒ, καὶ ὁ τῆς ΖΘ ἄρα πρὸς τὴν ΖΛ λόγος σύγκειται ἐκ τοῦ, ὃν ἔχει ἡ ΑΚ πρὸς ΚΓ καὶ ἡ ΑΚ πρὸς ΚΒ. ἀλλ' ὡς μὲν ἡ ΑΚ πρὸς ΚΓ, οὕτως ἡ ΘΗ πρὸς ΗΓ, τουτέστιν ἡ ΘΝ πρὸς ΝΣ, ὡς δὲ ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΖΗ πρὸς ΗΒ, τουτέστιν ἡ ΖΝ πρὸς ΝΡ. ὁ ἄρα τῆς ΘΖ πρὸς ΖΛ λόγος σύγκειται ἔκ τε τοῦ τῆς ΘΝ πρὸς ΝΣ καὶ τοῦ τῆς ΖΝ πρὸς ΝΡ. ὁ δὲ συγκείμενος λόγος ἐκ τοῦ τῆς ΘΝ πρὸς ΝΣ καὶ τοῦ τῆς ΖΝ πρὸς ΝΡ ὁ τοῦ ὑπὸ τῶν ΘΝΖ ἐστι πρὸς τὸ ὑπὸ τῶν ΣΝΡ· καὶ ὡς ἄρα τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΡ, οὕτως ἡ ΘΖ πρὸς ΖΛ, τουτέστιν ἡ ΘΝ πρὸς ΝΞ. ἀλλ' ὡς ἡ ΘΝ πρὸς ΝΞ, τῆς ΖΝ κοινοῦ ὕψους λαμβανομένης οὕτως τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΖΝΞ. καὶ ὡς ἄρα τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΡ, οὕτως τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΞΝΖ. τὸ ἄρα ὑπὸ ΣΝΡ ἴσον ἐστὶ τῷ ὑπὸ ΞΝΖ. τὸ δὲ ἀπὸ ΜΝ ἴσον ἐδείχθη τῷ ὑπὸ ΣΝΡ· καὶ τὸ ἀπὸ τῆς ΜΝ ἄρα ἴσον ἐστὶ τῷ ὑπὸ τῶν ΞΝΖ. τὸ δὲ ὑπὸ ΞΝΖ ἐστι τὸ ΞΖ παραλληλόγραμμον. ἡ ἄρα

MN, PΣ is parallel to the plane through BΓ, ΔE [Eucl. xi. 15], that is to the base of the cone. If, then, the plane through MN, PΣ be produced, the section will be a circle with diameter PNΣ [Prop. 4]. And MN is perpendicular to it; therefore

$$\mathrm{PN} \cdot \mathrm{N\Sigma} = \mathrm{MN}^2.$$

And since $\mathrm{AK}^2 : \mathrm{BK} \cdot \mathrm{K\Gamma} = \mathrm{Z\Theta} : \mathrm{Z\Lambda}$,

while $\mathrm{AK}^2 : \mathrm{BK} \cdot \mathrm{K\Gamma} = (\mathrm{AK} : \mathrm{K\Gamma})(\mathrm{AK} : \mathrm{KB})$,

therefore $\mathrm{Z\Theta} : \mathrm{Z\Lambda} = (\mathrm{AK} : \mathrm{K\Gamma})(\mathrm{AK} : \mathrm{KB})$.

But $\mathrm{AK} : \mathrm{K\Gamma} = \Theta\mathrm{H} : \mathrm{H\Gamma}$,

i.e., $= \Theta\mathrm{N} : \mathrm{N\Sigma}$, [Eucl. vi. 4

and $\mathrm{AK} : \mathrm{KB} = \mathrm{ZH} : \mathrm{HB}$,

i.e., $= \mathrm{ZN} : \mathrm{NP}$. [*ibid.*

Therefore $\Theta\mathrm{Z} : \mathrm{Z\Lambda} = (\Theta\mathrm{N} : \mathrm{N\Sigma})(\mathrm{ZN} : \mathrm{NP})$.

But $(\Theta\mathrm{N} : \mathrm{N\Sigma})(\mathrm{ZN} : \mathrm{NP}) = \Theta\mathrm{N} \cdot \mathrm{NZ} : \Sigma\mathrm{N} \cdot \mathrm{NP}$;

and therefore

$$\Theta\mathrm{N} \cdot \mathrm{NZ} : \Sigma\mathrm{N} \cdot \mathrm{NP} = \Theta\mathrm{Z} : \mathrm{Z\Lambda}$$

$= \Theta\mathrm{N} : \mathrm{N\Xi}$. [*ibid.*

But $\Theta\mathrm{N} : \mathrm{N\Xi} = \Theta\mathrm{N} \cdot \mathrm{NZ} : \mathrm{ZN} \cdot \mathrm{N\Xi}$,

by taking a common height ZN.

And therefore

$$\Theta\mathrm{N} \cdot \mathrm{NZ} : \Sigma\mathrm{N} \cdot \mathrm{NP} = \Theta\mathrm{N} \cdot \mathrm{NZ} : \Xi\mathrm{N} \cdot \mathrm{NZ}.$$

Therefore $\Sigma\mathrm{N} \cdot \mathrm{NP} = \Xi\mathrm{N} \cdot \mathrm{NZ}$. [Eucl. v. 9

But $\mathrm{MN}^2 = \Sigma\mathrm{N} \cdot \mathrm{NP}$,

as was proved;

and therefore $\mathrm{MN}^2 = \Xi\mathrm{N} \cdot \mathrm{NZ}$.

But the rectangle ΞN . NZ is the parallelogram ΞZ.

ΜΝ δύναται τὸ ΞΖ, ὃ παράκειται παρὰ τὴν ΖΛ, πλάτος ἔχον τὴν ΖΝ, ὑπερβάλλον τῷ ΛΞ ὁμοίῳ ὄντι τῷ ὑπὸ τῶν ΘΖΛ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ὑπερβολή, ἡ δὲ ΛΖ παρ' ἣν δύνανται αἱ ἐπὶ τὴν ΖΗ καταγόμεναι τεταγμένως· καλείσθω δὲ ἡ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΖΘ.

ιγ'

Ἐὰν κῶνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἑκατέρᾳ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν τοῦ κώνου ἠγμένῳ μήτε ὑπεναντίως, τὸ δὲ ἐπίπεδον, ἐν ᾧ ἐστιν ἡ βάσις τοῦ κώνου, καὶ τὸ τέμνον ἐπίπεδον συμπίπτῃ κατ' εὐθεῖαν πρὸς ὀρθὰς οὖσαν ἤτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἢ τῇ ἐπ' εὐθείας αὐτῇ, ἥτις ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τῶν ἐπιπέδων ἕως τῆς διαμέτρου τῆς τομῆς, δυνήσεταί τι χωρίον παρακείμενον παρά τινα εὐθεῖαν, πρὸς ἣν λόγον ἔχει ἡ διάμετρος τῆς τομῆς, ὃν τὸ τετράγωνον τὸ ἀπὸ τῆς ἠγμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διάμετρον τῆς τομῆς ἕως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν ἀπολαμβανομένων ὑπ' αὐτῆς πρὸς ταῖς τοῦ τριγώνου εὐθείαις, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ἐλλεῖπον εἴδει ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ περιεχομένῳ ὑπό τε τῆς διαμέτρου καὶ τῆς παρ' ἣν δύναται· καλείσθω δὲ ἡ τοιαύτη τομὴ ἔλλειψις.

Ἔστω κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον,

Therefore the square on MN is equal to ΞZ, which is applied to ZΛ, having ZN for its breadth, and exceeding by ΛΞ similar to the rectangle contained by ΘZ, ZΛ. Let such a section be called a *hyperbola*, let ΛZ be called *the parameter to the ordinates* to ZH; and let this line be also called the *erect side (latus rectum)*, and ZΘ the *transverse side*.[a]

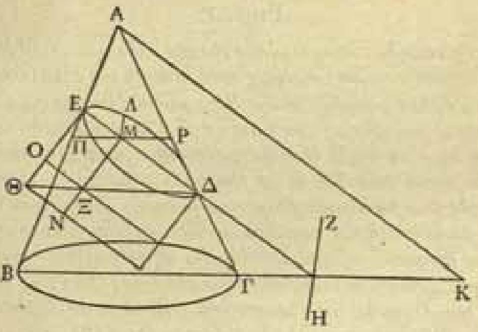
Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter; and let such a section be called an ellipse.

Let there be a cone, whose vertex is the point A

[a] The *erect* and *transverse* side, that is to say, of the figure (εἶδος) applied to the diameter. In the case of the parabola, the transverse side is infinite.

βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιείτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἑκατέρᾳ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παραλλήλῳ τῇ βάσει τοῦ κώνου μήτε ὑπεναντίως ἠγμένῳ, καὶ ποιείτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΕ γραμμήν· κοινὴ



δὲ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ, ἐν ᾧ ἐστιν ἡ βάσις τοῦ κώνου, ἔστω ἡ ΖΗ πρὸς ὀρθὰς οὖσα τῇ ΒΓ, ἡ δὲ διάμετρος τῆς τομῆς ἔστω ἡ ΕΔ, καὶ ἀπὸ τοῦ Ε τῇ ΕΔ πρὸς ὀρθὰς ἤχθω ἡ ΕΘ, καὶ διὰ τοῦ Α τῇ ΕΔ παράλληλος ἤχθω ἡ ΑΚ, καὶ πεποιήσθω ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τὸ Λ, καὶ διὰ τοῦ Λ τῇ ΖΗ παράλληλος ἤχθω ἡ ΛΜ. λέγω, ὅτι ἡ ΛΜ δύναταί τι χωρίον, ὃ παράκειται παρὰ τὴν ΕΘ, πλάτος ἔχον τὴν ΕΜ, ἐλλεῖπον εἴδει ὁμοίῳ τῷ ὑπὸ τῶν ΔΕΘ.

Ἐπεζεύχθω γὰρ ἡ ΔΘ, καὶ διὰ μὲν τοῦ Μ τῇ

and whose base is the circle ΒΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle ΑΒΓ, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve ΔΕ; let the common section of the cutting plane and of that containing the base of the cone be ΖΗ, perpendicular to ΒΓ, and let the diameter of the section be ΕΔ, and from Ε let ΕΘ be drawn perpendicular to ΕΔ, and through Α let ΑΚ be drawn parallel to ΕΔ, and let $\text{ΑΚ}^2 : \text{ΒΚ} \,.\, \text{ΚΓ} = \text{ΔΕ} : \text{ΕΘ}$, and let any point Λ be taken on the section, and through Λ let ΛΜ be drawn parallel to ΖΗ. I say that the square on ΛΜ is equal to an area applied to the straight line ΕΘ, having ΕΜ for its breadth, and being deficient by a figure similar to the rectangle contained by ΔΕ, ΕΘ.

For let ΔΘ be joined, and through Μ let ΜΞΝ be

ΘΕ παράλληλος ἤχθω ἡ ΜΞΝ, διὰ δὲ τῶν Θ, Ξ τῇ ΕΜ παράλληλοι ἤχθωσαν αἱ ΘΝ, ΞΟ, καὶ διὰ τοῦ Μ τῇ ΒΓ παράλληλος ἤχθω ἡ ΠΜΡ. ἐπεὶ οὖν ἡ ΠΡ τῇ ΒΓ παράλληλός ἐστιν, ἔστι δὲ καὶ ἡ ΛΜ τῇ ΖΗ παράλληλος, τὸ ἄρα διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον παράλληλόν ἐστι τῷ διὰ τῶν ΖΗ, ΒΓ ἐπιπέδῳ, τουτέστι τῇ βάσει τοῦ κώνου. ἐὰν ἄρα ἐκβληθῇ διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον, ἡ τομὴ κύκλος ἔσται, οὗ διάμετρος ἡ ΠΡ. καί ἐστι κάθετος ἐπ᾽ αὐτὴν ἡ ΛΜ· τὸ ἄρα ὑπὸ τῶν ΠΜΡ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΛΜ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ, οὕτως ἡ ΕΔ πρὸς τὴν ΕΘ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ λόγος σύγκειται ἐκ τοῦ, ὃν ἔχει ἡ ΑΚ πρὸς ΚΒ, καὶ ἡ ΑΚ πρὸς ΚΓ, ἀλλ᾽ ὡς μὲν ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΕΗ πρὸς ΗΒ, τουτέστιν ἡ ΕΜ πρὸς ΜΠ, ὡς δὲ ἡ ΑΚ πρὸς ΚΓ, οὕτως ἡ ΔΗ πρὸς ΗΓ, τουτέστιν ἡ ΔΜ πρὸς ΜΡ, ὁ ἄρα τῆς ΔΕ πρὸς τὴν ΕΘ λόγος σύγκειται ἔκ τε τοῦ τῆς ΕΜ πρὸς ΜΠ καὶ τοῦ τῆς ΔΜ πρὸς ΜΡ. ὁ δὲ συγκείμενος λόγος ἔκ τε τοῦ, ὃν ἔχει ἡ ΕΜ πρὸς ΜΠ, καὶ ἡ ΔΜ πρὸς ΜΡ, ὁ τοῦ ὑπὸ τῶν ΕΜΔ ἐστι πρὸς τὸ ὑπὸ τῶν ΠΜΡ. ἔστιν ἄρα ὡς τὸ ὑπὸ τῶν ΕΜΔ πρὸς τὸ ὑπὸ τῶν ΠΜΡ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, τουτέστιν ἡ ΔΜ πρὸς τὴν ΜΞ. ὡς δὲ ἡ ΔΜ πρὸς ΜΞ, τῆς ΜΕ κοινοῦ ὕψους λαμβανομένης, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΞΜΕ. καὶ ὡς ἄρα τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΠΜΡ, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΞΜΕ. ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΠΜΡ τῷ ὑπὸ ΞΜΕ. τὸ δὲ ὑπὸ ΠΜΡ ἴσον ἐδείχθη τῷ ἀπὸ τῆς ΛΜ· καὶ τὸ ὑπὸ ΞΜΕ ἄρα ἐστὶν ἴσον τῷ ἀπὸ τῆς ΛΜ. ἡ ΛΜ

drawn parallel to ΘE, and through Θ, Ξ, let ΘN, ΞO be drawn parallel to EM, and through M let ΠMP be drawn parallel to BΓ. Then since ΠP is parallel to BΓ, and ΛM is parallel to ZH, therefore the plane through ΛM, ΠP is parallel to the plane through ZH, BΓ [Eucl. xi. 15], that is to the base of the cone. If, therefore, the plane through ΛM, ΠP be produced, the section will be a circle with diameter ΠP [Prop. 4]. And ΛM is perpendicular to it; therefore

$$\Pi M \cdot MP = \Lambda M^2.$$

And since $\quad A K^2 : BK \cdot K\Gamma = E\Delta : E\Theta,$

and $\quad AK^2 : BK \cdot K\Gamma = (AK : KB)(AK : K\Gamma),$

while $\quad AK : KB = EH : HB$

$\quad = EM : M\Pi,$ [Eucl. vi. 4

and $\quad AK : K\Gamma = \Delta H : H\Gamma$

$\quad = \Delta M : MP,$ [*ibid.*

therefore $\quad \Delta E : E\Theta = (EM : M\Pi)(\Delta M : MP).$

But $(EM : M\Pi)(\Delta M : MP) \quad = EM \cdot M\Delta : \Pi M \cdot MP.$

Therefore

$$EM \cdot M\Delta : \Pi M \cdot MP = \Delta E : E\Theta$$

$\quad = \Delta M : M\Xi.$ [*ibid.*

But $\quad \Delta M : M\Xi = \Delta M \cdot ME : \Xi M \cdot ME,$

by taking a common height ME.

Therefore $\Delta M \cdot ME : \Pi M \cdot MP = \Delta M \cdot ME : \Xi M \cdot ME.$

Therefore $\quad \Pi M \cdot MP = \Xi M \cdot ME.$ [Eucl. v. 9

But $\quad \Pi M \cdot MP = \Lambda M^2,$

as was proved;

and therefore $\quad \Xi M \cdot ME = \Lambda M^2.$

ἄρα δύναται τὸ ΜΟ, ὃ παράκειται παρὰ τὴν ΘΕ, πλάτος ἔχον τὴν ΕΜ, ἐλλεῖπον εἴδει τῷ ΟΝ ὁμοίῳ ὄντι τῷ ὑπὸ ΔΕΘ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ἔλλειψις, ἡ δὲ ΕΘ παρ' ἣν δύνανται αἱ καταγόμεναι ἐπὶ τὴν ΔΕ τεταγμένως, ἡ δὲ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΕΔ.

ιδ'

Ἐὰν αἱ κατὰ κορυφὴν ἐπιφάνειαι ἐπιπέδῳ τμηθῶσι μὴ διὰ τῆς κορυφῆς, ἔσται ἐν ἑκατέρᾳ τῶν ἐπιφανειῶν τομὴ ἡ καλουμένη ὑπερβολή, καὶ τῶν δύο τομῶν ἥ τε διάμετρος ἡ αὐτὴ ἔσται, καὶ παρ' ἃς δύνανται αἱ ἐπὶ τὴν διάμετρον καταγόμεναι παράλληλοι τῇ ἐν τῇ βάσει τοῦ κώνου εὐθείᾳ ἴσαι, καὶ τοῦ εἴδους ἡ πλαγία πλευρὰ κοινὴ ἡ μεταξὺ τῶν κορυφῶν τῶν τομῶν· καλείσθωσαν δὲ αἱ τοιαῦται τομαὶ ἀντικείμεναι.

Ἔστωσαν αἱ κατὰ κορυφὴν ἐπιφάνειαι, ὧν κορυφὴ τὸ Α σημεῖον, καὶ τετμήσθωσαν ἐπιπέδῳ μὴ διὰ τῆς κορυφῆς, καὶ ποιείτω ἐν τῇ ἐπιφανείᾳ τομὰς τὰς ΔΕΖ, ΗΘΚ. λέγω, ὅτι ἑκατέρα τῶν ΔΕΖ, ΗΘΚ τομῶν ἐστιν ἡ καλουμένη ὑπερβολή.

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* Let p be the parameter of a conic section and d the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of co-ordinates (in general oblique). Then Props. 11-13 are equivalent to the Cartesian equations.

Therefore the square on ΛM is equal to MO, which is applied to ΘE, having EM for its breadth, and being deficient by the figure ON similar to the rectangle ΔE . EΘ. Let such a section be called an *eclipse*, let EΘ be called *the parameter to the ordinates* to ΔE, and let this line be called the *erect side* (*latus rectum*), and EΔ the *transverse side*.[a]

Prop. 14

If the vertically opposite surfaces [of a double cone] be cut by a plane not through the vertex, there will be formed on each of the surfaces the section called a hyperbola, and the diameter of both sections will be the same; and the parameter to the ordinates drawn parallel to the straight line in the base of the cone will be equal, and the transverse side of the figure will be common, being the straight line between the vertices of the sections; and let such sections be called opposite.

Let there be vertically opposite surfaces having the point A for vertex, and let them be cut by a plane not through the vertex, and let the sections so made on the surface be ΔEZ, HΘK. I say that each of the sections ΔEZ, HΘK is the so-called hyperbola.

$$y^2 = px \qquad \text{(the parabola),}$$

and

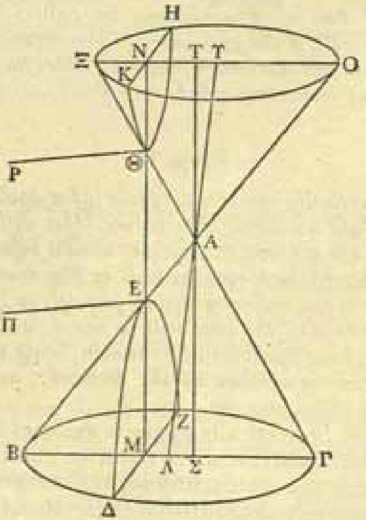
$$y^2 = px \pm \frac{p}{d}x^2 \text{ (the hyperbola and ellipse respectively).}$$

It is the essence of Apollonius's treatment to express the fundamental properties of the conics as equations between *areas*, whereas Archimedes had given the fundamental properties of the central conics as *proportions*

$$y^2 : (a^2 \pm x^2) = a^2 : b^2.$$

This form is, however, equivalent to the Cartesian equations referred to axes through the centre.

Ἔστω γὰρ ὁ κύκλος, καθ' οὗ φέρεται ἡ τὴν ἐπιφάνειαν γράφουσα εὐθεῖα, ὁ ΒΔΓΖ, καὶ ἤχθω



ἐν τῇ κατὰ κορυφὴν ἐπιφανείᾳ παράλληλον αὐτῷ ἐπίπεδον τὸ ΞΗΟΚ· κοιναὶ δὲ τομαὶ τῶν ΗΘΚ, ΖΕΔ τομῶν καὶ τῶν κύκλων αἱ ΖΔ, ΗΚ· ἔσονται δὴ παράλληλοι. ἄξων δὲ ἔστω τῆς κωνικῆς ἐπιφανείας ἡ ΛΑΥ εὐθεῖα, κέντρα δὲ τῶν κύκλων τὰ Λ, Υ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΖΔ κάθετος ἀχθεῖσα ἐκβεβλήσθω ἐπὶ τὰ Β, Γ σημεῖα, καὶ διὰ τῆς ΒΓ καὶ τοῦ ἄξονος ἐπίπεδον ἐκβεβλήσθω· ποιήσει δὴ τομὰς ἐν μὲν τοῖς κύκλοις παραλλήλους εὐθείας τὰς ΞΟ, ΒΓ, ἐν δὲ τῇ ἐπιφανείᾳ τὰς ΒΑΟ, ΓΑΞ·

For let ΒΔΓΖ be the circle round which revolves the straight line describing the surface, and in the vertically opposite surface let there be drawn parallel to it a plane ΞΗΟΚ; the common sections of the sections ΗΘΚ, ΖΕΔ and of the circles [Prop. 4] will be ΖΔ, ΗΚ; and they will be parallel [Eucl. xi. 16]. Let the axis of the conical surface be ΛΑΥ, let the centres of the circles be Λ, Υ, and from Λ let a perpendicular be drawn to ΖΔ and produced to the points Β, Γ, and let the plane through ΒΓ and the axis be produced; it will make in the circles the parallel straight lines ΞΟ, ΒΓ, and on the surface ΒΑΟ, ΓΑΞ;

ἔσται δὴ καὶ ἡ ΞΟ τῇ ΗΚ πρὸς ὀρθάς, ἐπειδὴ καὶ ἡ ΒΓ τῇ ΖΔ ἐστι πρὸς ὀρθάς, καί ἐστιν ἑκατέρα παράλληλος. καὶ ἐπεὶ τὸ διὰ τοῦ ἄξονος ἐπίπεδον ταῖς τομαῖς συμβάλλει κατὰ τὰ Μ, Ν σημεῖα ἐντὸς τῶν γραμμῶν, δῆλον, ὡς καὶ τὰς γραμμὰς τέμνει τὸ ἐπίπεδον. τεμνέτω κατὰ τὰ Θ, Ε· τὰ ἄρα Μ, Ε, Θ, Ν σημεῖα ἔν τε τῷ διὰ τοῦ ἄξονός ἐστιν ἐπιπέδῳ καὶ ἐν τῷ ἐπιπέδῳ, ἐν ᾧ εἰσιν αἱ γραμμαί· εὐθεῖα ἄρα ἐστὶν ἡ ΜΕΘΝ γραμμή. καὶ φανερόν, ὅτι τά τε Ξ, Θ, Α, Γ ἐπ' εὐθείας ἐστὶ καὶ τὰ Β, Ε, Α, Ο· ἔν τε γὰρ τῇ κωνικῇ ἐπιφανείᾳ ἐστὶ καὶ ἐν τῷ διὰ τοῦ ἄξονος ἐπιπέδῳ. ἤχθωσαν δὴ ἀπὸ μὲν τῶν Θ, Ε τῇ ΘΕ πρὸς ὀρθὰς αἱ ΘΡ, ΕΠ, διὰ δὲ τοῦ Α τῇ ΜΕΘΝ παράλληλος ἤχθω ἡ ΣΑΤ, καὶ πεποιήσθω, ὡς μὲν τὸ ἀπὸ τῆς ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ τῆς ΑΤ πρὸς τὸ ὑπὸ ΟΤΞ, οὕτως ἡ ΕΘ πρὸς ΘΡ. ἐπεὶ οὖν κῶνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, τέτμηται ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ πεποίηκε τομὴν τὸ ΑΒΓ τρίγωνον, τέτμηται δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ' εὐθεῖαν τὴν ΔΜΖ πρὸς ὀρθὰς οὖσαν τῇ ΒΓ, καὶ πεποίηκε τομὴν ἐν τῇ ἐπιφανείᾳ τὴν ΔΕΖ, ἡ δὲ διάμετρος ἡ ΜΕ ἐκβαλλομένη συμπέπτωκε μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς κορυφῆς τοῦ κώνου, καὶ διὰ τοῦ Α σημείου τῇ διαμέτρῳ τῆς τομῆς τῇ ΕΜ παράλληλος ἦκται ἡ ΑΣ, καὶ ἀπὸ τοῦ Ε τῇ ΕΜ πρὸς ὀρθὰς ἦκται ἡ ΕΠ, καί ἐστιν ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως ἡ ΕΘ πρὸς ΕΠ, ἡ μὲν ΔΕΖ ἄρα τομὴ ὑπερβολή ἐστιν, ἡ δὲ ΕΠ παρ' ἣν δύνανται αἱ ἐπὶ τὴν ΕΜ καταγόμεναι τεταγμένως, πλαγία

now ΞO will be perpendicular to HK, since $B\Gamma$ is perpendicular to $Z\Delta$, and each is parallel [Eucl. xi. 10]. And since the plane through the axis meets the sections at the points M, N within the curves, it is clear that the plane cuts the curves. Let it cut them at the points Θ, E; then the points M, E, Θ, N are both in the plane through the axis and in the plane containing the curves; therefore the line $ME\Theta N$ is a straight line [Eucl. xi. 3]. And it is clear that Ξ, Θ, A, Γ are on a straight line, and also B, E, A, O: for they are both on the conical surface and in the plane through the axis. Now let ΘP, $E\Pi$ be drawn from Θ, E perpendicular to ΘE, and through A let ΣAT be drawn parallel to $ME\Theta N$, and let

$$A\Sigma^2 : B\Sigma \,.\, \Sigma\Gamma = \Theta E : E\Pi,$$

and

$$AT^2 : OT \,.\, T\Xi = E\Theta : \Theta P.$$

Then since the cone, whose vertex is the point A and whose base is the circle $B\Gamma$, is cut by a plane through the axis, and the section so made is the triangle $AB\Gamma$, and it is cut by another plane cutting the base of the cone in the straight line ΔMZ perpendicular to $B\Gamma$, and the section so made on the surface is ΔEZ, and the diameter ME produced meets one side of the axial triangle beyond the vertex of the cone, and $A\Sigma$ is drawn through the point A parallel to the diameter of the section EM, and $E\Pi$ is drawn from E perpendicular to EM, and $A\Sigma^2 : B\Sigma \,.\, \Sigma\Gamma = E\Theta : E\Pi$, therefore the section ΔEZ is a hyperbola, in which $E\Pi$ is the parameter to the ordinates to EM, and ΘE is the

δὲ τοῦ εἴδους πλευρὰ ἡ ΘΕ. ὁμοίως δὲ καὶ ἡ ΗΘΚ ὑπερβολή ἐστιν, ἧς διάμετρος μὲν ἡ ΘΝ, ἡ δὲ ΘΡ παρ' ἣν δύνανται αἱ ἐπὶ τὴν ΘΝ καταγόμεναι τεταγμένως, πλαγία δὲ τοῦ εἴδους πλευρὰ ἡ ΘΕ.

Λέγω, ὅτι ἴση ἐστὶν ἡ ΘΡ τῇ ΕΠ. ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΒΓ τῇ ΞΟ, ἔστιν ὡς ἡ ΑΣ πρὸς ΣΓ, οὕτως ἡ ΑΤ πρὸς ΤΞ, καὶ ὡς ἡ ΑΣ πρὸς ΣΒ, οὕτως ἡ ΑΤ πρὸς ΤΟ. ἀλλ' ὁ τῆς ΑΣ πρὸς ΣΓ λόγος μετὰ τοῦ τῆς ΑΣ πρὸς ΣΒ ὁ τοῦ ἀπὸ ΑΣ ἐστι πρὸς τὸ ὑπὸ ΒΣΓ, ὁ δὲ τῆς ΑΤ πρὸς ΤΞ μετὰ τοῦ τῆς ΑΤ πρὸς ΤΟ ὁ τοῦ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ· ἔστιν ἄρα ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ. καί ἐστιν ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ, ἡ ΘΕ πρὸς ΘΡ· καὶ ὡς ἄρα ἡ ΘΕ πρὸς ΕΠ, ἡ ΕΘ πρὸς ΘΡ. ἴση ἄρα ἐστὶν ἡ ΕΠ τῇ ΘΡ.

(vi.) *Transition to New Diameter*

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

ν'

Ἐὰν ὑπερβολῆς ἢ ἐλλείψεως ἢ κύκλου περιφερείας εὐθεῖα ἐπιψαύουσα συμπίπτῃ τῇ διαμέτρῳ, καὶ διὰ τῆς ἁφῆς καὶ τοῦ κέντρου εὐθεῖα ἐκβληθῇ, ἀπὸ δὲ τῆς κορυφῆς ἀναχθεῖσα εὐθεῖα παρὰ τεταγμένως κατηγμένην συμπίπτῃ τῇ διὰ τῆς ἁφῆς καὶ

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[a] Apollonius is the first person known to have recognized the opposite branches of a hyperbola as portions of the same

transverse side of the figure [Prop. 12]. Similarly ΗΘΚ is a hyperbola, in which ΘΝ is a diameter, ΘΡ is the parameter to the ordinates to ΘΝ, and ΘΕ is the transverse side of the figure.

I say that ΘΡ = ΕΠ. For since ΒΓ is parallel to ΞΟ,

$$\mathrm{A\Sigma} : \Sigma\Gamma = \mathrm{AT} : \mathrm{T}\Xi,$$

and $$\mathrm{A\Sigma} : \Sigma\mathrm{B} = \mathrm{AT} : \mathrm{TO}.$$

But $$(\mathrm{A\Sigma} : \Sigma\Gamma)(\mathrm{A\Sigma} : \Sigma\mathrm{B}) = \mathrm{A\Sigma}^2 : \mathrm{B\Sigma} . \Sigma\Gamma,$$

and $$(\mathrm{AT} : \mathrm{T}\Xi)(\mathrm{AT} : \mathrm{TO}) = \mathrm{AT}^2 : \Xi\mathrm{T} . \mathrm{TO}.$$

Therefore $$\mathrm{A\Sigma}^2 : \mathrm{B\Sigma} . \Sigma\Gamma = \mathrm{AT}^2 : \Xi\mathrm{T} . \mathrm{TO}.$$

But $$\mathrm{A\Sigma}^2 : \mathrm{B\Sigma} . \Sigma\Gamma = \Theta\mathrm{E} : \mathrm{E}\Pi,$$

while $$\mathrm{AT}^2 : \Xi\mathrm{T} . \mathrm{TO} = \Theta\mathrm{E} : \Theta\mathrm{P};$$

therefore $$\Theta\mathrm{E} : \mathrm{E}\Pi = \mathrm{E}\Theta : \Theta\mathrm{P}.$$

Therefore $$\mathrm{E}\Pi = \Theta\mathrm{P}.$$ [a] [Eucl. v. 9

(vi.) *Transition to New Diameter*

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

Prop. 50

In a hyperbola, ellipse or circumference of a circle let a straight line be drawn to touch [the curve] and meet the diameter, and let the straight line through the point of contact and the centre be produced, and from the vertex let a straight line be drawn parallel to a straight line drawn ordinate-wise so as to meet the straight line drawn

curve. It is his practice, however, where possible to discuss the single-branch hyperbola (or the hyperbola *simpliciter* as he would call it) together with the ellipse and circle, and to deal with the opposite branches separately. But occasionally, as in i. 30, the double-branch hyperbola and the ellipse are included in one enunciation.

τοῦ κέντρου ἠγμένη εὐθεία, καὶ ποιηθῇ, ὡς τὸ τμῆμα τῆς ἐφαπτομένης τὸ μεταξὺ τῆς ἁφῆς καὶ τῆς ἀνηγμένης πρὸς τὸ τμῆμα τῆς ἠγμένης διὰ τῆς ἁφῆς καὶ τοῦ κέντρου τὸ μεταξὺ τῆς ἁφῆς καὶ τῆς ἀνηγμένης, εὐθεῖά τις πρὸς τὴν διπλασίαν τῆς ἐφαπτομένης, ἥτις ἂν ἀπὸ τῆς τομῆς ἀχθῇ ἐπὶ τὴν διὰ τῆς ἁφῆς καὶ τοῦ κέντρου ἠγμένην εὐθεῖαν παράλληλος τῇ ἐφαπτομένῃ, δυνήσεταί τι χωρίον ὀρθογώνιον παρακείμενον παρὰ τὴν πορισθεῖσαν, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς πρὸς τῇ ἁφῇ, ἐπὶ μὲν τῆς ὑπερβολῆς ὑπερβάλλον εἴδει ὁμοίῳ τῷ περιεχομένῳ ὑπὸ τῆς διπλασίας τῆς μεταξὺ τοῦ κέντρου καὶ τῆς ἁφῆς καὶ τῆς πορισθείσης εὐθείας, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἐλλεῖπον.

Ἔστω ὑπερβολὴ ἢ ἔλλειψις ἢ κύκλου περιφέρεια, ἧς διάμετρος ἡ ΑΒ, κέντρον δὲ τὸ Γ, ἐφαπτομένη δὲ ἡ ΔΕ, καὶ ἐπιζευχθεῖσα ἡ ΓΕ ἐκβεβλήσθω ἐφ' ἑκάτερα, καὶ κείσθω τῇ ΕΓ ἴση ἡ ΓΚ, καὶ διὰ τοῦ Β τεταγμένως ἀνήχθω ἡ ΒΖΗ, διὰ δὲ τοῦ Ε τῇ ΕΓ πρὸς ὀρθὰς ἤχθω ἡ ΕΘ, καὶ γινέσθω, ὡς ἡ ΖΕ πρὸς ΕΗ, οὕτως ἡ ΕΘ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐπιζευχθεῖσα ἡ ΘΚ ἐκβεβλήσθω, καὶ εἰλήφθω τι ἐπὶ τῆς τομῆς σημεῖον τὸ Λ, καὶ δι' αὐτοῦ τῇ ΕΔ παράλληλος ἤχθω ἡ ΛΜΞ, τῇ δὲ

[a] To save space, the figure is here given for the hyperbola only; in the mss. there are figures for the ellipse and circle as well.

The general enunciation is not easy to follow, but the particular enunciation will make it easier to understand. The

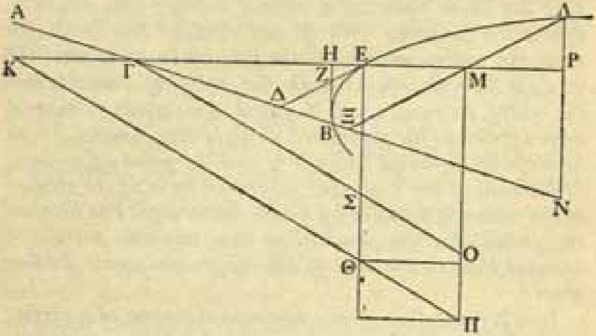
through the point of contact and the centre, and let the segment of the tangent between the point of contact and the line drawn ordinate-wise bear to the segment of the line drawn through the point of contact and the centre between the point of contact and the line drawn ordinate-wise the same ratio as a certain straight line bears to double the tangent; then if any straight line be drawn from the section parallel to the tangent so as to meet the straight line drawn through the point of contact and the centre, its square will be equal to a certain rectilineal area applied to the postulated straight line, having for its breadth the intercept between it and the point of contact, in the case of the hyperbola exceeding by a figure similar to the rectangle bounded by double the straight line between the centre and the point of contact and the postulated straight line, in the case of the ellipse and circle falling short.[a]

In a hyperbola, ellipse or circumference of a circle, with diameter AB and centre Γ, let ΔE be a tangent, and let ΓE be joined and produced in either direction, and let ΓK be placed equal to EΓ, and through B let BZH be drawn ordinate-wise, and through E let EΘ be drawn perpendicular to EΓ, and let ZE : EH = EΘ : 2EΔ, and let ΘK be joined and produced, and let any point Λ be taken on the section, and through it let ΛMΞ be drawn parallel to EΔ and

purpose of this important proposition is to show that, if any other diameter be taken, the ordinate-property of the conic with reference to this diameter has the same form as the ordinate-property with reference to the original diameter. The theorem amounts to a transformation of co-ordinates from the original diameter and the tangent at its extremity to any diameter and the tangent at its extremity. In succeeding propositions, showing how to construct conics from certain data, Apollonius introduces the axes for the first time as special cases of diameters.

ΒΗ ἡ ΛΡΝ, τῇ δὲ ΕΘ ἡ ΜΠ. λέγω, ὅτι τὸ ἀπὸ ΛΜ ἴσον ἐστὶ τῷ ὑπὸ ΕΜΠ.

Ἤχθω γὰρ διὰ τοῦ Γ τῇ ΚΠ παράλληλος ἡ ΓΣΟ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ τῇ ΓΚ, ὡς δὲ ἡ



ΕΓ πρὸς ΚΓ, ἡ ΕΣ πρὸς ΣΘ, ἴση ἄρα καὶ ἡ ΕΣ τῇ ΣΘ. καὶ ἐπεί ἐστιν, ὡς ἡ ΖΕ πρὸς ΕΗ, ἡ ΘΕ πρὸς τὴν διπλασίαν τῆς ΕΔ, καί ἐστι τῆς ΕΘ ἡμίσεια ἡ ΕΣ, ἔστιν ἄρα, ὡς ἡ ΖΕ πρὸς ΕΗ, ἡ ΣΕ πρὸς ΕΔ. ὡς δὲ ἡ ΖΕ πρὸς ΕΗ, ἡ ΛΜ πρὸς ΜΡ· ὡς ἄρα ἡ ΛΜ πρὸς ΜΡ, ἡ ΣΕ πρὸς ΕΔ. καὶ ἐπεὶ τὸ ΡΝΓ τρίγωνον τοῦ ΗΒΓ τριγώνου, τουτέστι τοῦ ΓΔΕ, ἐπὶ μὲν τῆς ὑπερβολῆς μεῖζον ἐδείχθη, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἔλασσον τῷ ΛΝΞ, κοινῶν ἀφαιρεθέντων ἐπὶ μὲν τῆς ὑπερβολῆς τοῦ τε ΕΓΔ τριγώνου καὶ τοῦ ΝΡΜΞ τετραπλεύρου, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου τοῦ ΜΞΓ τριγώνου, τὸ ΛΜΡ τρίγωνον τῷ ΜΕΔΞ τετραπλεύρῳ ἐστὶν ἴσον. καί ἐστι

$\Lambda P N$ parallel to BH, and let $M\Pi$ be drawn parallel to $E\Theta$. I say that $\Lambda M^2 = EM \cdot M\Pi$.

For through Γ let $\Gamma\Sigma O$ be drawn parallel to $K\Pi$. Then since

$$E\Gamma = \Gamma K$$

and $E\Gamma : \Gamma K = E\Sigma : \Sigma\Theta$, [Eucl. vi. 2

therefore $E\Sigma = \Sigma\Theta$.

And since $ZE : EH = \Theta E : 2E\Delta$,

and $E\Sigma = \frac{1}{2}E\Theta$,

therefore $ZE : EH = \Sigma E : E\Delta$.

But $ZE : EH = \Lambda M : MP$; [Eucl. vi. 4

therefore $\Lambda M : MP = \Sigma E : E\Delta$.

And since it has been proved [Prop. 43] that in the hyperbola

triangle $PN\Gamma$ = triangle $HB\Gamma$ + triangle $\Lambda N\Xi$,

i.e., triangle $PN\Gamma$ = triangle $\Gamma\Delta E$ + triangle $\Lambda N\Xi$,[a]

while in the ellipse and the circle

triangle $PN\Gamma$ = triangle $HB\Gamma$ − triangle $\Lambda N\Xi$,

i.e., triangle $PN\Gamma$ + triangle $\Lambda N\Xi$ = triangle $\Gamma\Delta E$,[b]

therefore by taking away the common elements—in the hyperbola the triangle $E\Gamma\Delta$ and the quadrilateral $NPM\Xi$, in the ellipse and the circle the triangle $M\Xi\Gamma$,

triangle ΛMP = quadrilateral $ME\Delta\Xi$.

[a] For this step *v.* Eutocius's comment on Prop. 43.
[b] See Eutocius.

παράλληλος ἡ ΜΞ τῇ ΔΕ, ἡ δὲ ὑπὸ ΛΜΡ τῇ ὑπὸ ΕΜΞ ἐστιν ἴση· ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΕΜ καὶ συναμφοτέρου τῆς ΕΔ, ΜΞ. καὶ ἐπεί ἐστιν, ὡς ἡ ΜΓ πρὸς ΓΕ, ἥ τε ΜΞ πρὸς ΕΔ καὶ ἡ ΜΟ πρὸς ΕΣ, ὡς ἄρα ἡ ΜΟ πρὸς ΕΣ, ἡ ΜΞ πρὸς ΔΕ. καὶ συνθέντι, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς ΕΣ, οὕτως συναμφότερος ἡ ΜΞ, ΕΔ πρὸς ΕΔ· ἐναλλάξ, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς συναμφότερον τὴν ΞΜ, ΕΔ ἡ ΣΕ πρὸς ΕΔ. ἀλλ' ὡς μὲν συναμφότερος ἡ ΜΟ, ΕΣ πρὸς συναμφότερον τὴν ΜΞ, ΔΕ, τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΕΜ πρὸς τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, ὡς δὲ ἡ ΣΕ πρὸς ΕΔ, ἡ ΖΕ πρὸς ΕΗ, τουτέστιν ἡ ΛΜ πρὸς ΜΡ, τουτέστι τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ· ὡς ἄρα τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ. καὶ ἐναλλάξ, ὡς τὸ ὑπὸ συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ἀπὸ ΜΛ, οὕτως τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ ΛΜΡ. ἴσον δὲ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΜΕ καὶ συναμφοτέρου τῆς ΜΞ, ΕΔ· ἴσον ἄρα καὶ τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜ καὶ συναμφοτέρου τῆς ΜΟ, ΕΣ. καί ἐστιν ἡ μὲν ΣΕ τῇ ΣΘ ἴση, ἡ δὲ ΣΘ τῇ ΟΠ· ἴσον ἄρα τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜΠ.

But $M\Xi$ is parallel to ΔE and angle ΛMP = angle $EM\Xi$ (Eucl. i. 15];

therefore $\Lambda M \,.\, MP = EM \,.\, (E\Delta + M\Xi)$.

And since $M\Gamma : \Gamma E = M\Xi : E\Delta$,

and $M\Gamma : \Gamma E = MO : E\Sigma$,

[Eucl. vi. 4

therefore $MO : E\Sigma = M\Xi : \Delta E$.

Componendo, $MO + \Sigma E : E\Sigma = M\Xi + E\Delta : E\Delta$;

and *permutando*

$$MO + \Sigma E : \Xi M + E\Delta = \Sigma E : E\Delta.$$

But $MO + \Sigma E : \Xi M + E\Delta = (MO + E\Sigma) \,.\, EM : (M\Xi + E\Delta) \,.\, EM$,

and $\Sigma E : E\Delta = ZE : EH$

$= \Lambda M : MP$

[Eucl. vi. 4

$= \Lambda M^2 : \Lambda M \,.\, MP$;

therefore

$$(MO + E\Sigma) \,.\, ME : (M\Xi + E\Delta) \,.\, EM = \Lambda M^2 : \Lambda M \,.\, MP.$$

And *permutando*

$$(MO + E\Sigma) \,.\, ME : M\Lambda^2 = (M\Xi + E\Delta) \,.\, ME : \Lambda M \,.\, MP.$$

But $\Lambda M \,.\, MP = ME \,.\, (M\Xi + E\Delta)$;

therefore $\Lambda M^2 = EM \,.\, (MO + E\Sigma)$.

And $\Sigma E = \Sigma\Theta$, while $\Sigma\Theta = O\Pi$ [Eucl. i. 34];

therefore $\Lambda M^2 = EM \,.\, M\Pi$.

(b) Other Works

(i.) *General*

Papp. *Coll.* vii. 3, ed. Hultsch 636. 18-23

Τῶν δὲ προειρημένων τοῦ Ἀναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη· Εὐκλείδου Δεδομένων βιβλίον ᾱ, Ἀπολλωνίου Λόγου ἀποτομῆς β̄, Χωρίου ἀποτομῆς β̄, Διωρισμένης τομῆς δύο, Ἐπαφῶν δύο, Εὐκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο, Κωνικῶν η̄.

(ii.) *On the Cutting-off of a Ratio*

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

Τῆς δ' Ἀποτομῆς τοῦ λόγου βιβλίων ὄντων β̄ πρότασίς ἐστιν μία ὑποδιῃρημένη, διὸ καὶ μίαν πρότασιν οὕτως γράφω· διὰ τοῦ δοθέντος σημείου εὐθεῖαν γραμμὴν ἀγαγεῖν τέμνουσαν ἀπὸ τῶν τῇ θέσει δοθεισῶν δύο εὐθειῶν πρὸς τοῖς ἐπ' αὐτῶν δοθεῖσι σημείοις λόγον ἐχούσας τὸν αὐτὸν τῷ δοθέντι. τὰς δὲ γραφὰς διαφόρους γενέσθαι καὶ πλῆθος λαβεῖν συμβέβηκεν ὑποδιαιρέσεως γενομένης ἕνεκα τῆς τε πρὸς ἀλλήλας θέσεως τῶν διδομένων εὐθειῶν καὶ τῶν διαφόρων πτώσεων τοῦ διδομένου σημείου καὶ διὰ τὰς ἀναλύσεις καὶ συνθέσεις αὐτῶν τε καὶ τῶν διορισμῶν. ἔχει γὰρ τὸ μὲν πρῶτον βιβλίον τῶν Λόγου ἀποτομῆς

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[a] Unhappily the only work by Apollonius which has survived, in addition to the *Conics*, is *On the Cutting-off of a*

(b) Other Works

(i.) *General*

Pappus, *Collection* vii. 3, ed. Hultsch 636. 18-23

The order of the aforesaid books in the *Treasury of Analysis* is as follows: the one book of Euclid's *Data*, the two books of Apollonius's *On the Cutting-off of a Ratio*, his two books *On the Cutting-off of an Area*, his two books *On Determinate Section*, his two books *On Tangencies*, the three books of Euclid's *Porisms*, the two books of Apollonius's *On Vergings*, the two books of the same writer *On Plane Loci*, his eight books of *Conics*.[a]

(ii.) *On the Cutting-off of a Ratio*

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

In the two books *On the Cutting-off of a Ratio* there is one enunciation which is subdivided, for which reason I state one enunciation thus: *Through a given point to draw a straight line cutting off from two straight lines given in position intercepts, measured from two given points on them, which shall have a given ratio.* When the subdivision is made, this leads to many different figures according to the position of the given straight lines in relation one to another and according to the different cases of the given point, and owing to the analysis and the synthesis both of these cases and of the propositions determining the limits of possibility. The first book of those *On the Cutting-off of a*

Ratio, and that only in Arabic. Halley published a Latin translation in 1706. But the contents of the other works are indicated fairly closely by Pappus's references.

τόπους ζ, πτώσεις $\overline{κδ}$, διορισμοὺς δὲ ε̄, ὧν τρεῖς μέν εἰσιν μέγιστοι, δύο δὲ ἐλάχιστοι. . . . τὸ δὲ δεύτερον βιβλίον Λόγου ἀποτομῆς ἔχει τόπους $\overline{ιδ}$, πτώσεις δὲ $\overline{ξγ}$, διορισμοὺς δὲ τοὺς ἐκ τοῦ πρώτου· ἀπάγεται γὰρ ὅλον εἰς τὸ πρῶτον.

(iii.) *On the Cutting-off of an Area*

Ibid. vii. 7, ed. Hultsch 640. 26–642. 5

Τῆς δ' Ἀποτομῆς τοῦ χωρίου βιβλία μέν ἐστιν δύο, πρόβλημα δὲ κἀν τούτοις ἓν ὑποδιαιρούμενον δίς, καὶ τούτων μία πρότασίς ἐστιν τὰ μὲν ἄλλα ὁμοίως ἔχουσα τῇ προτέρᾳ, μόνῳ δὲ τούτῳ διαφέρουσα τῷ δεῖν τὰς ἀποτεμνομένας δύο εὐθείας ἐν ἐκείνῃ μὲν λόγον ἐχούσας δοθέντα ποιεῖν, ἐν δὲ ταύτῃ χωρίον περιεχούσας δοθέν.

(iv.) *On Determinate Section*

Ibid. vii. 9, ed. Hultsch 642. 19–644. 16

Ἑξῆς τούτοις ἀναδέδονται τῆς Διωρισμένης τομῆς βιβλία $\overline{β}$, ὧν ὁμοίως τοῖς πρότερον μίαν πρότασιν πάρεστιν λέγειν, διεζευγμένην δὲ ταύτην·

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[a] The Arabic text shows that Apollonius first discussed the cases in which the lines are parallel, then the cases in which the lines intersect but one of the given points is at the point of intersection; in the second book he proceeds to the general case, but shows that it can be reduced to the case where one

Ratio contains seven loci, twenty-four cases and five determinations of the limits of possibility, of which three are maxima and two are minima. . . . The second book *On the Cutting-off of a Ratio* contains fourteen loci, sixty-three cases and the same determinations of the limits of possibility as the first; for they are all reduced to those in the first book.[a]

(iii.) *On the Cutting-off of an Area*

Ibid. vii. 7, ed. Hultsch 640. 26–642. 5

In the work *On the Cutting-off of an Area* there are two books, but in them there is only one problem, twice subdivided, and the one enunciation is similar in other respects to the preceding, differing only in this, that in the former work the intercepts on the two given lines were required to have a given ratio, in this to comprehend a given area.[b]

(iv.) *On Determinate Section*

Ibid. vii. 9, ed. Hultsch 642. 19–644. 16

Next in order after these are published the two books *On Determinate Section*, of which, as in the previous cases, it is possible to state one comprehen-

of the given points is at the intersection of the two lines. By this means the problem is reduced to the application of a rectangle. In all cases Apollonius works by analysis and synthesis.

[b] Halley attempted to restore this work in his edition of the *De sectione rationis*. As in that treatise, the general case can be reduced to the case where one of the given points is at the intersection of the two lines, and the problem is reduced to the application of a certain rectangle.

τὴν δοθεῖσαν ἄπειρον εὐθεῖαν ἑνὶ σημείῳ τεμεῖν, ὥστε τῶν ἀπολαμβανομένων εὐθειῶν πρὸς τοῖς ἐπ' αὐτῆς δοθεῖσι σημείοις ἤτοι τὸ ἀπὸ μιᾶς τετράγωνον ἢ τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθογώνιον δοθέντα λόγον ἔχειν ἤτοι πρὸς τὸ ἀπὸ μιᾶς τετράγωνον ἢ πρὸς τὸ ὑπὸ μιᾶς ἀπολαμβανομένης καὶ τῆς ἔξω δοθείσης ἢ πρὸς τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθογώνιον, ἐφ' ὁπότερα χρὴ τῶν δοθέντων σημείων. . . . ἔχει δὲ τὸ μὲν πρῶτον βιβλίον προβλήματα ϛ̄, ἐπιτάγματα ιϛ̄, διορισμοὺς ε̄, ὧν μεγίστους μὲν δ̄, ἐλάχιστον δὲ ἕνα. . . . τὸ δὲ δεύτερον Διωρισμένης τομῆς ἔχει προβλήματα γ̄, ἐπιτάγματα θ̄, διορισμοὺς γ̄.

(v.) *On Tangencies*

Ibid. vii. 11, ed. Hultsch 644. 23–646. 19

Ἑξῆς δὲ τούτοις τῶν Ἐπαφῶν ἐστιν βιβλία δύο. προτάσεις δὲ ἐν αὐτοῖς δοκοῦσιν εἶναι πλείονες, ἀλλὰ καὶ τούτων μίαν τίθεμεν οὕτως ἔχουσαν ἐξῆς· σημείων καὶ εὐθειῶν καὶ κύκλων τριῶν ὁποιωνοῦν θέσει δοθέντων κύκλον ἀγαγεῖν δι' ἑκάστου τῶν δοθέντων σημείων, εἰ δοθείη, ἢ ἐφαπτόμενον ἑκάστης τῶν δοθεισῶν γραμμῶν. ταύτης διὰ

[a] As the Greeks never grasped the conception of one point being two *coincident* points, it was not possible to enunciate this problem so concisely as we can do: *Given four points A, B, C, D on a straight line, of which A may coincide with C and B with D, to find another point P on the same straight line such that AP . CP : BP . DP has a given value.* If AP . CP = λ . BP . DP, where A, B, C, D, λ are given, the determination of P is equivalent to the solution of a quadratic equation, which the Greeks could achieve by means of the

sive enunciation thus: *To cut a given infinite straight line in a point so that the intercepts between this point and given points on the line shall furnish a given ratio, the ratio being that of the square on one intercept, or the rectangle contained by two, towards the square on the remaining intercept, or the rectangle contained by the remaining intercept and a given independent straight line, or the rectangle contained by two remaining intercepts, whichever way the given points [are situated].* . . . The first book contains six problems, sixteen subdivisions and five limits of possibility, of which four are maxima and one is a minimum. . . . The second book *On Determinate Section* contains three problems, nine subdivisions, and three limits of possibility.[a]

(v.) *On Tangencies*

Ibid. vii. 11, ed. Hultsch 644. 23–646. 19

Next in order are the two books *On Tangencies*. Their enunciations are more numerous, but we may bring these also under one enunciation thus stated: *Given three entities, of which any one may be a point or a straight line or a circle, to draw a circle which shall pass through each of the given points, so far as it is points which are given, or to touch each of the given lines.*[b] In

application of areas. But the fact that limits of possibility, and maxima and minima were discussed leads Heath (*H.G.M.* ii. 180-181) to conjecture that Apollonius investigated the *series* of point-pairs determined by the equation for different values of λ, and that "the treatise contained what amounts to a complete *Theory of Involution*." The importance of the work is shown by the large number of lemmas which Pappus collected.

[b] The word "lines" here covers both the straight lines and the circles.

πλήθη τῶν ἐν ταῖς ὑποθέσεσι δεδομένων ὁμοίων ἢ ἀνομοίων κατὰ μέρος διαφόρους προτάσεις ἀναγκαῖον γίνεσθαι δέκα· ἐκ τῶν τριῶν γὰρ ἀνομοίων γενῶν τριάδες διάφοροι ἄτακτοι γίνονται ῑ. ἤτοι γὰρ τὰ διδόμενα τρία σημεῖα ἢ τρεῖς εὐθεῖαι ἢ δύο σημεῖα καὶ εὐθεῖα ἢ δύο εὐθεῖαι καὶ σημεῖον ἢ δύο σημεῖα καὶ κύκλος ἢ δύο κύκλοι καὶ σημεῖον ἢ δύο εὐθεῖαι καὶ κύκλος ἢ δύο κύκλοι καὶ εὐθεῖα ἢ σημεῖον καὶ εὐθεῖα καὶ κύκλος ἢ τρεῖς κύκλοι. τούτων δύο μὲν τὰ πρῶτα δέδεικται ἐν τῷ δʹ βιβλίῳ τῶν πρώτων Στοιχείων, διὸ παρίει μὴ γράφων· τὸ μὲν γὰρ τριῶν δοθέντων σημείων μὴ ἐπ' εὐθείας ὄντων τὸ αὐτό ἐστιν τῷ περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι, τὸ δὲ γ̄ δοθεισῶν εὐθειῶν μὴ παραλλήλων οὐσῶν, ἀλλὰ τῶν τριῶν συμπιπτουσῶν, τὸ αὐτό ἐστιν τῷ εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι· τὸ δὲ δύο παραλλήλων οὐσῶν καὶ μιᾶς ἐμπιπτούσης ὡς μέρος ὂν τῆς βʹ ὑποδιαιρέσεως προγράφεται ἐν τούτοις πάντων. καὶ τὰ ἑξῆς ϛ̄ ἐν τῷ πρώτῳ βιβλίῳ τὰ δὲ λειπόμενα δύο, τὸ δύο δοθεισῶν εὐθειῶν καὶ κύκλου ἢ τριῶν δοθέντων κύκλων μόνον ἐν τῷ δευτέρῳ βιβλίῳ διὰ τὰς πρὸς ἀλλήλους θέσεις τῶν κύκλων τε καὶ εὐθειῶν πλείονας οὔσας καὶ πλειόνων διορισμῶν δεομένας.

[a] Eucl. iv. 5 and 4.

[b] The last problem, to describe a circle touching three

this problem, according to the number of like or unlike entities in the hypotheses, there are bound to be, when the problem is subdivided, ten enunciations. For the number of different ways in which three entities can be taken out of the three unlike sets is ten. For the given entities must be (1) three points or (2) three straight lines or (3) two points and a straight line or (4) two straight lines and a point or (5) two points and a circle or (6) two circles and a point or (7) two straight lines and a circle or (8) two circles and a straight line or (9) a point and a straight line and a circle or (10) three circles. Of these, the first two cases are proved in the fourth book of the first *Elements*,[a] for which reason they will not be described ; for to describe a circle through three points, not being in a straight line, is the same thing as to circumscribe a given triangle, and to describe a circle to touch three given straight lines, not being parallel but meeting each other, is the same thing as to inscribe a circle in a given triangle ; the case where two of the lines are parallel and one meets them is a subdivision of the second problem but is here given first place. The next six problems in order are investigated in the first book, while the remaining two, the case of two given straight lines and a circle and the case of three circles, are the sole subjects of the second book on account of the manifold positions of the circles and straight lines with respect one to another and the need for numerous investigations of the limits of possibility.[b]

given circles, has been investigated by many famous geometers, including Newton (*Arithmetica Universalis*, Prob. 47). The lemmas given by Pappus enable Heath (*H.G.M.* ii. 182-185) to restore Apollonius's solution—a " plane " solution depending only on the straight line and circle.

(vi.) *On Plane Loci*

Ibid. vii. 23, ed. Hultsch 662. 19–664. 7

Οἱ μὲν οὖν ἀρχαῖοι εἰς τὴν τῶν ἐπιπέδων τούτων[1] τόπων τάξιν ἀποβλέποντες ἐστοιχείωσαν· ἧς ἀμελήσαντες οἱ μετ' αὐτοὺς προσέθηκαν ἑτέρους, ὡς οὐκ ἀπείρων τὸ πλῆθος ὄντων, εἰ θέλοι τις προσγράφειν οὐ τῆς τάξεως ἐκείνης ἐχόμενα. θήσω οὖν τὰ μὲν προσκείμενα ὕστερα, τὰ δ' ἐκ τῆς τάξεως πρότερα μιᾷ περιλαβὼν προτάσει ταύτῃ·

Ἐὰν δύο εὐθεῖαι ἀχθῶσιν ἤτοι ἀπὸ ἑνὸς δεδομένου σημείου ἢ ἀπὸ δύο καὶ ἤτοι ἐπ' εὐθείας ἢ παράλληλοι ἢ δεδομένην περιέχουσαι γωνίαν καὶ ἤτοι λόγον ἔχουσαι πρὸς ἀλλήλας ἢ χωρίον περιέχουσαι δεδομένον, ἅπτηται δὲ τὸ τῆς μιᾶς πέρας ἐπιπέδου τόπου θέσει δεδομένου, ἅψεται καὶ τὸ τῆς ἑτέρας πέρας ἐπιπέδου τόπου θέσει δεδομένου ὁτὲ μὲν τοῦ ὁμογενοῦς, ὁτὲ δὲ τοῦ ἑτέρου, καὶ ὁτὲ μὲν ὁμοίως κειμένου πρὸς τὴν εὐθεῖαν, ὁτὲ δὲ ἐναντίως. ταῦτα δὲ γίνεται παρὰ τὰς διαφορὰς τῶν ὑποκειμένων.

(vii.) *On Vergings*

Ibid. vii. 27-28, ed. Hultsch 670. 4–672. 3

Νεύειν λέγεται γραμμὴ ἐπὶ σημεῖον, ἐὰν ἐπεκβαλλομένη ἐπ' αὐτὸ παραγίνηται [. . .]

[1] τούτων is attributed by Hultsch to dittography.

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[a] These words follow the passage (quoted *supra*, pp. 262-265) wherein Pappus divides loci into ἐφεκτικοί, διεξοδικοί and ἀναστροφικοί.

[b] It is not clear what straight line is meant—probably the most obvious straight line in each figure.

(vi.) *On Plane Loci*

Ibid. vii. 23, ed. Hultsch 662. 19–664. 7

The ancients had regard to the arrangement[a] of these plane loci with a view to instruction in the elements; heedless of this consideration, their successors have added others, as though the number could not be infinitely increased if one were to make additions from outside that arrangement. Accordingly I shall set out the additions later, giving first those in the arrangement, and including them in this single enunciation:

If two straight lines be drawn, from one given point or from two, which are in a straight line or parallel or include a given angle, and either bear a given ratio one towards the other or contain a given rectangle, then, if the locus of the extremity of one of the lines be a plane locus given in position, the locus of the extremity of the other will also be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of a different kind, and will sometimes be similarly situated with respect to the straight line,[b] sometimes contrariwise. These different cases arise according to the differences in the suppositions.[c]

(vii.) *On Vergings*[d]

Ibid. vii. 27-28, ed. Hultsch 670. 4–672. 3

A line is said to *verge* to a point if, when produced, it passes through the point. [. . .] The general

[c] Pappus proceeds to give seven other enunciations from the first book and eight from the second book. These have enabled reconstructions of the work to be made by Fermat, van Schooten and Robert Simson.

[d] Examples of vergings have already been encountered several times; v. pp. 186-189 and vol. i. p. 244 n. a.

προβλήματος δὲ ὄντος καθολικοῦ τούτου· δύο δοθεισῶν γραμμῶν θέσει θεῖναι μεταξὺ τούτων εὐθεῖαν τῷ μεγέθει δεδομένην νεύουσαν ἐπὶ δοθὲν σημεῖον, ἐπὶ τούτου τῶν ἐπὶ μέρους διάφορα τὰ ὑποκείμενα ἐχόντων, ἃ μὲν ἦν ἐπίπεδα, ἃ δὲ στερεά, ἃ δὲ γραμμικά, τῶν δ' ἐπιπέδων ἀποκληρώσαντες τὰ πρὸς πολλὰ χρησιμώτερα ἔδειξαν τὰ προβλήματα ταῦτα·

Θέσει δεδομένων ἡμικυκλίου τε καὶ εὐθείας πρὸς ὀρθὰς τῇ βάσει ἢ δύο ἡμικυκλίων ἐπ' εὐθείας ἐχόντων τὰς βάσεις θεῖναι δοθεῖσαν τῷ μεγέθει εὐθεῖαν μεταξὺ τῶν δύο γραμμῶν νεύουσαν ἐπὶ γωνίαν ἡμικυκλίου·

Καὶ ῥόμβου δοθέντος καὶ ἐπεκβεβλημένης μιᾶς πλευρᾶς ἁρμόσαι ὑπὸ τὴν ἐκτὸς γωνίαν δεδομένην τῷ μεγέθει εὐθεῖαν νεύουσαν ἐπὶ τὴν ἄντικρυς γωνίαν·

Καὶ θέσει δοθέντος κύκλου ἐναρμόσαι εὐθεῖαν μεγέθει δεδομένην νεύουσαν ἐπὶ δοθέν.

Τούτων δὲ ἐν μὲν τῷ πρώτῳ τεύχει δέδεικται τὸ ἐπὶ τοῦ ἑνὸς ἡμικυκλίου καὶ εὐθείας ἔχον πτώσεις δ̄ καὶ τὸ ἐπὶ τοῦ κύκλου ἔχον πτώσεις δύο καὶ τὸ ἐπὶ τοῦ ῥόμβου πτώσεις ἔχον β̄, ἐν δὲ τῷ δευτέρῳ τεύχει τὸ ἐπὶ τῶν δύο ἡμικυκλίων τῆς ὑποθέσεως πτώσεις ἐχούσης ῑ, ἐν δὲ ταύταις ὑποδιαιρέσεις πλείονες διοριστικαὶ ἕνεκα τοῦ δεδομένου μεγέθους τῆς εὐθείας.

problem is: *Two straight lines being given in position, to place between them a straight line of given length so as to verge to a given point.* When it is subdivided the subordinate problems are, according to differences in the suppositions, sometimes plane, sometimes solid, sometimes linear. Among the plane problems, a selection was made of those more generally useful, and these problems have been proved:

Given a semicircle and a straight line perpendicular to the base, or two semicircles with their bases in a straight line, to place a straight line of given length between the two lines and verging to an angle of the semicircle [or of one of the semicircles];

Given a rhombus with one side produced, to insert a straight line of given length in the external angle so that it verges to the opposite angle;

Given a circle, to insert a chord of given length verging to a given point.

Of these, there are proved in the first book four cases of the problem of one semicircle and a straight line, two cases of the circle, and two cases of the rhombus; in the second book there are proved ten cases of the problem in which two semicircles are assumed, and in these there are numerous subdivisions concerned with limits of possibility according to the given length of the straight line.[a]

[a] A restoration of Apollonius's work *On Vergings* has been attempted by several writers, most completely by Samuel Horsley (Oxford, 1770). A lemma by Pappus enables Apollonius's construction in the case of the rhombus to be restored with certainty; *v.* Heath, *H.G.M.* ii. 190-192.

(viii.) *On the Dodecahedron and the Icosahedron*

Hypsicl. [Eucl. *Elem.* xiv.], Eucl. ed. Heiberg v. 6. 19–8. 5

Ὁ αὐτὸς κύκλος περιλαμβάνει τό τε τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἰκοσαέδρου τρίγωνον τῶν εἰς τὴν αὐτὴν σφαῖραν ἐγγραφομένων. τοῦτο δὲ γράφεται ὑπὸ μὲν Ἀρισταίου ἐν τῷ ἐπιγραφομένῳ Τῶν ε̄ σχημάτων συγκρίσει, ὑπὸ δὲ Ἀπολλωνίου ἐν τῇ δευτέρᾳ ἐκδόσει τῆς Συγκρίσεως τοῦ δωδεκαέδρου πρὸς τὸ εἰκοσάεδρον, ὅτι ἐστίν, ὡς ἡ τοῦ δωδεκαέδρου ἐπιφάνεια πρὸς τὴν τοῦ εἰκοσαέδρου ἐπιφάνειαν, οὕτως καὶ αὐτὸ τὸ δωδεκάεδρον πρὸς τὸ εἰκοσάεδρον διὰ τὸ τὴν αὐτὴν εἶναι κάθετον ἀπὸ τοῦ κέντρου τῆς σφαίρας ἐπὶ τὸ τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἰκοσαέδρου τρίγωνον.

(ix.) *Principles of Mathematics*

Marin. *in Eucl. Dat.*, Eucl. ed. Heiberg vi. 234. 13-17

Διὸ τῶν ἁπλούστερον[1] καὶ μιᾷ τινι διαφορᾷ περιγράφειν τὸ δεδομένον προθεμένων οἱ μὲν τεταγμένον, ὡς Ἀπολλώνιος ἐν τῇ Περὶ νεύσεων καὶ

[1] ἁπλούστερον Heiberg, ἁπλουστέρων cod.

(viii.) *On the Dodecahedron and the Icosahedron*

Hypsicles [Euclid, *Elements* xiv.],[a] Eucl. ed. Heiberg v. 6. 19–8. 5

The pentagon of the dodecahedron and the triangle of the icosahedron [b] inscribed in the same sphere can be included in the same circle. For this is proved by Aristaeus in the work which he wrote *On the Comparison of the Five Figures*,[c] and it is proved by Apollonius in the second edition of his work *On the Comparison of the Dodecahedron and the Icosahedron* that the surface of the dodecahedron bears to the surface of the icosahedron the same ratio as the volume of the dodecahedron bears to the volume of the icosahedron, by reason of there being a common perpendicular from the centre of the sphere to the pentagon of the dodecahedron and the triangle of the icosahedron.

(ix.) *Principles of Mathematics*

Marinus, *Commentary on Euclid's Data*, Eucl. ed. Heiberg vi. 234. 13-17

Therefore, among those who made it their aim to define the datum more simply and with a single *differentia*, some called it the *assigned*, such as Apollonius in his book *On Vergings* and in his

[a] The so-called fourteenth book of Euclid's *Elements* is really the work of Hypsicles, for whom *v. infra*, pp. 394-397.

[b] For the regular solids *v.* vol. i. pp. 216-225. The face of the dodecahedron is a pentagon and the face of the icosahedron a triangle.

[c] A proof is given by Hypsicles as Prop. 2 of his book. Whether the Aristaeus is the same person as the author of the *Solid Loci* is not known.

ἐν τῇ Καθόλου πραγματείᾳ, οἱ δὲ γνώριμον, ὡς Διόδωρος.

(x.) *On the Cochlias*

Procl. *in Eucl.* i., ed. Friedlein 105. 1-6

Τὴν περὶ τὸν κύλινδρον ἕλικα γραφομένην, ὅταν εὐθείας κινουμένης περὶ τὴν ἐπιφάνειαν τοῦ κυλίνδρου σημεῖον ὁμοταχῶς ἐπ' αὐτῆς κινῆται. γίνεται γὰρ ἕλιξ, ἧς ὁμοιομερῶς πάντα τὰ μέρη πᾶσιν ἐφαρμόζει, καθάπερ Ἀπολλώνιος ἐν τῷ Περὶ τοῦ κοχλίου γράμματι δείκνυσιν.

(xi.) *On Unordered Irrationals*

Procl. *in Eucl.* i., ed. Friedlein 74. 23-24

Τὰ Περὶ τῶν ἀτάκτων ἀλόγων, ἃ ὁ Ἀπολλώνιος ἐπὶ πλέον ἐξειργάσατο.

Schol. i. in Eucl. *Elem.* x., Eucl. ed. Heiberg v. 414. 10-16

Ἐν μὲν οὖν τοῖς πρώτοις περὶ συμμέτρων καὶ ἀσυμμέτρων διαλαμβάνει πρὸς τὴν φύσιν αὐτῶν αὐτὰ ἐξετάζων, ἐν δὲ τοῖς ἑξῆς περὶ ῥητῶν καὶ ἀλόγων οὐ πασῶν· τινὲς γὰρ αὐτῷ ὡς ἐνιστάμενοι ἐγκαλοῦσιν· ἀλλὰ τῶν ἁπλουστάτων εἰδῶν, ὧν

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[a] Heath (*H.G.M.* ii. 192-193) conjectures that this work must have dealt with the fundamental principles of mathematics, and to it he assigns various remarks on such subjects attributed to Apollonius by Proclus, and in particular his attempts to prove the axioms. The different ways in which entities are said to be given are stated in the definitions quoted from Euclid's *Data* in vol. i. pp. 478-479.

General Treatise,[a] others the *known*, such as Diodorus.[b]

(x.) *On the Cochlias*

Proclus, *On Euclid* i., ed. Friedlein 105. 1-6

The cylindrical helix is described when a point moves uniformly along a straight line which itself moves round the surface of a cylinder. For in this way there is generated a helix which is *homoeomeric*, any part being such that it will coincide with any other part, as is shown by Apollonius in his work *On the Cochlias*.

(xi.) *On Unordered Irrationals*

Proclus, *On Euclid* i., ed. Friedlein 74. 23-24

The theory of *unordered irrationals*, which Apollonius fully investigated.

Euclid, *Elements* x., Scholium i.,[c] ed. Heiberg v. 414. 10-16

Therefore in the first [theorems of the tenth book] he treats of symmetrical and asymmetrical magnitudes, investigating them according to their nature, and in the succeeding theorems he deals with rational and irrational quantities, but not all, which is held up against him by certain detractors; for he dealt only with the simplest kinds, by the combination of which

[b] Possibly Diodorus of Alexandria, for whom *v.* vol. i. p. 300 and p. 301 n. *b*.

[c] In *Studien über Euklid*, p. 170, Heiberg conjectured that this scholium was extracted from Pappus's commentary, and he has established his conjecture in *Videnskabernes Selskabs Skrifter*, 6 Raekke, hist.-philos. Afd. ii. p. 236 *seq.* (1888).

συντιθεμένων γίνονται ἄπειροι ἄλογοι, ὧν τινας καὶ ὁ Ἀπολλώνιος ἀναγράφει.

(xii.) *Measurement of a Circle*

Eutoc. *Comm. in Archim. Dim. Circ.*, Archim. ed. Heiberg iii. 258. 16-22

Ἰστέον δέ, ὅτι καὶ Ἀπολλώνιος ὁ Περγαῖος ἐν τῷ Ὠκυτοκίῳ ἀπέδειξεν αὐτὸ δι' ἀριθμῶν ἑτέρων ἐπὶ τὸ σύνεγγυς μᾶλλον ἀγαγών. τοῦτο δὲ ἀκριβέστερον μὲν εἶναι δοκεῖ, οὐ χρήσιμον δὲ πρὸς τὸν Ἀρχιμήδους σκοπόν· ἔφαμεν γὰρ αὐτὸν σκοπὸν ἔχειν ἐν τῷδε τῷ βιβλίῳ τὸ σύνεγγυς εὑρεῖν διὰ τὰς ἐν τῷ βίῳ χρείας.

(xiii.) *Continued Multiplications*

Papp. *Coll.* ii. 17-21, ed. Hultsch 18. 23-24. 20[1]

Τούτου δὴ προτεθεωρημένου πρόδηλον, πῶς ἔστιν τὸν δοθέντα στίχον πολλαπλασιάσαι καὶ εἰπεῖν τὸν γενόμενον ἀριθμὸν ἐκ τοῦ τὸν πρῶτον ἀριθμὸν ὃν εἴληφε τὸ πρῶτον τῶν γραμμάτων ἐπὶ τὸν δεύτερον ἀριθμὸν ὃν εἴληφε τὸ δεύτερον τῶν γραμμάτων πολλαπλασιασθῆναι καὶ τὸν γενόμενον ἐπὶ τὸν τρίτον ἀριθμὸν ὃν εἴληφε τὸ τρίτον γράμμα

[1] The extensive interpolations are omitted.

* Pappus's commentary on Eucl. *Elem.* x. was discovered in an Arabic translation by Woepcke (*Mémoires présentés par divers savans à l'Académie des sciences*, 1856, xiv.). It contains several references to Apollonius's work, of which one is thus translated by Woepcke (p. 693): "Enfin, Apollonius distingua les espèces des irrationnelles ordonnées, et

an infinite number of irrationals are formed, of which latter Apollonius also describes some.[a]

(xii.) *Measurement of a Circle*

Eutocius, *Commentary on Archimedes' Measurement of a Circle*, Archim. ed. Heiberg iii. 258. 16-22

It should be noticed, however, that Apollonius of Perga proved the same thing (*sc.* the ratio of the circumference of a circle to the diameter) in the *Quick-deliverer* by a different calculation leading to a closer approximation. This appears to be more accurate, but it is of no use for Archimedes' purpose; for we have stated that his purpose in this book was to find an approximation suitable for the everyday needs of life.[b]

(xiii.) *Continued Multiplications*[c]

Pappus, *Collection* ii. 17-21, ed. Hultsch 18. 23–24. 20[d]

This theorem having first been proved, it is clear how to multiply together a given verse and to tell the number which results when the number represented by the first letter is multiplied into the number represented by the second letter and the product is multiplied into the number represented by the third

découvrit la science des quantités appelées (irrationnelles) inordonnées, dont il produisit un très-grand nombre par des méthodes exactes."

[b] We do not know what the approximation was.

[c] Heiberg (Apollon. Perg. ed. Heiberg ii. 124, n. 1) suggests that these calculations were contained in the Ὠκυτόκιον, but there is no definite evidence.

[d] The passages, chiefly detailed calculations, adjudged by Hultsch to be interpolations are omitted.

καὶ κατὰ τὸ ἑξῆς περαίνεσθαι μέχρι τοῦ διεξοδεύεσθαι τὸν στίχον, ὃν εἶπεν Ἀπολλώνιος ἐν ἀρχῇ οὕτως·

Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι

(τὸ δὲ κλεῖτέ φησιν ἀντὶ τοῦ ὑπομνήσατε).

Ἐπεὶ οὖν γράμματά ἐστιν λ̅η̅ τοῦ στίχου, ταῦτα δὲ περιέχει ἀριθμοὺς δέκα τοὺς ρ̅ τ̅ σ̅ τ̅ ρ̅ τ̅ σ̅ χ̅ υ̅ ρ̅, ὧν ἕκαστος ἐλάσσων μέν ἐστιν χιλιάδος μετρεῖται δὲ ὑπὸ ἑκατοντάδος, καὶ ἀριθμοὺς ι̅ζ̅ τοὺς μ̅ ι̅ ο̅ κ̅ λ̅ ι̅ κ̅ ο̅ ξ̅ ο̅ ο̅ ν̅ ν̅ ν̅ κ̅ ο̅ ι̅, ὧν ἕκαστος ἐλάσσων μέν ἐστιν ἑκατοντάδος μετρεῖται δὲ ὑπὸ δεκάδος, καὶ τοὺς λοιποὺς ι̅α̅ τοὺς α̅ ε̅ δ̅ ε̅ ε̅ α̅ ε̅ ε̅ ε̅ α̅ α̅, ὧν ἕκαστος ἐλάσσων δεκάδος, ἐὰν ἄρα τοῖς μὲν δέκα ἀριθμοῖς ὑποτάξωμεν ἰσαρίθμους δέκα κατὰ τάξιν ἑκατοντάδος, τοῖς δὲ ι̅ζ̅ ὁμοίως ὑποτάξωμεν δεκάδας ι̅ζ̅, φανερὸν ἐκ τοῦ ἀνώτερου λογιστικοῦ θεωρήματος ιβʹ ὅτι δέκα ἑκατοντάδες μετὰ τῶν ι̅ζ̅ δεκάδων ποιοῦσι μυριάδας ἐνναπλᾶς δέκα.

Ἐπεὶ δὲ καὶ πυθμένες ὁμοῦ τῶν μετρουμένων ἀριθμῶν ὑπὸ ἑκατοντάδος καὶ τῶν μετρουμένων ὑπὸ δεκάδος εἰσὶν οἱ ὑποκείμενοι κ̅ζ̅

α̅ γ̅ β̅ γ̅ α̅ γ̅ β̅ ϛ̅ δ̅ α̅
δ̅ α̅ ζ̅ β̅ γ̅ α̅ β̅ ζ̅ ϛ̅ ζ̅ ζ̅ ε̅ ε̅ ε̅ β̅ ζ̅ α̅,

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[a] Apollonius, it is clear from Pappus, had a system of *tetrads* for calculations involving big numbers, the unit being the *myriad* or fourth power of 10. The tetrads are called μυριάδες ἁπλαῖ, μυριάδες διπλαῖ, μυριάδες τριπλαῖ, *simple myriads*, *double myriads*, *triple myriads* and so on, by which are meant 10000, 10000^2, 10000^3 and so on. In the text of

letter and so on in order until the end of the verse which Apollonius gave in the beginning, that is

Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι

(where he says κλεῖτε for ὑπομνήσατε, *recall to mind*).

Since there are thirty-eight letters in the verse, of which ten, namely $\bar{\rho}\ \bar{\tau}\ \bar{\sigma}\ \bar{\tau}\ \bar{\rho}\ \bar{\tau}\ \bar{\sigma}\ \bar{\chi}\ \bar{\upsilon}\ \bar{\rho}$ (=100, 300, 200 300, 100, 300, 200, 600, 400, 100), represent numbers less than 1000 and divisible by 100, and seventeen, namely $\bar{\mu}\ \bar{\iota}\ \bar{o}\ \bar{\kappa}\ \bar{\lambda}\ \bar{\iota}\ \bar{\kappa}\ \bar{o}\ \bar{\xi}\ \bar{o}\ \bar{o}\ \bar{\nu}\ \bar{\nu}\ \bar{\nu}\ \bar{\kappa}\ \bar{o}\ \bar{\iota}$ (=40, 10, 70, 20, 30, 10, 20, 70, 60, 70, 70, 50, 50, 50, 20, 70, 10), represent numbers less than 100 and divisible by 10, while the remaining eleven, namely, $\bar{\alpha}\ \bar{\epsilon}\ \bar{\delta}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\alpha}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\alpha}\ \bar{\alpha}$ (=1, 5, 4, 5, 5, 1, 5, 5, 5, 1, 1), represent numbers less than 10, then if for those ten numbers we substitute an equal number of hundreds, and if for the seventeen numbers we similarly substitute seventeen tens, it is clear from the above arithmetical theorem, the twelfth, that the ten hundreds together with the seventeen tens make $10 \cdot 10000^9$.[a]

And since the bases of the numbers divisible by 100 and those divisible by 10 are the following twenty-seven

1, 3, 2, 3, 1, 3, 2, 6, 4, 1

4, 1, 7, 2, 3, 1, 2, 7, 6, 7, 7, 5, 5, 5, 2, 7, 1,

Pappus they are sometimes abbreviated to μ^{α}, μ^{β}, μ^{γ} and so on.

[a] From Pappus, though the text is defective, Apollonius's procedure in multiplying together powers of 10 can be seen to be equivalent to adding the indices of the separate powers of 10, and then dividing by 4 to obtain the power of the myriad which the product contains. If the division is exact, the number is the n-myriad, say, meaning 10000^n. If there is a remainder, 3, 2 or 1, the number is 1000, 100 or 10 times the n-myriad as the case may be.

ἀλλὰ καὶ τῶν ἐλασσόνων δεκάδος εἰσὶν $\overline{\iota\alpha}$, τουτέστιν ἀριθμοὶ οἱ

$$\bar{\alpha}\ \bar{\epsilon}\ \bar{\delta}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\alpha}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\epsilon}\ \bar{\alpha}\ \bar{\alpha},$$

ἐὰν τὸν ἐκ τούτων τῶν $\overline{\iota\alpha}$ καὶ τὸν ἐκ τῶν $\overline{\kappa\zeta}$ πυθμένων στερεὸν δι' ἀλλήλων πολλαπλασιάσωμεν, ἔσται ὁ στερεὸς μυριάδων τετραπλῶν $\overline{\iota\theta}$ καὶ τριπλῶν $\overline{,\digamma\lambda\digamma}$ καὶ διπλῶν $\overline{,\eta\upsilon\pi}$.

Αὗται δὴ συμπολλαπλασιαζόμεναι ἐπὶ τὸν ἐκ τῶν ἑκατοντάδων καὶ δεκάδων στερεόν, τουτέστι τὰς προκειμένας μυριάδας ἐνναπλᾶς δέκα, ποιοῦσιν μυριάδας τρισκαιδεκαπλᾶς $\overline{\rho\digamma\digamma}$, δωδεκαπλᾶς $\overline{\tau\xi\eta}$, ἑνδεκαπλᾶς $\overline{,\delta\omega}$.

(xiv.) *On the Burning Mirror*

Fragmentum mathematicum Bobiense 113. 28-33, ed. Belger, *Hermes*, xvi., 1881, 279-280 [1]

Οἱ μὲν οὖν παλαιοὶ ὑπέλαβον τὴν ἔξαψιν ποιεῖσθαι περὶ τὸ κέντρον τοῦ κατόπτρου, τοῦτο δὲ ψεῦδος Ἀπολλώνιος μάλα δεόντως . . . (ἐν τῷ) πρὸς τοὺς κατοπτρικοὺς ἔδειξεν, καὶ περὶ τίνα δὲ τόπον ἡ ἐκπύρωσις ἔσται, διασεσάφηκεν ἐν τῷ Περὶ τοῦ πυρίου.

[1] As amended by Heiberg, *Zeitschrift für Mathematik und Physik*, xxviii., 1883, hist. Abth. 124-125.

while there are eleven less than ten, that is the numbers

1, 5, 4, 5, 5, 1, 5, 5, 5, 1, 1,

if we multiply together the solid number formed by these eleven with the solid number formed by the twenty-seven the result will be the solid number

$$19 \,.\, 10000^4 + 6036 \,.\, 10000^3 + 8480 \,.\, 10000^2.$$

When these numbers are multiplied into the solid number formed by the hundreds and the tens, that is with $10 \,.\, 10000^9$ as calculated above, the result is

$$196 \,.\, 10000^{13} + 368 \,.\, 10000^{12} + 4800 \,.\, 10000^{11}.$$

(xiv.) *On the Burning Mirror*

Fragmentum mathematicum Bobiense 113. 28-33,[a] ed. Belger, *Hermes*, xvi., 1881, 279-280

The older geometers thought that the burning took place at the centre of the mirror, but Apollonius very suitably showed this to be false . . . in his work on mirrors, and he explained clearly where the kindling takes place in his works *On the Burning Mirror*.[b]

[a] This fragment is attributed to Anthemius by Heiberg, but its antiquated terminology leads Heath (*H.G.M.* ii. 194) to suppose that it is much earlier.

[b] Of Apollonius's other achievements, his solution of the problem of finding two mean proportionals has already been mentioned (vol. i. p. 267 n. *b*) and sufficiently indicated; for his astronomical work the reader is referred to Heath, *H.G.M.* ii. 195-196.

XX. LATER DEVELOPMENTS IN GEOMETRY

XX. LATER DEVELOPMENTS IN GEOMETRY

(a) Classification of Curves

Procl. *in Eucl.* i., ed. Friedlein 111. 1–112. 11

Διαιρεῖ δ' αὖ τὴν γραμμὴν ὁ Γέμινος[1] πρῶτον μὲν εἰς τὴν ἀσύνθετον καὶ τὴν σύνθετον—καλεῖ δὲ σύνθετον τὴν κεκλασμένην καὶ γωνίαν ποιοῦσαν—ἔπειτα τὴν ἀσύνθετον[2] εἴς τε τὴν σχηματοποιοῦσαν καὶ τὴν ἐπ' ἄπειρον ἐκβαλλομένην, σχῆμα λέγων ποιεῖν τὴν κυκλικήν, τὴν τοῦ θυρεοῦ, τὴν κιττοειδῆ, μὴ ποιεῖν δὲ τὴν τοῦ ὀρθογωνίου κώνου τομήν, τὴν τοῦ ἀμβλυγωνίου, τὴν κογχοειδῆ, τὴν εὐθεῖαν, πάσας τὰς τοιαύτας. καὶ πάλιν κατ' ἄλλον τρόπον τῆς ἀσυνθέτου γραμμῆς τὴν μὲν ἁπλῆν εἶναι, τὴν δὲ μικτήν, καὶ τῆς ἁπλῆς τὴν μὲν σχῆμα ποιεῖν ὡς τὴν κυκλικήν, τὴν δὲ ἀόριστον εἶναι ὡς τὴν εὐθεῖαν, τῆς δὲ μικτῆς τὴν μὲν ἐν τοῖς ἐπιπέδοις εἶναι, τὴν δὲ ἐν τοῖς στερεοῖς, καὶ τῆς ἐν ἐπιπέδοις τὴν μὲν ἐν αὑτῇ συμπίπτειν ὡς τὴν κιττοειδῆ, τὴν δ' ἐπ' ἄπειρον ἐκβάλλεσθαι, τῆς δὲ ἐν στερεοῖς

[1] Γέμινος Tittel, Γεμῖνος Friedlein.
[2] σύνθετον codd., correxi.

[a] No great new developments in geometry were made by the Greeks after the death of Apollonius, probably through

XX. LATER DEVELOPMENTS IN GEOMETRY [a]

(a) Classification of Curves

Proclus, *On Euclid* i., ed. Friedlein 111. 1–112. 11

Geminus first divides lines into the *incomposite* and the *composite*, meaning by composite the broken line forming an angle; and then he divides the incomposite into those *forming a figure* and those *extending without limit*, including among those forming a figure the circle, the ellipse and the cissoid, and among those not forming a figure the parabola, the hyperbola, the conchoid, the straight line, and all such lines. Again, in another manner he says that some incomposite lines are *simple*, others *mixed*, and among the simple are some *forming a figure*, such as the circle, and others *indeterminate*, such as the straight line, while the mixed include both *lines on planes* and *lines on solids*, and among the lines on planes are lines *meeting themselves*, such as the cissoid, and others *extending without limit*, and among lines on solids are

the limits imposed by their methods, and the recorded additions to the corpus of Greek mathematics may be described as reflections upon existing work or "stock-taking." On the basis of geometry, however, the new sciences of trigonometry and mensuration were founded, as will be described, and the revival of geometry by Pappus will also be reserved for separate treatment.

τὴν μὲν κατὰ τὰς τομὰς ἐπινοεῖσθαι τῶν στερεῶν, τὴν δὲ περὶ τὰ στερεὰ ὑφίστασθαι. τὴν μὲν γὰρ ἕλικα τὴν περὶ σφαῖραν ἢ κῶνον περὶ τὰ στερεὰ ὑφεστάναι, τὰς δὲ κωνικὰς τομὰς ἢ τὰς σπειρικὰς ἀπὸ τοιᾶσδε τομῆς γεννᾶσθαι τῶν στερεῶν. ἐπινενοῆσθαι δὲ ταύτας τὰς τομὰς τὰς μὲν ὑπὸ Μεναίχμου τὰς κωνικάς, ὃ καὶ Ἐρατοσθένης ἱστορῶν λέγει· "μὴ δὲ Μεναιχμίους κωνοτομεῖν τριάδας". τὰς δὲ ὑπὸ Περσέως, ὃς καὶ τὸ ἐπίγραμμα ἐποίησεν ἐπὶ τῇ εὑρέσει—

Τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς εὑρὼν ἑλικώδεις[1]
  Περσεὺς τῶν δ' ἕνεκεν δαίμονας ἱλάσατο.

αἱ μὲν δὴ τρεῖς τομαὶ τῶν κώνων εἰσὶν παραβολὴ καὶ ὑπερβολὴ καὶ ἔλλειψις, τῶν δὲ σπειρικῶν τομῶν ἡ μέν ἐστιν ἐμπεπλεγμένη, ἐοικυῖα τῇ τοῦ ἵππου πέδῃ, ἡ δὲ κατὰ τὰ μέσα πλατύνεται, ἐξ ἑκατέρου δὲ ἀπολήγει μέρους, ἡ δὲ παραμήκης οὖσα τῷ μὲν μέσῳ διαστήματι ἐλάττονι χρῆται, εὐρύνεται δὲ ἐφ' ἑκάτερα. τῶν δὲ ἄλλων μίξεων τὸ πλῆθος ἀπέραντόν ἐστιν· καὶ γὰρ στερεῶν σχημάτων πλῆθός ἐστιν ἄπειρον καὶ τομαὶ αὐτῶν συνίστανται πολυειδεῖς.

Ibid., ed. Friedlein 356. 8-12

Καὶ γὰρ Ἀπολλώνιος ἐφ' ἑκάστης τῶν κωνικῶν γραμμῶν τί τὸ σύμπτωμα δείκνυσι, καὶ ὁ Νικομήδης ἐπὶ τῶν κογχοειδῶν, καὶ ὁ Ἱππίας ἐπὶ

[1] ἑλικώδεις Knoche, εὑρὼν τὰς σπειρικὰς λέγων codd.

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[a] v. vol. i. pp. 296-297.
[b] For Perseus, v. p. 364 n. a and p. 365 n. b.

lines conceived as *formed by sections* of the solids and lines *formed round the solids*. The helix round the sphere or cone is an example of the lines formed round solids, and the conic sections or the spiric curves are generated by various sections of solids. Of these sections, the conic sections were discovered by Menaechmus, and Eratosthenes in his account says: "Cut not the cone in the triads of Menaechmus"[a]; and the others were discovered by Perseus,[b] who wrote an epigram on the discovery—

Three spiric lines upon five sections finding,
Perseus thanked the gods therefor.

Now the three conic sections are the parabola, the hyperbola and the ellipse, while of the spiric sections one is *interlaced*, resembling the horse-fetter, another is widened out in the middle and contracts on each side, a third is elongated and is narrower in the middle, broadening out on either side. The number of the other mixed lines is unlimited; for the number of solid figures is infinite and there are many different kinds of section of them.

Ibid., ed. Friedlein 356. 8-12

For Apollonius shows for each of the conic curves what is its property, as does Nicomedes for the

τῶν τετραγωνιζουσῶν, καὶ ὁ Περσεὺς ἐπὶ τῶν σπειρικῶν.

Ibid., ed. Friedlein 119. 8-17

Ὃ δὲ συμβαίνειν φαμὲν κατὰ τὴν σπειρικὴν ἐπιφάνειαν· κατὰ γὰρ κύκλου νοεῖται στροφὴν ὀρθοῦ διαμένοντος καὶ στρεφομένου περὶ τὸ αὐτὸ σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύκλου, διὸ καὶ τριχῶς ἡ σπεῖρα γίνεται, ἢ γὰρ ἐπὶ τῆς περιφερείας ἐστὶ τὸ κέντρον ἢ ἐντὸς ἢ ἐκτός. καὶ εἰ μὲν ἐπὶ τῆς περιφερείας ἐστὶ τὸ κέντρον, γίνεται σπεῖρα συνεχής, εἰ δὲ ἐντός, ἡ ἐμπεπλεγμένη, εἰ δὲ ἐκτός, ἡ διεχής. καὶ τρεῖς αἱ σπειρικαὶ τομαὶ κατὰ τὰς τρεῖς ταύτας διαφοράς.

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[a] Obviously the work of Perseus was on a substantial scale to be associated with these names, but nothing is known of him beyond these two references. He presumably flourished after Euclid (since the conic sections were probably well developed before the spiric sections were tackled) and before Geminus (since Proclus relies on Geminus for his knowledge of the spiric curves). He may therefore be placed between 300 and 75 B.C.

Nicomedes appears to have flourished between Eratosthenes and Apollonius. He is known only as the inventor of the conchoid, which has already been fully described (vol. i. pp. 298-309).

It is convenient to recall here that about a century later flourished Diocles, whose discovery of the cissoid has already been sufficiently noted (vol. i. pp. 270-279). He has also been referred to as the author of a brilliant solution of the problem of dividing a cone in a given ratio, which is equivalent to the solution of a cubic equation (*supra*, p. 162 n. *a*). The Dionysodorus who solved the same problem (*ibid.*) may have been the Dionysodorus of Caunus mentioned in the Herculaneum Roll, No. 1044 (so W. Schmidt in *Bibliotheca mathematica*, iv. pp. 321-325), a younger contemporary of Apollonius; he is presumably the same person as the

conchoid and Hippias for the quadratices and Perseus for the spiric curves.[a]

Ibid., ed. Friedlein 119. 8-17

We say that this is the case with the spiric surface; for it is conceived as generated by the revolution of a circle remaining perpendicular [to a given plane] and turning about a fixed point which is not its centre. Hence there are three forms of spire according as the centre is on the circumference, or within it, or without. If the centre is on the circumference, the spire generated is said to be *continuous*, if within *interlaced*, and if without *open*. And there are three spiric sections according to these three differences.[b]

Dionysodorus mentioned by Heron, *Metrica* ii. 13 (cited *infra*, p. 481), as the author of a book *On the Spire*.

[b] This last sentence is believed to be a slip, perhaps due to too hurried transcription from Geminus. At any rate, no satisfactory meaning can be obtained from the sentence as it stands. Tannery (*Mémoires scientifiques* ii. pp. 24-28) interprets Perseus' epigram as meaning "three curves *in addition to* five sections." He explains the passages thus: Let a be the radius of the generating circle, c the distance of the centre of the generating circle from the axis of revolution, d the perpendicular distance of the plane of section (assumed to be parallel to the axis of revolution) from the axis of revolution. Then in the *open* spire, in which $c>a$, there are five different cases:

(1) $c+a>d>c$. The curve is an oval.

(2) $d=c$. Transition to (3).

(3) $c>d>c-a$. The curve is a closed curve narrowest in the middle.

(4) $d=c-a$. The curve is the *hippopede* (horse-fetter), which is shaped like the figure of 8 (*v*. vol. i. pp. 414-415 for the use of this curve by Eudoxus).

(5) $c-a>d>0$. The section consists of two symmetrical ovals.

Tannery identifies the "five sections" of Perseus with these five types of section of the open spire; the three curves

(b) Attempts to Prove the Parallel Postulate

(i.) *General*

Procl. *in Eucl.* i., ed. Friedlein 191. 16–193. 9

"Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάττονας ποιῇ, ἐκβαλλομένας τὰς εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάττονες."

Τοῦτο καὶ παντελῶς διαγράφειν χρὴ τῶν αἰτημάτων· θεώρημα γάρ ἐστι, πολλὰς μὲν ἀπορίας ἐπιδεχόμενον, ἃς καὶ ὁ Πτολεμαῖος ἔν τινι βιβλίῳ διαλῦσαι προύθετο, πολλῶν δὲ εἰς ἀπόδειξιν δεόμενον καὶ ὅρων καὶ θεωρημάτων. καὶ τό γε ἀντιστρέφον καὶ ὁ Εὐκλείδης ὡς θεώρημα δείκνυσιν. ἴσως δὲ ἄν τινες ἀπατώμενοι καὶ τοῦτο τάττειν ἐν τοῖς αἰτήμασιν ἀξιώσειαν, ὡς διὰ τὴν ἐλάττωσιν τῶν δύο ὀρθῶν αὐτόθεν τὴν πίστιν παρεχόμενον

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described by Proclus are (1), (3) and (4). When the spire is *continuous* or closed, $c=a$ and there are only three sections corresponding to (1), (2) and (3); (4) and (5) reduce to two equal circles touching one another. But the *interlaced* spire, in which $c<a$, gives three new types of section, and in these Tannery sees his "three curves in addition to five sections." There are difficulties in the way of accepting this interpretation, but no better has been proposed.

Further passages on the spire by Heron, including a formula for its volume, are given *infra*, pp. 476–483.

[a] Eucl. i. Post. 5, for which *v.* vol. i. pp. 442–443, especially n. *c.*

Aristotle (*Anal. Prior.* ii. 16, 65 a 4) alludes to a *petitio principii* current in his day among those who "think they establish the theory of parallels"—τὰς παραλλήλους γράφειν. As Heath notes (*The Thirteen Books of Euclid's Elements*,

(b) Attempts to Prove the Parallel Postulate

(i.) *General*

Proclus, *On Euclid* i., ed. Friedlein 191. 16–193. 9

"*If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.*" [a]

This ought to be struck right out of the Postulates; for it is a theorem, and one involving many difficulties, which Ptolemy set himself to resolve in one of his books, and for its proof it needs a number of definitions as well as theorems. Euclid actually proves its converse as a theorem. Possibly some would erroneously consider it right to place this assumption among the Postulates, arguing that, as the angles are less than two right angles, there is

vol. i. pp. 191-192). Philoponus's comment on this passage suggests that the *petitio principii* lay in a *direction* theory of parallels. Euclid appears to have admitted the validity of the criticism and, by *assuming* his famous postulate once and for all, to have countered any logical objections.

Nevertheless, as the extracts here given will show, ancient geometers were not prepared to accept the undemonstrable character of the postulate. Attempts to prove it continued to be made until recent times, and are summarized by R. Bonola, "Sulla teoria delle parallele e sulle geometrie non-euclidee" in *Questioni riguardanti la geometria elementare*, and by Heath, *loc. cit.*, pp. 204-219. The chapter on the subject in W. Rouse Ball's *Mathematical Essays and Recreations*, pp. 307-326, may also be read with profit. Attempts to prove the postulate were abandoned only when it was shown that, by not conceding it, alternative geometries could be built.

τῆς τῶν εὐθειῶν συνεύσεως καὶ συμπτώσεως. πρὸς οὓς ὁ Γεμῖνος ὀρθῶς ἀπήντησε λέγων ὅτι παρ' αὐτῶν ἐμάθομεν τῶν τῆς ἐπιστήμης ταύτης ἡγεμόνων μὴ πάνυ προσέχειν τὸν νοῦν ταῖς πιθαναῖς φαντασίαις εἰς τὴν τῶν λόγων τῶν ἐν γεωμετρίᾳ παραδοχήν. ὅμοιον γάρ φησι καὶ Ἀριστοτέλης ῥητορικὸν ἀποδείξεις ἀπαιτεῖν καὶ γεωμέτρου πιθανολογοῦντος ἀνέχεσθαι, καὶ ὁ παρὰ τῷ Πλάτωνι Σιμμίας, ὅτι "τοῖς ἐκ τῶν εἰκότων τὰς ἀποδείξεις ποιουμένοις σύνοιδα οὖσιν ἀλαζόσι." κἀνταῦθα τοίνυν τὸ μὲν ἠλαττωμένων τῶν ὀρθῶν συνεύειν τὰς εὐθείας ἀληθὲς καὶ ἀναγκαῖον, τὸ δὲ συνευούσας ἐπὶ πλέον ἐν τῷ ἐκβάλλεσθαι συμπεσεῖσθαί ποτε πιθανόν, ἀλλ' οὐκ ἀναγκαῖον, εἰ μή τις ἀποδείξειεν λόγος, ὅτι ἐπὶ τῶν εὐθειῶν τοῦτο ἀληθές. τὸ γὰρ εἶναί τινας γραμμὰς συννευούσας μὲν ἐπ' ἄπειρον, ἀσυμπτώτους δὲ ὑπαρχούσας, καίτοι δοκοῦν ἀπίθανον εἶναι καὶ παράδοξον, ὅμως ἀληθές ἐστι καὶ πεφώραται ἐπ' ἄλλων εἰδῶν τῆς γραμμῆς. μήποτε οὖν τοῦτο καὶ ἐπὶ τῶν εὐθειῶν δυνατόν, ὅπερ ἐπ' ἐκείνων τῶν γραμμῶν; ἕως γὰρ ἂν δι' ἀποδείξεως αὐτὸ καταδησώμεθα, περισπᾷ τὴν φαντασίαν τὰ ἐπ' ἄλλων δεικνύμενα γραμμῶν. εἰ δὲ καὶ οἱ διαμφισβητοῦντες λόγοι πρὸς τὴν σύμπτωσιν πολὺ τὸ πληκτικὸν ἔχοιεν, πῶς οὐχὶ πολλῷ πλέον ἂν τὸ πιθανὸν τοῦτο καὶ τὸ ἄλογον ἐκβάλλοιμεν τῆς ἡμετέρας παραδοχῆς;

Ἀλλ' ὅτι μὲν ἀπόδειξιν χρὴ ζητεῖν τοῦ προκειμένου θεωρήματος δῆλον ἐκ τούτων, καὶ ὅτι

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[a] For Geminus, v. *infra*, p. 370 n. c.

immediate reason for believing that the straight lines converge and meet. To such, Geminus [a] rightly rejoined that we have learnt from the pioneers of this science not to incline our mind to mere plausible imaginings when it is a question of the arguments to be used in geometry. For Aristotle [b] says it is as reasonable to demand scientific proof from a rhetorician as to accept mere plausibilities from a geometer, and Simmias is made to say by Plato [c] that he "recognizes as quacks those who base their proofs on probabilities." In this case the convergence of the straight lines by reason of the lessening of the right angles is true and necessary, but the statement that, since they converge more and more as they are produced, they will some time meet is plausible but not necessary, unless some argument is produced to show that this is true in the case of straight lines. For the fact that there are certain lines which converge indefinitely but remain non-secant, although it seems improbable and paradoxical, is nevertheless true and well-established in the case of other species of lines. May not this same thing be possible in the case of straight lines as happens in the case of those other lines? For until it is established by rigid proof, the facts shown in the case of other lines may turn our minds the other way. And though the controversial arguments against the meeting of the two lines should contain much that is surprising, is that not all the more reason for expelling this merely plausible and irrational assumption from our accepted teaching?

It is clear that a proof of the theorem in question must be sought, and that it is alien to the special

[b] *Eth. Nic.* i. 3. 4, 1094 b 25-27.   [c] *Phaedo* 92 D.

τῆς τῶν αἰτημάτων ἐστὶν ἀλλότριον ἰδιότητος, πῶς δὲ ἀποδεικτέον αὐτὸ καὶ διὰ ποίων λόγων ἀναιρετέον τὰς πρὸς αὐτὸ φερομένας ἐνστάσεις, τηνικαῦτα λεκτέον, ἡνίκα ἂν καὶ ὁ στοιχειωτὴς αὐτοῦ μέλλῃ ποιεῖσθαι μνήμην ὡς ἐναργεῖ προσχρώμενος. τότε γὰρ ἀναγκαῖον αὐτοῦ δεῖξαι τὴν ἐνάργειαν οὐκ ἀναποδείκτως προφαινομένην ἀλλὰ δι᾽ ἀποδείξεων γνώριμον γιγνομένην.

(ii.) *Posidonius and Geminus*

Ibid., ed. Friedlein 176. 5-10

Καὶ ὁ μὲν Εὐκλείδης τοῦτον ὁρίζεται τὸν τρόπον τὰς παραλλήλους εὐθείας, ὁ δὲ Ποσειδώνιος, παράλληλοι, φησίν, εἰσὶν αἱ μήτε συνεύουσαι μήτε ἀπονεύουσαι ἐν ἑνὶ ἐπιπέδῳ, ἀλλ᾽ ἴσας ἔχουσαι

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[a] *i.e.*, Eucl. i. 28.

[b] Posidonius was a Stoic and the teacher of Cicero; he was born at Apamea and taught at Rhodes, flourishing 151–135 B.C. He contributed a number of definitions to elementary geometry, as we know from Proclus, but is more famous for a geographical work *On the Ocean* (lost but copiously quoted by Strabo) and for an astronomical work Περὶ μετεώρων. In this he estimated the circumference of the earth (*v. supra*, p. 267) and he also wrote a separate work on the size of the sun.

[c] As with so many of the great mathematicians of antiquity, we know practically nothing about Geminus's life, not even his date, birthplace or the correct spelling of his name. As he wrote a commentary on Posidonius's Περὶ μετεώρων, we have an upper limit for his date, and " the view most generally accepted is that he was a Stoic philosopher, born probably in the island of Rhodes, and a pupil of Posidonius, and that he wrote about 73–67 B.C. " (Heath, *H.G.M.* ii. 223). Further details may be found in Manitius's edition of the so-called *Gemini elementa astronomiae*.

Geminus wrote an encyclopaedic work on mathematics

character of the Postulates. But how it should be proved, and by what sort of arguments the objections made against it may be removed, must be stated at the point where the writer of the *Elements* is about to recall it and to use it as obvious.[a] Then it will be necessary to prove that its obvious character does not appear independently of proof, but by proof is made a matter of knowledge.

(ii.) *Posidonius*[b] *and Geminus*[c]

Ibid., ed. Friedlein 176. 5-10

Such is the manner in which Euclid defines parallel straight lines, but Posidonius says that parallels are lines in one plane which neither converge nor diverge

which is referred to by ancient writers under various names, but that used by Eutocius (Τῶν μαθημάτων θεωρία, *v. supra*, pp. 280-281) was most probably the actual title. It is unfortunately no longer extant, but frequent references are made to it by Proclus, and long extracts are preserved in an Arabic commentary by an-Nairīzī.

It is from this commentary that Geminus is known to have attempted to prove the parallel-postulate by a definition of parallels similar to that of Posidonius. The method is reproduced in Heath, *H.G.M.* ii. 228-230. It tacitly assumes "Playfair's axiom," that through a given point only one parallel can be drawn to a given straight line; this axiom —which was explicitly stated by Proclus in his commentary on Eucl. i. 30 (Procl. *in Eucl.* i., ed. Friedlein 374. 18–375. 3)—is, in fact, equivalent to Euclid's Postulate 5. Saccheri noted an even more fundamental objection, that, before Geminus's definition of parallels can be used, it has to be proved that the locus of points equidistant from a straight line is a straight line; and this cannot be done without some equivalent postulate. Nevertheless, Geminus deserves to be held in honour as the author of the first known attempt to prove the parallel-postulate, a worthy predecessor to Lobachewsky and Riemann.

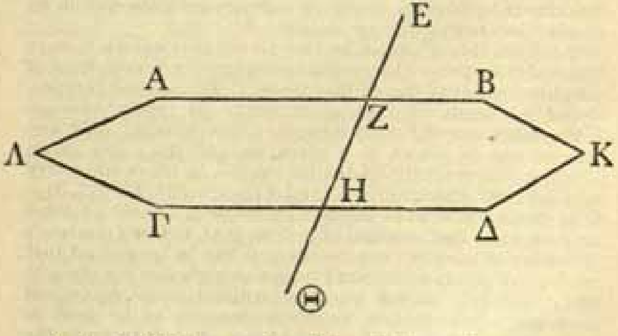
πάσας τὰς καθέτους τὰς ἀγομένας ἀπὸ τῶν τῆς ἑτέρας σημείων ἐπὶ τὴν λοιπήν.

(iii.) *Ptolemy*

Ibid., ed. Friedlein 362. 12–363. 18

Ἀλλ' ὅπως μὲν ὁ Στοιχειωτὴς δείκνυσιν ὅτι δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς αἱ εὐθεῖαι παράλληλοί εἰσι, φανερὸν ἐκ τῶν γεγραμμένων. Πτολεμαῖος δὲ ἐν οἷς ἀποδεῖξαι προέθετο τὰς ἀπ' ἐλαττόνων ἢ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες, τοῦτο πρὸ πάντων δεικνὺς τὸ θεώρημα τὸ δυεῖν ὀρθαῖς ἴσων ὑπαρχουσῶν τῶν ἐντὸς παραλλήλους εἶναι τὰς εὐθείας οὕτω πως δείκνυσιν.

Ἔστωσαν δύο εὐθεῖαι αἱ ΑΒ, ΓΔ, καὶ τεμνέτω τις αὐτὰς εὐθεῖα ἡ ΕΖΗΘ, ὥστε τὰς ὑπὸ ΒΖΗ



καὶ ὑπὸ ΖΗΔ γωνίας δύο ὀρθαῖς ἴσας ποιεῖν. λέγω ὅτι παράλληλοί εἰσιν αἱ εὐθεῖαι, τουτέστιν

but the perpendiculars drawn from points on one of the lines to the other are all equal.

(iii.) *Ptolemy* [a]

Ibid., ed. Friedlein 362. 12–363. 18

How the writer of the *Elements* proves that, if the interior angles be equal to two right angles, the straight lines are parallel is clear from what has been written. But Ptolemy, in the work [b] in which he attempted to prove that straight lines produced from angles less than two right angles will meet on the side on which the angles are less than two right angles, first proved this theorem, that if *the interior angles be equal to two right angles the lines are parallel*, and he proves it somewhat after this fashion.

Let the two straight lines be AB, ΓΔ, and let any straight line EZHΘ cut them so as to make the angles BZH and ZHΔ equal to two right angles. I say that the straight lines are parallel, that is they are non-

[a] For the few details known about Ptolemy, *v. infra*, p. 408 and n. *b*.
[b] This work is not otherwise known.

ἀσύμπτωτοί εἰσιν. εἰ γὰρ δυνατόν, συμπιπτέτωσαν ἐκβαλλόμεναι αἱ ΒΖ, ΗΔ κατὰ τὸ Κ. ἐπεὶ οὖν εὐθεῖα ἡ ΗΖ ἐφέστηκεν ἐπὶ τὴν ΑΒ, δύο ὀρθαῖς ἴσας ποιεῖ τὰς ὑπὸ ΑΖΗ, ΒΖΗ γωνίας. ὁμοίως δέ, ἐπεὶ ἡ ΗΖ ἐφέστηκεν ἐπὶ τὴν ΓΔ, δύο ὀρθαῖς ἴσας ποιεῖ τὰς ὑπὸ ΓΗΖ, ΔΗΖ γωνίας. αἱ τέσσαρες ἄρα αἱ ὑπὸ ΑΖΗ, ΒΖΗ, ΓΗΖ, ΔΗΖ τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ὧν αἱ δύο αἱ ὑπὸ ΒΖΗ, ΖΗΔ δύο ὀρθαῖς ὑπόκεινται ἴσαι. λοιπαὶ ἄρα αἱ ὑπὸ ΑΖΗ, ΓΗΖ καὶ αὗται δύο ὀρθαῖς ἴσαι. εἰ οὖν αἱ ΖΒ, ΗΔ δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς ἐκβαλλόμεναι συνέπεσον κατὰ τὸ Κ, καὶ αἱ ΖΑ, ΗΓ ἐκβαλλόμεναι συμπεσοῦνται. δύο γὰρ ὀρθαῖς καὶ αἱ ὑπὸ ΑΖΗ, ΓΗΖ ἴσαι εἰσίν. ἢ γὰρ κατ' ἀμφότερα συμπεσοῦνται αἱ εὐθεῖαι, ἢ κατ' οὐδέτερα, εἴπερ καὶ αὗται κἀκεῖναι δύο ὀρθαῖς εἰσιν ἴσαι. συμπιπτέτωσαν οὖν αἱ ΖΑ, ΗΓ κατὰ τὸ Λ. αἱ ἄρα ΛΑΒΚ, ΛΓΔΚ εὐθεῖαι χωρίον περιέχουσιν, ὅπερ ἀδύνατον. οὐκ ἄρα δυνατόν ἐστιν δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς συμπίπτειν τὰς εὐθείας. παράλληλοι ἄρα εἰσίν.

Ibid., ed. Friedlein 365. 5–367. 27

Ἤδη μὲν οὖν καὶ ἄλλοι τινὲς ὡς θεώρημα προτάξαντες τοῦτο αἴτημα παρὰ τῷ Στοιχειωτῇ ληφθὲν ἀποδείξεως ἠξίωσαν. δοκεῖ δὲ καὶ ὁ Πτολεμαῖος

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[a] There is a Common Notion to this effect interpolated in the text of Euclid; *v.* vol. i. pp. 444 and 445 n. *a*.

[b] The argument would have been clearer if it had been proved that the two interior angles on one side of ΖΗ were *severally* equal to the two interior angles on the other side, that is ΒΖΗ = ΓΗΖ and ΔΗΖ = ΑΖΗ; whence, if ΖΑ, ΗΓ meet at Λ, the triangle ΖΗΛ can be rotated about the mid-

secant. For, if it be possible, let BZ, HΔ, when produced, meet at K. Then since the straight line HZ stands on AB, it makes the angles AZH, BZH equal to two right angles [Eucl. i. 13]. Similarly, since HZ stands on ΓΔ, it makes the angles ΓHZ, ΔHZ equal to two right angles [*ibid.*]. Therefore the four angles AZH, BZH, ΓHZ, ΔHZ are equal to four right angles, and of them two, BZH, ZHΔ, are by hypothesis equal to two right angles. Therefore the remaining angles AZH, ΓHZ are also themselves equal to two right angles. If then, the interior angles being equal to two right angles, ZB, HΔ meet at K when produced, ZA, HΓ will also meet when produced. For the angles AZH, ΓHZ are also equal to two right angles. Therefore the straight lines will either meet on both sides or on neither, since these angles also are equal to two right angles. Let ZA, HΓ meet, then, at Λ. Then the straight lines ΛABK, ΛΓΔK enclose a space, which is impossible.[a] Therefore it is not possible that, if the interior angles be equal to two right angles, the straight lines should meet. Therefore they are parallel.[b]

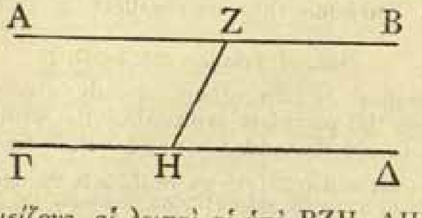
Ibid., ed. Friedlein 365. 5–367. 27

Therefore certain others already classed as a theorem this postulate assumed by the writer of the *Elements* and demanded a proof. Ptolemy appears

point of ZH so that ZH lies where HZ is in the figure, while ZK, HK lie along the sides HΓ, ZA respectively; and therefore HΓ, ZA must meet at the point where K falls.

The proof is based on the assumption that two straight lines cannot enclose a space. But Riemann devised a geometry in which this assumption does not hold good, for all straight lines having a common point have another point common also.

αὐτὸ δεικνύναι ἐν τῷ περὶ τοῦ τὰς ἀπ' ἐλαττόνων ἢ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, καὶ δείκνυσι πολλὰ προλαβὼν τῶν μέχρι τοῦδε τοῦ θεωρήματος ὑπὸ τοῦ Στοιχειωτοῦ προαποδεδειγμένων. καὶ ὑποκείσθω πάντα εἶναι ἀληθῆ, ἵνα μὴ καὶ ἡμεῖς ὄχλον ἐπεισάγωμεν ἄλλον, καὶ ὡς λημμάτιον τοῦτο δείκνυσθαι διὰ τῶν προειρημένων· ἓν δὲ καὶ τοῦτο τῶν προδεδειγμένων τὸ τὰς ἀπὸ δυεῖν ὀρθαῖς ἴσων ἐκβαλλομένας μηδαμῶς συμπίπτειν. λέγω τοίνυν ὅτι καὶ τὸ ἀνάπαλιν ἀληθές, καὶ τὸ παραλλήλων οὐσῶν τῶν εὐθειῶν καὶ τεμνομένων ὑπὸ μιᾶς εὐθείας τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθαῖς ἴσας εἶναι. ἀνάγκη γὰρ τὴν τέμνουσαν τὰς παραλλήλους ἢ δύο ὀρθαῖς ἴσας ποιεῖν τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας ἢ δύο ὀρθῶν ἐλάσσους ἢ μείζους. ἔστωσαν οὖν παράλληλοι αἱ ΑΒ, ΓΔ, καὶ ἐμπιπτέτω εἰς αὐτὰς ἡ ΗΖ· λέγω ὅτι οὐ ποιεῖ δύο ὀρθῶν μείζους τὰς ἐντὸς καὶ ἐπὶ τὰ αυτά. εἰ γὰρ αἱ ὑπὸ ΑΖΗ, ΓΗΖ δύο



ὀρθῶν μείζους, αἱ λοιπαὶ αἱ ὑπὸ ΒΖΗ, ΔΗΖ δύο ὀρθῶν ἐλάσσους. ἀλλὰ καὶ δύο ὀρθῶν μείζους αἱ αὐταί· οὐδὲν γὰρ μᾶλλον αἱ ΑΖ, ΓΗ παράλληλοι ἢ ΖΒ, ΗΔ, ὥστε εἰ ἡ ἐμπεσοῦσα εἰς τὰς ΑΖ, ΓΗ δύο ὀρθῶν μείζους ποιεῖ τὰς ἐντός, καὶ

to have proved it in his book on the proposition that *straight lines drawn from angles less than two right angles meet if produced*, and he uses in the proof many of the propositions proved by the writer of the *Elements* before this theorem. Let all these be taken as true, in order that we may not introduce another mass of propositions, and by means of the aforesaid propositions this theorem is proved as a lemma, that *straight lines drawn from two angles together equal to two right angles do not meet when produced*[a]—for this is common to both sets of preparatory theorems. I say then that the converse is also true, that *if parallel straight lines be cut by one straight line the interior angles on the same side are equal to two right angles.*[b] For the straight line cutting the parallel straight lines must make the interior angles on the same side equal to two right angles or less or greater. Let AB, ΓΔ be parallel straight lines, and let HZ cut them ; I say that it does not make the interior angles on the same side greater than two right angles. For if the angles AZH, ΓHZ are greater than two right angles, the remaining angles BZH, ΔHZ are less than two right angles.[c] But these same angles are greater than two right angles ; for AZ, ΓH are not more parallel than ZB, HΔ, so that if the straight line falling on AZ, ΓH make the interior angles greater than two right angles, the same straight line falling

[a] This is equivalent to Eucl. i. 28.
[b] This is equivalent to Eucl. i. 29.
[c] By Eucl. i. 13, for the angles AZH, BZH are together equal to two right angles and so are the angles ΓHZ, ΔHZ.

ἡ εἰς τὰς ΖΒ, ΗΔ ἐμπίπτουσα δύο ὀρθῶν ποιήσει μείζους τὰς ἐντός· ἀλλ' αἱ αὐταὶ καὶ δύο ὀρθῶν ἐλάσσους· αἱ γὰρ τέσσαρες αἱ ὑπὸ ΑΖΗ, ΓΗΖ, ΒΖΗ, ΔΗΖ τέτρασιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον. ὁμοίως δὴ δείξομεν ὅτι εἰς τὰς παραλλήλους ἐμπίπτουσα οὐ ποιεῖ δύο ὀρθῶν ἐλάσσους τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας. εἰ δὲ μήτε μείζους μήτε ἐλάσσους ποιεῖ τῶν δύο ὀρθῶν, λείπεται τὴν ἐμπίπτουσαν δύο ὀρθαῖς ἴσας ποιεῖν τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας.

Τούτου δὴ οὖν προδεδειγμένου τὸ προκείμενον ἀναμφισβητήτως ἀποδείκνυται. λέγω γὰρ ὅτι ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, συμπεσοῦνται αἱ εὐθεῖαι ἐκβαλλόμεναι, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες. μὴ γὰρ συμπιπτέτωσαν. ἀλλ' εἰ ἀσύμπτωτοί εἰσιν, ἐφ' ἃ μέρη αἱ τῶν δύο ὀρθῶν ἐλάσσονες, πολλῷ μᾶλλον ἔσονται ἀσύμπτωτοι ἐπὶ θάτερα, ἐφ' ἃ τῶν δύο εἰσὶν ὀρθῶν αἱ μείζονες, ὥστε ἐφ' ἑκάτερα ἂν εἶεν ἀσύμπτωτοι αἱ εὐθεῖαι. εἰ δὲ τοῦτο, παράλληλοί εἰσιν. ἀλλὰ δέδεικται ὅτι ἡ εἰς τὰς παραλλήλους ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθαῖς ἴσας ποιήσει γωνίας. αἱ αὐταὶ ἄρα καὶ δύο ὀρθαῖς ἴσαι καὶ δύο ὀρθῶν ἐλάσσονες, ὅπερ ἀδύνατον.

Ταῦτα προδεδειχὼς ὁ Πτολεμαῖος καὶ καταν-

[a] See note c on p. 377.

[b] The fallacy lies in the assumption that "ΑΖ, ΓΗ are not more parallel than ΖΒ, ΗΔ," so that the angles ΒΖΗ, ΔΗΖ must also be greater than two right angles. This assump-

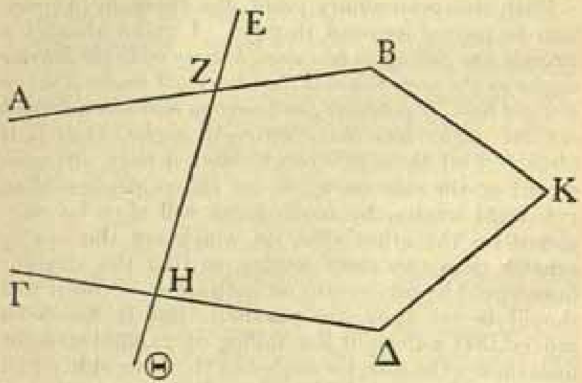
on ZB, HΔ also makes the interior angles greater than two right angles; but these same angles are less than two right angles, for the four angles AZH, ΓHZ, BZH, ΔHZ are equal to four right angles [a]; which is impossible. Similarly we may prove that a straight line falling on parallel straight lines does not make the interior angles on the same side less than two right angles. But if it make them neither greater nor less than two right angles, the only conclusion left is that the transversal makes the interior angles on the same side equal to two right angles.[b]

With this preliminary proof, the theorem in question is proved beyond dispute. I mean that *if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced, will meet on that side on which are the angles less than two right angles.* For [, if possible,] let them not meet. But if they are non-secant on the side on which are the angles less than two right angles, by much more will they be non-secant on the other side, on which are the angles greater than two right angles, so that the straight lines would be non-secant on both sides. Now if this should be so, they are parallel. But it has been proved that a straight line falling on parallel straight lines makes the interior angles on the same side equal to two right angles. Therefore the same angles are both equal to and less than two right angles, which is impossible.

Having first proved these things and squarely faced

tion is equivalent to the hypothesis that through a given point only one parallel can be drawn to a given straight line; but this hypothesis can be proved equivalent to Euclid's postulate. It is known as "Playfair's Axiom," but is, in fact, stated by Proclus in his note on Eucl. i. 31.

τήσας εἰς τὸ προκείμενον ἀκριβέστερόν τι προσθεῖναι βούλεται καὶ δεῖξαι ὅτι, ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθῶν ποιῇ ἐλάσσονας, οὐ μόνον οὐκ εἰσὶν ἀσύμπτωτοι αἱ εὐθεῖαι, ὡς δέδεικται, ἀλλὰ καὶ ἡ σύμπτωσις αὐτῶν κατ᾽ ἐκεῖνα γίνεται τὰ μέρη, ἐφ᾽ ἃ αἱ τῶν δύο ὀρθῶν ἐλάσσονες, οὐκ ἐφ᾽ ἃ αἱ μείζονες. ἔστωσαν γὰρ δύο εὐθεῖαι αἱ ΑΒ, ΓΔ καὶ ἐμπίπτουσα εἰς αὐτὰς ἡ ΕΖΗΘ ποιείτω τὰς ὑπὸ ΑΖΗ καὶ ὑπὸ ΓΗΖ δύο ὀρθῶν ἐλάσσους.



αἱ λοιπαὶ ἄρα μείζους δύο ὀρθῶν. ὅτι μὲν [οὖν][1] οὐκ ἀσύμπτωτοι αἱ εὐθεῖαι δέδεικται. εἰ δὲ συμπίπτουσιν, ἢ ἐπὶ τὰ Α, Γ συμπεσοῦνται, ἢ ἐπὶ τὰ Β, Δ. συμπιπτέτωσαν ἐπὶ τὰ Β, Δ κατὰ τὸ Κ. ἐπεὶ οὖν αἱ μὲν ὑπὸ ΑΖΗ καὶ ΓΗΖ δύο ὀρθῶν εἰσιν ἐλάσσους, αἱ δὲ ὑπὸ ΑΖΗ, ΒΖΗ δύο ὀρθαῖς ἴσαι, κοινῆς ἀφαιρεθείσης τῆς ὑπὸ ΑΖΗ,

the theorem in question, Ptolemy tries to make a more precise addition and to prove that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, not only are the straight lines not non-secant, as has been proved, but their meeting takes place on that side on which the angles are less than two right angles, and not on the side on which they are greater. For let AB, ΓΔ be two straight lines and let EZHΘ fall on them and make the angles AZH, ΓHZ less than two right angles. Then the remaining angles are greater than two right angles [Eucl. i. 13]. Now it has been proved that the straight lines are not non-secant. If they meet, they will meet either on the side of A, Γ or on the side of B, Δ. Let them meet on the side of B, Δ at K. Then since the angles AZH, ΓHZ are less than two right angles, while the angles AZH, BZH are equal to two right angles, when the common angle AZH is taken away, the angle ΓHZ will be less

[1] *οὖν* is clearly out of place.

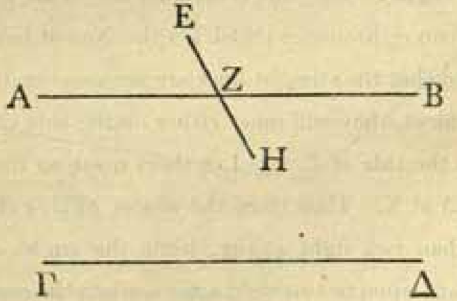
ἡ ὑπὸ ΓΗΖ ἐλάσσων ἔσται τῆς ὑπὸ ΒΖΗ. τριγώνου ἄρα τοῦ ΚΖΗ ἡ ἐκτὸς τῆς ἐντὸς καὶ ἀπεναντίον ἐλάσσων, ὅπερ ἀδύνατον. οὐκ ἄρα κατὰ ταῦτα συμπίπτουσιν. ἀλλὰ μὴν συμπίπτουσι. κατὰ θάτερα ἄρα ἡ σύμπτωσις αὐτῶν ἔσται, καθ' ἃ αἱ τῶν δύο ὀρθῶν εἰσιν ἐλάσσονες.

(iv.) *Proclus*

Ibid., ed. Friedlein 371. 23–373. 2

Τούτου δὴ προυποτεθέντος λέγω ὅτι, ἐὰν παραλλήλων εὐθειῶν τὴν ἑτέραν τέμνει τις εὐθεῖα, τεμεῖ καὶ τὴν λοιπήν.

Ἔστωσαν γὰρ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ τεμνέτω τὴν ΑΒ ἡ ΕΖΗ. λέγω ὅτι τὴν ΓΔ τεμεῖ.



Ἐπεὶ γὰρ δύο εὐθεῖαί εἰσιν ἀφ' ἑνὸς σημείου τοῦ Ζ, εἰς ἄπειρον ἐκβαλλόμεναι αἱ ΒΖ, ΖΗ, παντὸς μεγέθους μείζονα ἔχουσι διάστασιν, ὥστε καὶ τούτου, ὅσον ἐστὶ τὸ μεταξὺ τῶν παραλλήλων.

than the angle BZH. Therefore the exterior angle of the triangle KZH will be less than the interior and opposite angle, which is impossible [Eucl. i. 16]. Therefore they will not meet on this side. But they do meet. Therefore their meeting will be on the other side, on which the angles are less than two right angles.

(iv.) *Proclus*

Ibid., ed. Friedlein 371. 23–373. 2

This having first been assumed, I say that, *if any straight line cut one of parallel straight lines, it will cut the other also.*

For let AB, ΓΔ be parallel straight lines, and let EZH cut AB. I say that it will cut ΓΔ.

For since BZ, ZH are two straight lines drawn from one point Z, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than that between the parallels.

ὅταν οὖν μεῖζον ἀλλήλων διαστῶσιν τῆς τούτων διαστάσεως τεμεῖ ἡ ΖΗ τὴν ΓΔ. ἐὰν ἄρα παραλλήλων τὴν ἑτέραν τέμνῃ τις εὐθεῖα, τεμεῖ καὶ τὴν λοιπήν.

Τούτου προαποδειχθέντος ἀκολούθως δείξομεν τὸ προκείμενον. ἔστωσαν γὰρ δύο εὐθεῖαι αἱ ΑΒ, ΓΔ, καὶ ἐμπιπτέτω εἰς αὐτὰς ἡ ΕΖ ἐλάσσονας δύο ὀρθῶν ποιοῦσα τὰς ὑπὸ ΒΕΖ, ΔΖΕ.[1] λέγω ὅτι συμπεσοῦνται αἱ εὐθεῖαι κατὰ ταῦτα τὰ μέρη, ἐφ' ἃ αἱ τῶν δύο ὀρθῶν εἰσιν ἐλάσσους.

Ἐπειδὴ γὰρ αἱ ὑπὸ ΒΕΖ, ΔΖΕ ἐλάσσους εἰσὶ δύο ὀρθῶν, τῇ ὑπεροχῇ τῶν δύο ὀρθῶν ἔστω ἴση ἡ ὑπὸ ΘΕΒ. καὶ ἐκβεβλήσθω ἡ ΘΕ ἐπὶ τὸ Κ. ἐπεὶ οὖν εἰς τὰς ΚΘ, ΓΔ ἐμπέπτωκεν ἡ ΕΖ καὶ ποιεῖ τὰς ἐντὸς δύο ὀρθαῖς ἴσας τὰς ὑπὸ ΘΕΖ, ΔΖΕ, παράλληλοί εἰσιν αἱ ΘΚ, ΓΔ εὐθεῖαι. καὶ τέμνει τὴν ΚΘ ἡ ΑΒ· τεμεῖ ἄρα καὶ τὴν ΓΔ διὰ τὸ προδεδειγμένον. συμπεσοῦνται ἄρα αἱ ΑΒ, ΓΔ κατὰ τὰ μέρη ἐκεῖνα, ἐφ' ἃ αἱ τῶν δύο ὀρθῶν ἐλάσσονες, ὥστε δέδεικται τὸ προκείμενον.

[1] ΔΕΖ codd., correxi.

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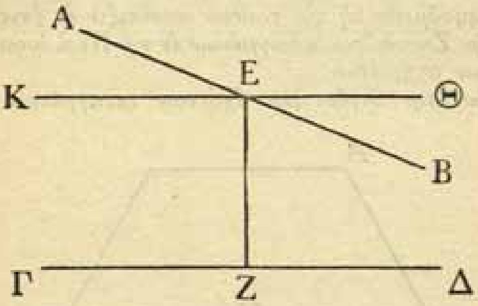
[a] The method is ingenious, but Clavius detected the flaw, which lies in the initial assumption, taken from Aristotle, that two divergent straight lines will eventually be so far apart that a perpendicular drawn from a point on one to the other will be greater than any assigned distance; Clavius draws attention to the conchoid of Nicomedes (v. vol. i. pp. 298-301), which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; but the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote.

Whenever, therefore, they are at a distance from one another greater than the distance between the parallels, ZH will cut ΓΔ. If, therefore, any straight line cuts one of parallels, it will cut the other also.

This having first been established, we shall prove in turn the theorem in question. For let AB, ΓΔ be two straight lines, and let EZ fall on them so as to



make the angles BEZ, ΔZE less than two right angles. I say that the straight lines will meet on that side on which are the angles less than two right angles.

For since the angles BEZ, ΔZE are less than two right angles, let the angle ΘEB be equal to the excess of the two right angles. And let ΘE be produced to K. Then since EZ falls on KΘ, ΓΔ and makes the interior angles ΘEZ, ΔZE equal to two right angles, the straight lines ΘK, ΓΔ are parallel. And AB cuts KΘ; therefore, by what was before shown, it will also cut ΓΔ. Therefore AB, ΓΔ will meet on that side on which are the angles less than two right angles, so that the theorem in question is proved.[a]

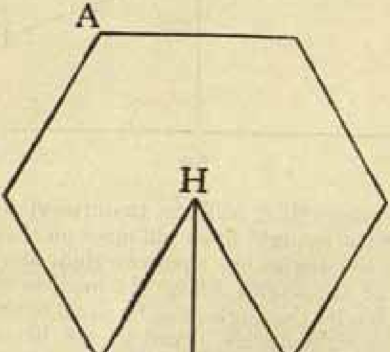
(c) Isoperimetric Figures

Theon. Alex. *in Ptol. Math. Syn. Comm.* i. 3, ed. Rome, *Studi e Testi*, lxxii. (1936), 354. 19–357. 22

" Ὡσαύτως δ' ὅτι, τῶν ἴσην περίμετρον ἐχόντων σχημάτων διαφόρων, ἐπειδὴ μείζονά ἐστιν τὰ πολυγωνότερα, τῶν μὲν ἐπιπέδων ὁ κύκλος γίνεται μείζων, τῶν δὲ στερεῶν ἡ σφαῖρα."

Ποιησόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἐπιτομῇ ἐκ τῶν Ζηνοδώρῳ δεδειγμένων ἐν τῷ Περὶ ἰσοπεριμέτρων σχημάτων.

Τῶν ἴσην περίμετρον ἐχόντων τεταγμένων εὐ-



[a] Ptolemy, *Math. Syn.* i. 3, ed. Heiberg i. pars i. 13. 16-19.

[b] Zenodorus, as will shortly be seen, cites a proposition by Archimedes, and therefore must be later in date than Archimedes; as he follows the style of Archimedes closely, he is generally put not much later. Zenodorus's work is not

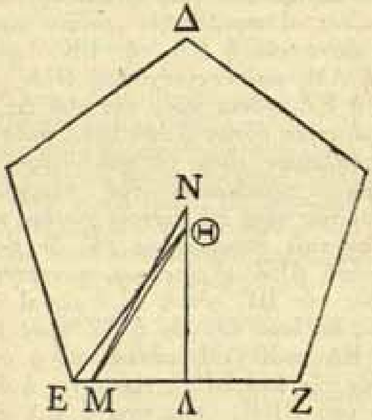
(c) Isoperimetric Figures

Theon of Alexandria, *Commentary on Ptolemy's Syntaxis* i. 3, ed. Rome, *Studi e Testi*, lxxii. (1936), 354. 19–357. 22

" In the same way, since the greatest of the various figures having an equal perimeter is that which has most angles, the circle is the greatest among plane figures and the sphere among solid.[a] "

We shall give the proof of these propositions in a summary taken from the proofs by Zenodorus [b] in his book *On Isoperimetric Figures.*

Of all rectilinear figures having an equal perimeter—



extant, but Pappus also quotes from it extensively (*Coll.* v. *ad init.*), and so does the passage edited by Hultsch (Papp. *Coll.*, ed. Hultsch 1138–1165) which is extracted from an introduction to Ptolemy's *Syntaxis* of uncertain authorship (*v.* Rome, *Studi e Testi*, liv., 1931, pp. xiii-xvii). It is disputed which of these versions is the most faithful.

θυγράμμων σχημάτων, λέγω δὴ ἰσοπλεύρων τε καὶ ἰσογωνίων, τὸ πολυγωνότερον μεῖζόν ἐστιν.

Ἔστω γὰρ ἰσοπερίμετρα ἰσόπλευρά τε καὶ ἰσογώνια τὰ ΑΒΓ, ΔΕΖ, πολυγωνότερον δὲ ἔστω τὸ ΑΒΓ. λέγω, ὅτι μεῖζόν ἐστιν τὸ ΑΒΓ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν περὶ τὰ ΑΒΓ, ΔΕΖ πολύγωνα περιγραφομένων κύκλων τὰ Η, Θ, καὶ ἐπεζεύχθωσαν αἱ ΗΒ, ΗΓ, ΘΕ, ΘΖ. καὶ ἔτι ἀπὸ τῶν Η, Θ ἐπὶ τὰς ΒΓ, ΕΖ κάθετοι ἤχθωσαν αἱ ΗΚ, ΘΛ. ἐπεὶ οὖν πολυγωνότερόν ἐστιν τὸ ΑΒΓ τοῦ ΔΕΖ, πλεονάκις ἡ ΒΓ τὴν τοῦ ΑΒΓ περίμετρον καταμετρεῖ ἤπερ ἡ ΕΖ τὴν τοῦ ΔΕΖ. καί εἰσιν ἴσαι αἱ περίμετροι. μείζων ἄρα ἡ ΕΖ τῆς ΒΓ· ὥστε καὶ ἡ ΕΛ τῆς ΒΚ. κείσθω τῇ ΒΚ ἴση ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ ΘΜ. καὶ ἐπεί ἐστιν ὡς ἡ ΕΖ εὐθεῖα πρὸς τὴν τοῦ ΔΕΖ πολυγώνου περίμετρον οὕτως ἡ ὑπὸ ΕΘΖ πρὸς δ̄ ὀρθάς, διὰ τὸ ἰσόπλευρον εἶναι τὸ πολύγωνον καὶ ἴσας ἀπολαμβάνειν περιφερείας τοῦ περιγραφομένου κύκλου καὶ τὰς πρὸς τῷ κέντρῳ γωνίας τὸν αὐτὸν ἔχειν λόγον ταῖς περιφερείαις ἐφ' ὧν βεβήκασιν, ὡς δὲ ἡ τοῦ ΔΕΖ περίμετρος, τουτέστιν ἡ τοῦ ΑΒΓ, πρὸς τὴν ΒΓ οὕτως αἱ δ̄ ὀρθαὶ πρὸς τὴν ὑπὸ ΒΗΓ, δι' ἴσου ἄρα ὡς ἡ ΕΖ πρὸς ΒΓ, τουτέστιν ἡ ΕΛ πρὸς ΛΜ, οὕτως καὶ ἡ ὑπὸ ΕΘΖ γωνία πρὸς τὴν ὑπὸ ΒΗΓ, τουτέστιν ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ. καὶ ἐπεὶ ἡ ΕΛ πρὸς ΛΜ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΕΘΛ γωνία πρὸς τὴν ὑπὸ ΜΘΛ, ὡς ἑξῆς δείξομεν, ὡς δὲ ἡ ΕΛ

[a] ΘΖ is not, in fact, joined in the ms. figures.

[b] This is proved in a lemma immediately following the proposition by drawing an arc of a circle with Θ as centre

I mean equilateral and equiangular figures—the greatest is that which has most angles.

For let ΑΒΓ, ΔΕΖ be equilateral and equiangular figures having equal perimeters, and let ΑΒΓ have the more angles. I say that ΑΒΓ is the greater.

For let Η, Θ be the centres of the circles circumscribed about the polygons ΑΒΓ, ΔΕΖ, and let ΗΒ, ΗΓ, ΘΕ, ΘΖ [a] be joined. And from Η, Θ let ΗΚ, ΘΛ be drawn perpendicular to ΒΓ, ΕΖ. Then since ΑΒΓ has more angles than ΔΕΖ, ΒΓ is contained more often in the perimeter of ΑΒΓ than ΕΖ is contained in the perimeter of ΔΕΖ. And the perimeters are equal. Therefore ΕΖ > ΒΓ; and therefore ΕΛ > ΒΚ. Let ΛΜ be placed equal to ΒΚ, and let ΘΜ be joined. Then since the straight line ΕΖ bears to the perimeter of the polygon ΔΕΖ the same ratio as the angle ΕΘΖ bears to four right angles—owing to the fact that the polygon is equilateral and the sides cut off equal arcs from the circumscribing circle, while the angles at the centre are in the same ratio as the arcs on which they stand [Eucl. iii. 26]—and the perimeter of ΔΕΖ, that is the perimeter of ΑΒΓ, bears to ΒΓ the same ratio as four right angles bears to the angle ΒΗΓ, therefore *ex aequali* [Eucl. v. 17]

ΕΖ : ΒΓ = angle ΕΘΖ : angle ΒΗΓ,

i.e., ΕΛ : ΛΜ = angle ΕΘΖ : angle ΒΗΓ,

i.e., ΕΛ : ΛΜ = angle ΕΘΛ : angle ΒΗΚ.

And since ΕΛ : ΛΜ > angle ΕΘΛ : angle ΜΘΛ,

as we shall prove in due course,[b]

and ΘΜ as radius cutting ΘΕ and ΘΛ produced, as in Eucl. *Optic.* 8 (*v.* vol. i. pp. 502-505); the proposition is equivalent to the formula $\tan\alpha : \tan\beta > \alpha : \beta$ if $\frac{1}{2}\pi > \alpha > \beta$.

πρὸς ΛΜ ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ, ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ μείζονα λόγον ἔχει ἤπερ πρὸς τὴν ὑπὸ ΜΘΛ. μείζων ἄρα ἡ ὑπὸ ΜΘΛ γωνία τῆς ὑπὸ ΒΗΚ. ἔστιν δὲ καὶ ὀρθὴ ἡ πρὸς τῷ Λ ὀρθῇ τῇ πρὸς τῷ Κ ἴση. λοιπὴ ἄρα ἡ ὑπὸ ΗΒΚ μείζων ἔσται τῆς ὑπὸ ΘΜΛ. κείσθω τῇ ὑπὸ ΗΒΚ ἴση ἡ ὑπὸ ΛΜΝ καὶ διήχθω ἡ ΛΘ ἐπὶ τὸ Ν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΗΒΚ τῇ ὑπὸ ΝΜΛ, ἀλλὰ καὶ ἡ πρὸς τῷ Λ ἴση τῇ πρὸς τῷ Κ, ἔστι δὲ καὶ ἡ ΒΚ πλευρᾷ τῇ ΜΛ ἴση, ἴση ἄρα καὶ ἡ ΗΚ τῇ ΝΛ. μείζων ἄρα ἡ ΗΚ τῆς ΘΛ. μεῖζον ἄρα καὶ τὸ ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ τοῦ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ. καὶ ἔστιν τὸ μὲν ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ διπλάσιον τοῦ ΑΒΓ πολυγώνου, ἐπεὶ καὶ τὸ ὑπὸ τῆς ΒΓ καὶ τῆς ΗΚ διπλάσιόν ἐστιν τοῦ ΗΒΓ τριγώνου. τὸ δὲ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μεῖζον ἄρα τὸ ΑΒΓ πολύγωνον τοῦ ΔΕΖ.

Ibid. 358. 12–360. 3

Τούτου δεδειγμένου λέγω, ὅτι ἐὰν κύκλος εὐθυγράμμῳ ἰσοπλεύρῳ τε καὶ ἰσογωνίῳ ἰσοπερίμετρος ᾖ, μείζων ἔσται ὁ κύκλος.

Κύκλος γὰρ ὁ ΑΒΓ ἰσοπλεύρῳ τε καὶ ἰσογωνίῳ τῷ ΔΕΖ εὐθυγράμμῳ ἰσοπερίμετρος ἔστω· λέγω, ὅτι μείζων ἐστὶν ὁ κύκλος.

Εἰλήφθω τοῦ μὲν ΑΒΓ κύκλου κέντρον τὸ Η, τοῦ δὲ περὶ τὸ ΔΕΖ πολύγωνον περιγραφομένου τὸ Θ, καὶ περιγεγράφθω περὶ τὸν ΑΒΓ κύκλον

and $\quad E\Lambda : \Lambda M = \text{angle } E\Theta\Lambda : \text{angle } BHK,$

$\therefore$ angle $E\Theta\Lambda$: angle $BHK >$ angle $E\Theta\Lambda : M\Theta\Lambda$.

$\therefore \quad$ angle $M\Theta\Lambda >$ angle BHK.

Now the right angle at Λ is equal to the right angle at K. Therefore the remaining angle HBK is greater than the angle $\Theta M\Lambda$ [by Eucl. i. 32]. Let the angle ΛMN be placed equal to the angle HBK, and let $\Lambda\Theta$ be produced to N. Then since the angle HBK is equal to the angle $NM\Lambda$, and the angle at Λ is equal to the angle at K, while BK is equal to the side $M\Lambda$, therefore HK is equal to $N\Lambda$ [Eucl. i. 26]. Therefore $HK > \Theta\Lambda$. Therefore the rectangle contained by the perimeter of $AB\Gamma$ and HK is greater than the rectangle contained by the perimeter of ΔEZ and $\Theta\Lambda$. But the rectangle contained by the perimeter of $AB\Gamma$ and HK is double of the polygon $AB\Gamma$, since the rectangle contained by $B\Gamma$ and HK is double of the triangle $HB\Gamma$ [Eucl. i. 41]; and the rectangle contained by the perimeter of ΔEZ and $\Theta\Lambda$ is double of the polygon ΔEZ. Therefore the polygon $AB\Gamma$ is greater than ΔEZ.

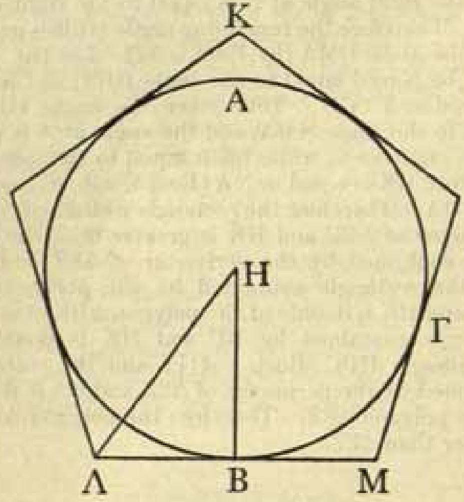
Ibid. 358. 12–360. 3

This having been proved, I say that *if a circle have an equal perimeter with an equilateral and equiangular rectilineal figure, the circle shall be the greater.*

For let $AB\Gamma$ be a circle having an equal perimeter with the equilateral and equiangular rectilineal figure ΔEZ. I say that the circle is the greater.

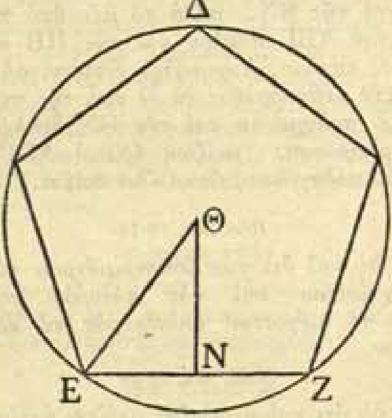
Let H be the centre of the circle $AB\Gamma$, Θ the centre of the circle circumscribing the polygon ΔEZ; and let there be circumscribed about the circle $AB\Gamma$ the

πολύγωνον ὅμοιον τῷ ΔΕΖ τὸ ΚΛΜ, καὶ ἐπεζεύχθω ἡ ΗΒ, καὶ κάθετος ἀπὸ τοῦ Θ ἐπὶ τὴν ΕΖ ἤχθω ἡ ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΛ, ΘΕ.



ἐπεὶ οὖν ἡ τοῦ ΚΛΜ πολυγώνου περίμετρος μείζων ἐστὶν τῆς τοῦ ΑΒΓ κύκλου περιμέτρου ὡς ἐν τῷ Περὶ σφαίρας καὶ κυλίνδρου Ἀρχιμήδης, ἴση δὲ ἡ τοῦ ΑΒΓ κύκλου περίμετρος τῇ τοῦ ΔΕΖ πολυγώνου περιμέτρῳ, μείζων ἄρα καὶ ἡ τοῦ ΚΛΜ πολυγώνου περίμετρος τῆς τοῦ ΔΕΖ πολυγώνου περιμέτρου. καί εἰσιν ὅμοια τὰ πολύγωνα· μείζων ἄρα ἡ ΒΛ τῆς ΝΕ. καὶ ὅμοιον τὸ ΗΛΒ τρίγωνον τῷ ΘΕΝ τριγώνῳ, ἐπεὶ καὶ τὰ

polygon ΚΛΜ similar to ΔΕΖ, and let ΗΒ be joined, and from Θ let ΘΝ be drawn perpendicular to ΕΖ, and let ΗΛ, ΘΕ be joined. Then since the perimeter



of the polygon ΚΛΜ is greater than the perimeter of the circle ΑΒΓ, as Archimedes proves in his work *On the Sphere and Cylinder*,[a] while the perimeter of the circle ΑΒΓ is equal to the perimeter of the polygon ΔΕΖ, therefore the perimeter of the polygon ΚΛΜ is greater than the perimeter of the polygon ΔΕΖ. And the polygons are similar; therefore ΒΛ > ΝΕ. And the triangle ΗΛΒ is similar to the triangle ΘΕΝ,

[a] Prop. 1, *v. supra*, pp. 48-49.

ὅλα πολύγωνα. μείζων ἄρα καὶ ἡ ΗΒ τῆς ΘΝ. καὶ ἔστιν ἴση ἡ τοῦ ΑΒΓ κύκλου περίμετρος τῇ τοῦ ΔΕΖ πολυγώνου περιμέτρῳ. τὸ ἄρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ΗΒ μεῖζόν ἐστιν τοῦ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ. ἀλλὰ τὸ μὲν ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ΗΒ διπλάσιον τοῦ ΑΒΓ κύκλου Ἀρχιμήδης ἔδειξεν, οὗ καὶ τὴν δεῖξιν ἑξῆς ἐκθησόμεθα· τὸ δὲ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μείζων ἄρα ὁ ΑΒΓ κύκλος τοῦ ΔΕΖ πολυγώνου, ὅπερ ἔδει δεῖξαι.

Ibid. 364. 12-14

Λέγω δὴ καὶ ὅτι τῶν ἰσοπεριμέτρων εὐθυγράμμων σχημάτων καὶ τὰς πλευρὰς ἰσοπληθεῖς ἐχόντων τὸ μέγιστον ἰσόπλευρόν τέ ἐστιν καὶ ἰσογώνιον.

Ibid. 374. 12-14

Λέγω δὴ ὅτι καὶ ἡ σφαῖρα μείζων ἐστὶν πάντων τῶν ἴσην ἐπιφάνειαν ἐχόντων στερεῶν σχημάτων, προσχρησάμενος τοῖς ὑπὸ Ἀρχιμήδους δεδειγμένοις ἐν τῷ Περὶ σφαίρας καὶ κυλίνδρου.

(*d*) Division of Zodiac Circle into 360 Parts: Hypsicles

Hypsicl. *Anaph.*, ed. Manitius 5. 25-31

Τοῦ τῶν Ζῳδίων κύκλου εἰς τξ̄ περιφερείας ἴσας

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[a] *Dim. Circ.* Prop. 1, v. vol. i. pp. 316-321.
[b] The proofs of these two last propositions are worked out by similar methods.

since the whole polygons are similar; therefore HB > ΘN. And the perimeter of the circle ΑΒΓ is equal to the perimeter of the polygon ΔΕΖ. Therefore the rectangle contained by the perimeter of the circle ΑΒΓ and HB is greater than the rectangle contained by the perimeter of the polygon ΔΕΖ and ΘN. But the rectangle contained by the perimeter of the circle ΑΒΓ and HB is double of the circle ΑΒΓ as was proved by Archimedes,[a] whose proof we shall set out next; and the rectangle contained by the perimeter of the polygon ΔΕΖ and ΘN is double of the polygon ΔΕΖ [by Eucl. i. 41]. Therefore the circle ΑΒΓ is greater than the polygon ΔΕΖ, which was to be proved.

Ibid. 364. 12-14

Now I say that, *of all rectilineal figures having an equal number of sides and equal perimeter, the greatest is that which is equilateral and equiangular.*

Ibid. 374. 12-14

Now I say that, *of all solid figures having an equal surface, the sphere is the greatest*; and I shall use the theorems proved by Archimedes in his work *On the Sphere and Cylinder.*[b]

(*d*) Division of Zodiac Circle into 360 Parts: Hypsicles

Hypsicles, *On Risings*, ed. Manitius[a] 5. 25-31

The circumference of the zodiac circle having been

[a] *Des Hypsikles Schrift Anaphorikos nach Überlieferung und Inhalt kritisch behandelt*, in *Programm des Gymnasiums zum Heiligen Kreuz in Dresden* (Dresden, 1888), I[e] Abt.

διηρημένου, ἑκάστη τῶν περιφερειῶν μοῖρα τοπικὴ καλείσθω. ὁμοίως δὴ καὶ τοῦ χρόνου, ἐν ᾧ ὁ ζῳδιακὸς ἀφ' οὗ ἔτυχε σημείου ἐπὶ τὸ αὐτὸ σημεῖον παραγίγνεται, εἰς τξ̄ χρόνους ἴσους διηρημένου, ἕκαστος τῶν χρόνων μοῖρα χρονικὴ καλείσθω.

(e) Handbooks

(i.) *Cleomedes*

Cleom. *De motu circ.* ii. 6, ed. Ziegler 218. 8–224. 8

Τοιούτων δὲ τῶν περὶ τὴν ἔκλειψιν τῆς σελήνης εἶναι ἐπιδεδειγμένων δοκεῖ ἐναντιοῦσθαι τῷ λόγῳ τῷ κατασκευάζοντι ἐκλείπειν τὴν σελήνην εἰς τὴν σκιὰν ἐμπίπτουσαν τῆς γῆς τὰ λεγόμενα κατὰ τὰς παραδόξους τῶν ἐκλείψεων. φασὶ γάρ τινες, ὅτι γίνεται σελήνης ἔκλειψις καὶ ἀμφοτέρων τῶν φωτῶν ὑπὲρ τὸν ὁρίζοντα θεωρουμένων. τοῦτον δὲ δῆλον ποιεῖ, διότι μὴ ἐκλείπει ἡ σελήνη τῇ σκιᾷ

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[a] Hypsicles, who flourished in the second half of the second century B.C., is the author of the continuation of Euclid's *Elements* known as Book xiv. Diophantus attributed to him a definition of a polygonal number which is equivalent to the formula $\frac{1}{2}n\{2+(n-1)(a-2)\}$ for the nth a-gonal number.

The passage here cited is the earliest known reference in Greek to the division of the ecliptic into 360 degrees. This number appears to have been adopted by the Greeks from the Chaldaeans, among whom the zodiac was divided into twelve signs and each sign into thirty parts according to one system, sixty according to another (v. Tannery, *Mémoires scientifiques*, ii. pp. 256-268). The Chaldaeans do not, however, seem to have applied this system to other circles; Hipparchus is believed to have been the first to divide the

divided into 360 equal arcs, let each of the arcs be called a *degree in space*, and similarly, if the time in which the zodiac circle returns to any position it has left be divided into 360 equal times, let each of the times be called a *degree in time*.[a]

(e) Handbooks

(i.) *Cleomedes*[b]

Cleomedes, *On the Circular Motion of the Heavenly Bodies* ii. 6, ed. Ziegler 218. 8–224. 8

Although these facts have been proved with regard to the eclipse of the moon, the argument that the moon suffers eclipse by falling into the shadow of the earth seems to be refuted by the stories told about paradoxical eclipses. For some say that an eclipse of the moon may take place even when both luminaries are seen above the horizon. This should make it clear that the moon does not suffer eclipse by

circle in general into 360 degrees. The problem which Hypsicles sets himself in his book is: *Given the ratio between the length of the longest day and the length of the shortest day at any given place, to find how many time-degrees it takes any given sign to rise.* A number of arithmetical lemmas are proved.

[b] Cleomedes is known only as the author of the two books Κυκλικὴ θεωρία μετεώρων. This work is almost wholly based on Posidonius. He must therefore have lived after Posidonius and presumably before Ptolemy, as he appears to know nothing of Ptolemy's works. In default of better evidence, he is generally assigned to the middle of the first century B.C.

The passage explaining the measurement of the earth by Eratosthenes has already been cited (*supra*, pp. 266-273). This is the only other passage calling for notice.

τῆς γῆς περιπίπτουσα, ἀλλ' ἕτερον τρόπον. . . . οἱ παλαιότεροι τῶν μαθηματικῶν οὕτως ἐπεχείρουν λύειν τὴν ἀπορίαν ταύτην. ἔφασαν γάρ, ὅτι . . . οἱ δ' ἐπὶ γῆς ἑστῶτες οὐδὲν ἂν κωλύοιντο ὁρᾶν ἀμφοτέρους αὐτοὺς ἐπὶ τοῖς κυρτώμασι τῆς γῆς ἑστῶτες. . . . τοιαύτην μὲν οὖν οἱ παλαιότεροι τῶν μαθηματικῶν τὴν τῆς προσαγομένης ἀπορίας λύσιν ἐποιήσαντο. μή ποτε δ' οὐχ ὑγιῶς εἰσιν ἐνηνεγμένοι. ἐφ' ὕψους μὲν γὰρ ἡ ὄψις ἡμῶν γενομένη δύναιτ' ἂν τοῦτο παθεῖν, κωνοειδοῦς τοῦ ὁρίζοντος γινομένου πολὺ ἀπὸ τῆς γῆς ἐκ τὸν ἀέρα ἡμῶν ἐξαρθέντων, ἐπὶ δὲ τῆς γῆς ἑστώτων οὐδαμῶς. εἰ γὰρ καὶ κύρτωμά ἐστιν, ἐφ' οὗ βεβήκαμεν, ἀφανίζεται ἡμῶν ἡ ὄψις ὑπὸ τοῦ μεγέθους τῆς γῆς. . . . ἀλλὰ πρῶτον μὲν ἀπαντητέον λέγοντας, ὅτι πέπλασται ὁ λόγος οὗτος ὑπό τινων ἀπορίαν βουλομένων ἐμποιῆσαι τοῖς περὶ ταῦτα καταγινομένοις τῶν ἀστρολόγων καὶ φιλοσόφων. . . . πολλῶν δὲ καὶ παντοδαπῶν περὶ τὸν ἀέρα παθῶν συνίστασθαι πεφυκότων οὐκ ἂν εἴη ἀδύνατον, ἤδη καταδεδυκότος τοῦ ἡλίου καὶ ὑπὸ τὸν ὁρίζοντα ὄντος φαντασίαν ἡμῖν προσπεσεῖν ὡς μηδέπω καταδεδυκότος αὐτοῦ, ἢ νέφους παχυτέρου πρὸς τῇ δύσει ὄντος καὶ λαμπρυνομένου ὑπὸ τῶν ἡλιακῶν ἀκτίνων καὶ ἡλίου ἡμῖν φαντασίαν ἀποπέμποντος ἢ ἀνθηλίου γενομένου. καὶ γὰρ

[a] *i.e.*, the horizon would form the base of a cone whose vertex would be at the eye of the observer. He could thus look down on both the sun and moon as along the generators of a cone, even though they were diametrically opposite each other.

falling into the shadow of the earth, but in some other way. . . . The more ancient of the mathematicians tried to explain this difficulty after this fashion. They said that persons standing on the earth would not be prevented from seeing them both because they would be standing on the convexities of the earth. . . . Such is the solution of the alleged difficulty given by the more ancient of the mathematicians. But its soundness may be doubted. For, if our eye were situated on a height, the phenomenon in question might take place, the horizon becoming conical [a] if we were raised sufficiently far above the earth, but it could in no wise happen if we stood on the earth. For though there might be some convexity where we stood, our sight itself becomes evanescent owing to the size of the earth. . . . The fundamental objection must first be made, that this story has been invented by certain persons wishing to make difficulty for the astronomers and philosophers who busy themselves with such matters. . . . Nevertheless, as the conditions which naturally arise in the air are many and various, it would not be impossible that, when the sun has just set and is below the horizon, we should receive the impression of its not having yet set, if there were a cloud of considerable density at the place of setting and if it were illumined by the solar rays and transmitted to us an image of the sun, or if there were a mock sun.[b] For such images are often

[b] Lit. "anthelion," defined in the Oxford English Dictionary as "a luminous ring or nimbus seen (chiefly in alpine or polar regions) surrounding the shadow of the observer's head projected on a cloud or fog bank opposite the sun." The explanation here tentatively put forward by Cleomedes is, of course, the true one.

τοιαῦτα πολλὰ φαντάζεται ἐν τῷ ἀέρι, καὶ μάλιστα περὶ τὸν Πόντον.

(ii.) *Theon of Smyrna*

Ptol. *Math. Syn.* x. 1, ed. Heiberg i. pars ii. 296. 14-16

Ἐν μὲν γὰρ ταῖς παρὰ Θέωνος τοῦ μαθηματικοῦ δοθείσαις ἡμῖν εὕρομεν ἀναγεγραμμένην τήρησιν τῷ ιϛ´ ἔτει Ἀδριανοῦ.

Theon Smyr., ed. Hiller 1. 1-2. 2

Ὅτι μὲν οὐχ οἷόν τε συνεῖναι τῶν μαθηματικῶς λεγομένων παρὰ Πλάτωνι μὴ καὶ αὐτὸν ἠσκημένον ἐν τῇ θεωρίᾳ ταύτῃ, πᾶς ἂν που ὁμολογήσειεν· ὡς δὲ οὐδὲ τὰ ἄλλα ἀνωφελὴς οὐδὲ ἀνόνητος ἡ περὶ ταῦτα ἐμπειρία, διὰ πολλῶν αὐτὸς ἐμφανίζειν ἔοικε. τὸ μὲν οὖν συμπάσης γεωμετρίας καὶ συμπάσης μουσικῆς καὶ ἀστρονομίας ἔμπειρον γενόμενον τοῖς Πλάτωνος συγγράμμασιν ἐντυγχάνειν μακαριστὸν μὲν εἴ τῳ γένοιτο, οὐ μὴν εὔπορον οὐδὲ ῥᾴδιον ἀλλὰ πάνυ πολλοῦ τοῦ ἐκ παίδων πόνου δεόμενον. ὥστε δὲ τοὺς διημαρτηκότας τοῦ ἐν τοῖς μαθήμασιν ἀσκηθῆναι, ὀρεγομένους δὲ τῆς γνώσεως τῶν συγγραμμάτων αὐτοῦ μὴ παντάπασιν ὧν ποθοῦσι διαμαρτεῖν, κεφαλαιώδη καὶ σύντομον ποιησόμεθα τῶν ἀναγκαίων καὶ ὧν δεῖ μάλιστα τοῖς ἐντευξομένοις Πλάτωνι μαθηματικῶν θεωρημάτων παράδοσιν, ἀριθμητικῶν τε καὶ μουσικῶν καὶ γεωμετρικῶν τῶν τε κατὰ στερεομετρίαν καὶ ἀστρονομίαν, ὧν χωρὶς οὐχ

seen in the air, and especially in the neighbourhood of Pontus.

(ii.) *Theon of Smyrna*

Ptolemy, *Syntaxis* x. 1, ed. Heiberg i. pars ii. 296. 14-16

For in the account given to us by Theon the mathematician we find recorded an observation made in the sixteenth year of Hadrian.[a]

Theon of Smyrna, ed. Hiller 1. 1-2. 2

Everyone would agree that he could not understand the mathematical arguments used by Plato unless he were practised in this science; and that the study of these matters is neither unintelligent nor unprofitable in other respects Plato himself would seem to make plain in many ways. One who had become skilled in all geometry and all music and astronomy would be reckoned most happy on making acquaintance with the writings of Plato, but this cannot be come by easily or readily, for it calls for a very great deal of application from youth upwards. In order that those who have failed to become practised in these studies, but aim at a knowledge of his writings, should not wholly fail in their desires, I shall make a summary and concise sketch of the mathematical theorems which are specially necessary for readers of Plato, covering not only arithmetic and music and geometry, but also their application to stereometry and astronomy, for

[a] *i.e.*, in A.D. 132. Ptolemy mentions other observations made by Theon in the years A.D. 127, 129, and 130. In three places Theon of Alexandria refers to his namesake as "the old Theon," ὁ Θέων παλαιός (ed. Basil. pp. 390, 395, 396).

οἷόν τε εἶναί φησι τυχεῖν τοῦ ἀρίστου βίου, διὰ πολλῶν πάνυ δηλώσας ὡς οὐ χρὴ τῶν μαθημάτων ἀμελεῖν.

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[a] By way of example, Theon proceeds to relate Plato's reply to the craftsmen about the doubling of the cube (v.

without these studies, as he says, it is not possible to attain the best life, and in many ways he makes clear that mathematics should not be ignored.[a]

vol. i. p. 257), and also the *Epinomis*. Theon's work, which has often been cited in these volumes, is a curious hotch-potch, containing little of real value to the study of Plato and no original work.

XXI. TRIGONOMETRY

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1. HIPPARCHUS AND MENELAUS

Theon Alex. *in Ptol. Math. Syn. Comm.* i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 451. 4-5

Δέδεικται μὲν οὖν καὶ Ἱππάρχῳ πραγματεία τῶν ἐν κύκλῳ εὐθειῶν ἐν ιβ̅ βιβλίοις, ἔτι τε καὶ Μενελάῳ ἐν ϛ̅.

Heron, *Metr.* i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

Ἔστω ἐννάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕΖΗΘΚ, οὗ ἑκάστη τῶν πλευρῶν μονάδων ι̅. εὑρεῖν αὐτοῦ τὸ ἐμβαδόν. περιγεγράφθω περὶ αὐτὸ κύκλος, οὗ κέντρον ἔστω τὸ Λ, καὶ ἐπε-

[a] The beginnings of Greek trigonometry may be found in the science of *sphaeric*, the geometry of the sphere, for which v. vol. i. p. 5 n. *b*. It reached its culminating point in the *Sphaerica* of Theodosius.

Trigonometry in the strict sense was founded, so far as we know, by Hipparchus, the great astronomer, who was born at Nicaea in Bithynia and is recorded by Ptolemy to have made observations between 161 and 126 B.C., the most important of them at Rhodes. His greatest achievement was the discovery of the precession of the equinoxes, and he made a calculation of the mean lunar month which differs by less than a second from the present accepted figure. Unfortunately the only work of his which has survived is his early *Commentary on the Phenomena of Eudoxus and Aratus*. It

XXI. TRIGONOMETRY

1. HIPPARCHUS AND MENELAUS

Theon of Alexandria, *Commentary on Ptolemy's Syntaxis* i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 451. 4-5

An investigation of the chords in a circle is made by Hipparchus in twelve books and again by Menelaus in six.[a]

Heron, *Metrics* i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

Let ΑΒΓΔΕΖΗΘΚ be an equilateral and equiangular enneagon,[b] whose sides are each equal to 10. *To find its area.* Let there be described about it a circle with centre Λ, and let ΕΛ be joined and pro-

is clear, however, from the passage here cited, that he drew up, as did Ptolemy, a table of chords, or, as we should say, a table of sines; and Heron may have used this table (*v.* the next passage cited and the accompanying note).

Menelaus, who also drew up a table of chords, is recorded by Ptolemy to have made an observation in the first year of Trajan's reign (A.D. 98). He has already been encountered (vol. i. pp. 348-349 and n. *c*) as the discoverer of a curve called "paradoxical." His trigonometrical work *Sphaerica* has fortunately been preserved, but only in Arabic, which will prevent citation here. A proof of the famous theorem in spherical trigonometry bearing his name can, however, be given in the Greek of Ptolemy (*infra*, pp. 458-463); and a summary from the Arabic is provided by Heath, *H.G.M.* ii. 262-273.

[b] *i.e.*, a figure of nine sides.

ζεύχθω ἡ ΕΛ καὶ ἐκβεβλήσθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχθω ἡ ΜΖ. τὸ ἄρα ΕΖΜ τρίγωνον δοθέν ἐστιν τοῦ ἐνναγώνου. δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν, ὅτι ἡ ΖΕ τῆς ΕΜ τρίτον μέρος ἐστὶν ὡς ἔγγιστα.

2. PTOLEMY

(a) General

Suidas, s.v. Πτολεμαῖος

Πτολεμαῖος, ὁ Κλαύδιος χρηματίσας, Ἀλεξανδρεύς, φιλόσοφος, γεγονὼς ἐπὶ τῶν χρόνων Μάρκου τοῦ βασιλέως. οὗτος ἔγραψε Μηχανικὰ βιβλία γ̄, Περὶ φάσεων καὶ ἐπισημασιῶν ἀστέρων ἀπλανῶν βιβλία β̄, Ἅπλωσιν ἐπιφανείας σφαίρας, Κανόνα πρόχειρον, τὸν Μέγαν ἀστρονόμον ἤτοι Σύνταξιν· καὶ ἄλλα.

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[a] A similar passage (i. 24, ed. H. Schöne 62. 11-20) asserts that the ratio of the side of a regular hendecagon to the diameter of the circumscribing circle is approximately $\frac{7}{25}$; and of this assertion also it is said δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν. These are presumably the works of Hipparchus and Menelaus, though this opinion is controverted by A. Rome, "Premiers essais de trigonométrie rectiligne chez les Grecs" in *L'Antiquité classique*, t. 2 (1933), pp. 177-192. The assertions are equivalent to saying that sin 20° is approximately 0·333... and sin 16° 21′ 49″ is approximately 0·28.

[b] Nothing else is certainly known of the life of Ptolemy except, as can be gleaned from his own works, that he made observations between A.D. 125 and 141 (or perhaps 151). Arabian traditions add details on which too much reliance should not be placed. Suidas's statement that he was born

duced to M, and let MZ be joined. Then the triangle EZM is given in the enneagon. But it has been proved in the works on chords in a circle that ZE : EM is approximately $\frac{1}{3}$.[a]

2. PTOLEMY

(a) General

Suidas, *s.v. Ptolemaeus*

Ptolemy, called Claudius, an Alexandrian, a philosopher, born in the time of the Emperor Marcus. He wrote *Mechanics*, three books, *On the Phases and Seasons of the Fixed Stars*, two books, *Explanation of the Surface of a Sphere*, *A Ready Reckoner*, the *Great Astronomy* or *Syntaxis*; and others.[b]

in the time of the Emperor Marcus [Aurelius] is not accurate as Marcus reigned from A.D. 161 to 180.

Ptolemy's *Mechanics* has not survived in any form; but the books *On Balancings* and *On the Elements* mentioned by Simplicius may have been contained in it. The lesser astronomical works of Ptolemy published in the second volume of Heiberg's edition of Ptolemy include, in Greek, Φάσεις ἀπλανῶν ἀστέρων καὶ συναγωγὴ ἐπισημασιῶν and Προχείρων κανόνων διάταξις καὶ ψηφοφορία, which can be identified with two titles in Suidas's notice. In the same edition is the *Planisphaerium*, a Latin translation from the Arabic, which can be identified with the Ἅπλωσις ἐπιφανείας σφαίρας of Suidas; it is an explanation of the *stereographic* system of projection by which points on the heavenly sphere are represented on the equatorial plane by projection from a pole—circles are projected into circles, as Ptolemy notes, except great circles through the poles, which are projected into straight lines.

Allied to this, but not mentioned by Suidas, is Ptolemy's *Analemma*, which explains how points on the heavenly sphere can be represented as points on a plane by means of *orthogonal* projection upon three planes mutually at right angles—

Simpl. *in De caelo* iv. 4 (Aristot. 311 b 1), ed. Heiberg 710. 14-19

Πτολεμαῖος δὲ ὁ μαθηματικὸς ἐν τῷ Περὶ ῥοπῶν τὴν ἐναντίαν ἔχων τῷ Ἀριστοτέλει δόξαν πειρᾶται κατασκευάζειν καὶ αὐτός, ὅτι ἐν τῇ ἑαυτῶν χώρᾳ οὔτε τὸ ὕδωρ οὔτε ὁ ἀὴρ ἔχει βάρος. καὶ ὅτι μὲν τὸ ὕδωρ οὐκ ἔχει, δείκνυσιν ἐκ τοῦ τοὺς καταδύοντας μὴ αἰσθάνεσθαι βάρους τοῦ ἐπικειμένου ὕδατος, καίτοι τινὰς εἰς πολὺ καταδύοντας βάθος.

Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11

Πτολεμαῖος ἐν τῷ Περὶ τῶν στοιχείων βιβλίῳ καὶ ἐν τοῖς Ὀπτικοῖς . . .

Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27

Ὁ δὲ θαυμαστὸς Πτολεμαῖος ἐν τῷ Περὶ διαστάσεως μονοβίβλῳ ἀπέδειξεν, ὅτι οὐκ εἰσὶ

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the meridian, the horizontal and the "prime vertical." Only fragments of the Greek and a Latin version from the Arabic have survived; they are given in Heiberg's second volume.

Among the "other works" mentioned by Suidas are presumably the *Inscription in Canobus* (a record of some of Ptolemy's discoveries), which exists in Greek; the Ὑποθέσεις τῶν πλανωμένων, of which the first book is extant in Greek and the second in Arabic; and the *Optics* and the book *On Dimension* mentioned by Simplicius.

But Ptolemy's fame rests most securely on his *Great Astronomy* or *Syntaxis* as it is called by Suidas. Ptolemy himself called this majestic astronomical work in thirteen books the Μαθηματικὴ σύνταξις or *Mathematical Collection*. In due course the lesser astronomical works came to be called the Μικρὸς ἀστρονομούμενος (τόπος), the *Little Astronomy*, and the *Syntaxis* came to be called the Μεγάλη σύνταξις, or *Great Collection*. Later still the Arabs, combining their article Al

Simplicius, *Commentary on Aristotle's De caelo* iv. 4 (311 b 1), ed. Heiberg 710. 14-19

Ptolemy the mathematician in his work *On Balancings* maintains an opinion contrary to that of Aristotle and tries to show that in its own place neither water nor air has weight. And he proves that water has not weight from the fact that divers do not feel the weight of the water above them, even though some of them dive into considerable depths.

Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11

Ptolemy in his book *On the Elements* and in his *Optics* . . .[a]

Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27

The gifted Ptolemy in his book *On Dimension* showed that there are not more than three dimen-

with the Greek superlative μέγιστος, called it Al-majisti; corrupted into *Almagest*, this has since been the favourite name for the work.

The *Syntaxis* was the subject of commentaries by Pappus and Theon of Alexandria. The trigonometry in it appears to have been abstracted from earlier treatises, but condensed and arranged more systematically.

Ptolemy's attempt to prove the parallel-postulate has already been noticed (*supra*, pp. 372-383).

[a] Ptolemy's *Optics* exists in an Arabic version, which was translated into Latin in the twelfth century by Admiral Eugenius Siculus (*v.* G. Govi, *L' ottica di Claudio Tolomeo di Eugenio Ammiraglio di Sicilia*): but of the five books the first and the end of the last are missing. Until the Arabic text was discovered, Ptolemy's *Optics* was commonly supposed to be identical with the Latin work known as *De Speculis*: but this is now thought to be a translation of Heron's *Catoptrica* by William of Moerbeke (*v. infra*, p. 502 n. *a*).

πλείονες τῶν τριῶν διαστάσεις, ἐκ τοῦ δεῖν μὲν τὰς διαστάσεις ὡρισμένας εἶναι, τὰς δὲ ὡρισμένας διαστάσεις κατ' εὐθείας λαμβάνεσθαι καθέτους, τρεῖς δὲ μόνας πρὸς ὀρθὰς ἀλλήλαις εὐθείας δυνατὸν εἶναι λαβεῖν, δύο μὲν καθ' ἃς τὸ ἐπίπεδον ὁρίζεται, τρίτην δὲ τὴν τὸ βάθος μετροῦσαν· ὥστε, εἴ τις εἴη μετὰ τὴν τριχῆ διάστασιν ἄλλη, ἄμετρος ἂν εἴη παντελῶς καὶ ἀόριστος.

(b) Table of Sines

(i.) *Introduction*

Ptol. *Math. Syn.* i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

ι΄. Περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθειῶν

Πρὸς μὲν οὖν τὴν ἐξ ἑτοίμου χρῆσιν κανονικήν τινα μετὰ ταῦτα ἔκθεσιν ποιησόμεθα τῆς πηλικότητος αὐτῶν τὴν μὲν περίμετρον εἰς $\overline{\tau\xi}$ τμήματα διελόντες, παρατιθέντες δὲ τὰς ὑπὸ τὰς καθ' ἡμιμοίριον παραυξήσεις τῶν περιφερειῶν ὑποτεινομένας εὐθείας, τουτέστι πόσων εἰσὶν τμημάτων ὡς τῆς διαμέτρου διὰ τὸ ἐξ αὐτῶν τῶν ἐπιλογισμῶν φανησόμενον ἐν τοῖς ἀριθμοῖς εὔχρηστον εἰς $\overline{\rho\kappa}$ τμήματα διῃρημένης. πρότερον δὲ δείξομεν, πῶς ἂν ὡς ἔνι μάλιστα δι' ὀλίγων καὶ τῶν αὐτῶν θεωρημάτων εὐμεθόδευτον καὶ ταχεῖαν τὴν ἐπιβολὴν τὴν πρὸς τὰς πηλικότητας αὐτῶν ποιοίμεθα, ὅπως μὴ μόνον ἐκτεθειμένα τὰ μεγέθη τῶν εὐθειῶν ἔχωμεν ἀνεπιστάτως, ἀλλὰ καὶ διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως τὸν ἔλεγχον ἐξ εὐχεροῦς μεταχειριζώμεθα. καθόλου

sions; for dimensions must be determinate, and determinate dimensions are along perpendicular straight lines, and it is not possible to find more than three straight lines at right angles one to another, two of them determining a plane and the third measuring depth; therefore, if any other were added after the third dimension, it would be completely unmeasurable and undetermined.

(b) Table of Sines

(i.) *Introduction*

Ptolemy, *Syntaxis* i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

10. On the lengths of the chords in a circle

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of these [chords in a circle], dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set out correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considera-

μέντοι χρησόμεθα ταῖς τῶν ἀριθμῶν ἐφόδοις κατὰ τὸν τῆς ἑξηκοντάδος τρόπον διὰ τὸ δύσχρηστον τῶν μοριασμῶν ἔτι τε τοῖς πολυπλασιασμοῖς καὶ μερισμοῖς ἀκολουθήσομεν τοῦ συνεγγίζοντος ἀεὶ καταστοχαζόμενοι, καὶ καθʼ ὅσον ἂν τὸ παραλειπόμενον μηδενὶ ἀξιολόγῳ διαφέρῃ τοῦ πρὸς αἴσθησιν ἀκριβοῦς.

(ii.) *sin* 18° *and sin* 36°

Ibid. 32. 10–35. 16

Ἔστω δὴ πρῶτον ἡμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΔΓ περὶ κέντρον τὸ Δ, καὶ ἀπὸ τοῦ Δ τῇ ΑΓ πρὸς ὀρθὰς γωνίας ἤχθω ἡ ΔΒ, καὶ τετμήσθω δίχα ἡ ΔΓ κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΒ, καὶ κείσθω αὐτῇ ἴση ἡ ΕΖ, καὶ ἐπεζεύχθω ἡ ΖΒ. λέγω, ὅτι ἡ μὲν ΖΔ δεκαγώνου ἐστὶν πλευρά, ἡ δὲ ΒΖ πενταγώνου.

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[a] By διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς συστάσεως Ptolemy meant more than a graphical method: the phrase indicates a *rigorous proof* by means of geometrical considerations, as will be seen when the argument proceeds; *cf.* the use of διὰ τῶν γραμμῶν *infra*, p. 434. It may be inferred, therefore, that when Hipparchus proved "by means of lines" (διὰ τῶν γραμμῶν, *On the Phaenomena of Eudoxus and Aratus*, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations: in other words, he was familiar with the main formulae of spherical trigonometry.

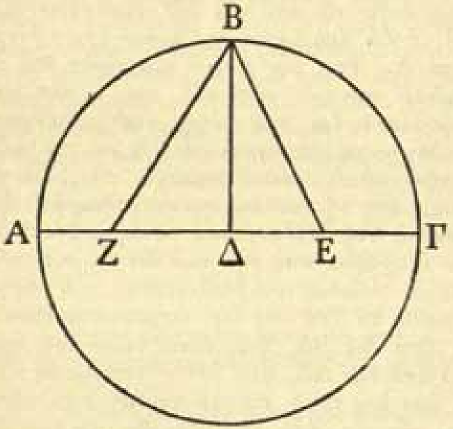
[b] *i.e.*, ΖΔ is equal to the side of a regular decagon, and ΒΖ to the side of a regular pentagon, inscribed in the circle ΑΒΓ.

tions.[a] In general we shall use the sexagesimal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

(ii.) *sin* 18° *and sin* 36°

Ibid. 32. 10–35. 16

First, let ΑΒΓ be a semicircle on the diameter ΑΔΓ and with centre Δ, and from Δ let ΔΒ be drawn per-



pendicular to ΑΓ, and let ΔΓ be bisected at Ε, and let ΕΒ be joined, and let ΕΖ be placed equal to it, and let ΖΒ be joined. I say that ΖΔ is the side of a decagon, and ΒΖ of a pentagon.[b]

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΔΓ τέτμηται δίχα κατὰ τὸ Ε, καὶ πρόσκειταί τις αὐτῇ εὐθεῖα ἡ ΔΖ, τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶν τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ, τουτέστιν τῷ ἀπὸ τῆς ΒΕ, ἐπεὶ ἴση ἐστὶν ἡ ΕΒ τῇ ΖΕ. ἀλλὰ τῷ ἀπὸ τῆς ΕΒ τετραγώνῳ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΔ καὶ ΔΒ τετράγωνα· τὸ ἄρα ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΔΕ τετραγώνου ἴσον ἐστὶν τοῖς ἀπὸ τῶν ΕΔ, ΔΒ τετραγώνοις. καὶ κοινοῦ ἀφαιρεθέντος τοῦ ἀπὸ τῆς ΕΔ τετραγώνου λοιπὸν τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ ἴσον ἐστὶν τῷ ἀπὸ τῆς ΔΒ, τουτέστιν τῷ ἀπὸ τῆς ΔΓ· ἡ ΖΓ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Δ. ἐπεὶ οὖν ἡ τοῦ ἑξαγώνου καὶ ἡ τοῦ δεκαγώνου πλευρὰ τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων ἐπὶ τῆς αὐτῆς εὐθείας ἄκρον καὶ μέσον λόγον τέμνονται, ἡ δὲ ΓΔ ἐκ τοῦ κέντρου οὖσα τὴν τοῦ ἑξαγώνου περιέχει πλευράν, ἡ ΔΖ ἄρα ἐστὶν ἴση τῇ τοῦ δεκαγώνου πλευρᾷ. ὁμοίως δέ, ἐπεὶ ἡ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων, τοῦ δὲ ΒΔΖ ὀρθογωνίου τὸ ἀπὸ τῆς ΒΖ τετράγωνον ἴσον ἐστὶν τῷ τε ἀπὸ τῆς ΒΔ, ἥτις ἐστὶν ἑξαγώνου πλευρά, καὶ τῷ ἀπὸ τῆς ΔΖ, ἥτις ἐστὶν δεκαγώνου πλευρά, ἡ ΒΖ ἄρα ἴση ἐστὶν τῇ τοῦ πενταγώνου πλευρᾷ.

Ἐπεὶ οὖν, ὡς ἔφην, ὑποτιθέμεθα τὴν τοῦ κύκλου διάμετρον τμημάτων $\overline{\rho\kappa}$, γίνεται διὰ τὰ προκείμενα ἡ μὲν ΔΕ ἡμίσεια οὖσα τῆς ἐκ τοῦ κέντρου

[a] Following the usual practice, I shall denote *segments* (τμήματα) of the diameter by ^{p}, sixtieth parts of a τμῆμα by

For since the straight line $\Delta\Gamma$ is bisected at E, and the straight line ΔZ is added to it,

$$\Gamma Z \cdot Z\Delta + E\Delta^2 = EZ^2 \qquad \text{[Eucl. ii. 6}$$
$$= BE^2,$$

since $EB = ZE$.

But $\quad E\Delta^2 + \Delta B^2 = EB^2$; $\qquad$ [Eucl. i. 47

therefore $\quad \Gamma Z \cdot Z\Delta + E\Delta^2 = E\Delta^2 + \Delta B^2$.

When the common term $E\Delta^2$ is taken away,

the remainder $\quad \Gamma Z \cdot Z\Delta = \Delta B^2$

i.e., $\qquad = \Delta\Gamma^2$;

therefore $Z\Gamma$ is divided in extreme and mean ratio at Δ [Eucl. vi., Def. 3]. Therefore, since the side of the hexagon and the side of the decagon inscribed in the same circle when placed in one straight line are cut in extreme and mean ratio [Eucl. xiii. 9], and $\Gamma\Delta$, being a radius, is equal to the side of the hexagon [Eucl. iv. 15, coroll.], therefore ΔZ is equal to the side of the decagon. Similarly, since the square on the side of the pentagon is equal to the rectangle contained by the side of the hexagon and the side of the decagon inscribed in the same circle [Eucl. xiii. 10], and in the right-angled triangle $B\Delta Z$ the square on BZ is equal [Eucl. i. 47] to the sum of the squares on $B\Delta$, which is a side of the hexagon, and ΔZ, which is a side of the decagon, therefore BZ is equal to the side of the pentagon.

Then since, as I said, we made the diameter [a] consist of 120^p, by what has been stated ΔE, being half

the numeral with a single accent, and second-sixtieths by the numeral with two accents. As the circular associations of the system tend to be forgotten, and it is used as a general system of enumeration, the same notation will be used for the *squares* of parts.

τμημάτων λ καὶ τὸ ἀπ' αὐτῆς ϡ, ἡ δὲ ΒΔ ἐκ τοῦ κέντρου οὖσα τμημάτων ξ καὶ τὸ ἀπὸ αὐτῆς ,γχ, τὸ δὲ ἀπὸ τῆς ΕΒ, τουτέστιν τὸ ἀπὸ τῆς ΕΖ, τῶν ἐπὶ τὸ αὐτὸ ,δφ· μήκει ἄρα ἔσται ἡ ΕΖ τμημάτων ξζ δ νε ἔγγιστα, καὶ λοιπὴ ἡ ΔΖ τῶν αὐτῶν λζ δ νε. ἡ ἄρα τοῦ δεκαγώνου πλευρά, ὑποτείνουσα δὲ περιφέρειαν τοιούτων λϛ, οἵων ἐστὶν ὁ κύκλος τξ, τοιούτων ἔσται λζ δ νε, οἵων ἡ διάμετρος ρκ. πάλιν ἐπεὶ ἡ μὲν ΔΖ τμημάτων ἐστὶ λζ δ νε, τὸ δὲ ἀπὸ αὐτῆς ,ατοε δ ιε, ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΔΒ τῶν αὐτῶν ,γχ, ἃ συντεθέντα ποιεῖ τὸ ἀπὸ τῆς ΒΖ τετράγωνον ,δϡοε δ ιε, μήκει ἄρα ἔσται ἡ ΒΖ τμημάτων ο λβ γ ἔγγιστα. καὶ ἡ τοῦ πενταγώνου ἄρα πλευρά, ὑποτείνουσα δὲ μοίρας οβ, οἵων ἐστὶν ὁ κύκλος τξ, τοιούτων ἐστὶν ο λβ γ, οἵων ἡ διάμετρος ρκ.

Φανερὸν δὲ αὐτόθεν, ὅτι καὶ ἡ τοῦ ἑξαγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ξ, καὶ ἴση οὖσα τῇ ἐκ τοῦ κέντρου, τμημάτων ἐστὶν ξ. ὁμοίως δέ, ἐπεὶ ἡ μὲν τοῦ τετραγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ϙ, δυνάμει διπλασία ἐστὶν τῆς ἐκ τοῦ κέντρου, ἡ δὲ τοῦ τριγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ρκ, δυνάμει τῆς αὐτῆς ἐστιν τριπλασίων, τὸ δὲ ἀπὸ τῆς ἐκ τοῦ κέντρου τμημάτων ἐστὶν ,γχ, συναχθήσεται τὸ μὲν ἀπὸ τῆς τοῦ τετραγώνου πλευρᾶς ,ζσ, τὸ δὲ ἀπὸ τῆς τοῦ τριγώνου Μ ω. ὥστε καὶ μήκει ἡ μὲν τὰς ϙ μοίρας ὑποτείνουσα εὐθεῖα τοιούτων ἔσται πδ να ι ἔγγιστα, οἵων ἡ

of the radius, consists of 30^p and its square of 900^p, and $B\Delta$, being the radius, consists of 60^p and its square of 3600^p, while EB^2, that is EZ^2, consists of 4500^p; therefore EZ is approximately $67^p\ 4'\ 55''$,[a] and the remainder ΔZ is $37^p\ 4'\ 55''$. Therefore the side of the decagon, subtending an arc of 36° (the whole circle consisting of 360°), is $37^p\ 4'\ 55''$ (the diameter being 120^p). Again, since ΔZ is $37^p\ 4'\ 55''$, its square is $1375^p\ 4'\ 15''$, and the square on ΔB is 3600^p, which added together make the square on BZ $4975^p\ 4'\ 15''$, so that BZ is approximately $70^p\ 32'\ 3''$. And therefore the side of the pentagon, subtending 72° (the circle consisting of 360°), is $70^p\ 32'\ 3''$ (the diameter being 120^p).

Hence it is clear that the side of the hexagon, subtending 60° and being equal to the radius, is 60^p. Similarly, since the square on the side of the square,[b] subtending 90°, is double of the square on the radius, and the square on the side of the triangle, subtending 120°, is three times the square on the radius, while the square on the radius is 3600^p, the square on the side of the square is 7200^p and the square on the side of the triangle is 10800^p. Therefore the chord subtending 90° is approximately $84^p\ 51'\ 10''$ (the diameter

[a] Theon's proof that $\sqrt{4500}$ is approximately $67^p\ 4'\ 55''$ has already been given (vol. i. pp. 56-61).

[b] This is, of course, the square itself; the Greek phrase is not so difficult. We could translate, "the second power of the side of the square," but the notion of powers was outside the ken of the Greek mathematician.

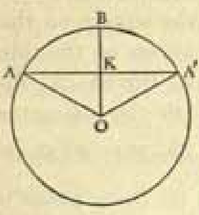
διάμετρος $\overline{\rho\kappa}$, ἡ δὲ τὰς $\overline{\rho\kappa}$ τῶν αὐτῶν $\overline{\rho\gamma}$ $\overline{\nu\epsilon}$ $\overline{\kappa\gamma}$.

(iii.) $\sin^2\theta + \cos^2\theta = 1$

Ibid. 35. 17–36. 12

Αἵδε μὲν οὕτως ἡμῖν ἐκ προχείρου καὶ καθ' αὑτὰς εἰλήφθωσαν, καὶ ἔσται φανερὸν ἐντεῦθεν, ὅτι τῶν διδομένων εὐθειῶν ἐξ εὐχεροῦς δίδονται καὶ αἱ ὑπὸ τὰς λειπούσας εἰς τὸ ἡμικύκλιον περιφερείας ὑποτείνουσαι διὰ τὸ τὰ ἀπ' αὐτῶν συντιθέμενα ποιεῖν τὸ ἀπὸ τῆς διαμέτρου τετράγωνον· οἷον, ἐπειδὴ ἡ ὑπὸ τὰς $\overline{\lambda\varsigma}$ μοίρας εὐθεῖα τμημάτων ἐδείχθη $\lambda\zeta$ $\overline{\delta}$ $\overline{\nu\epsilon}$ καὶ τὸ ἀπ' αὐτῆς $\overline{͵\alpha\tau o\epsilon}$ $\overline{\delta}$ $\overline{\iota\epsilon}$, τὸ δὲ ἀπὸ τῆς διαμέτρου τμημάτων ἐστὶν $\overset{\alpha}{\mathrm{M}}$ $\overline{͵\delta\upsilon}$, ἔσται καὶ τὸ μὲν ἀπὸ τῆς ὑποτεινούσης τὰς λειπούσας εἰς τὸ ἡμικύκλιον μοίρας $\overline{\rho\mu\delta}$ τῶν λοιπῶν

ᵃ Let AB be a chord of a circle subtending an angle α at the centre O, and let AKA′ be drawn perpendicular to OB so as to meet OB in K and the circle again in A′. Then



$$\sin\alpha\ (=\sin AB) = \frac{AK}{AO} = \frac{\frac{1}{2}AA'}{AO}.$$

And AA′ is the chord subtended by double of the arc AB, while Ptolemy expresses the lengths of chords as so many 120th parts of the diameter; therefore $\sin\alpha$ is half the chord subtended by an angle 2α at the centre, which is conveniently abbreviated by Heath to $\frac{1}{2}$(crd. 2α), or, as we may alternatively express the relationship, sin AB is "half the chord subtended by

consisting of 120^p), and the chord subtending 120° is $103^p\ 55'\ 23''$.[a]

(iii.) $\sin^2\theta + \cos^2\theta = 1$

Ibid. 35. 17–36. 12

The lengths of these chords have thus been obtained immediately and by themselves,[b] and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending 36° was shown to be $37^p\ 4'\ 55''$ and its square $1375^p\ 4'\ 15''$, while the square on the diameter is 14400^p, therefore the square on the chord subtending the remaining 144° in the semicircle is

double of the arc AB," which is the Ptolemaic form; as Ptolemy means by this expression precisely what we mean by sin AB, I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that $\cos a$ $[=\sin(90-a)]=\frac{1}{2}$ crd. $(180°-2a)$, or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." Tan a and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended Ptolemy proves that

side of decagon (=crd. $36° = 2\sin 18°) = 37^p\ 4'\ 55''$,
side of pentagon (=crd. $72° = 2\sin 36°) = 70^p\ 32'\ 3''$,
side of hexagon (=crd. $60° = 2\sin 30°) = 60^p$,
side of square (=crd. $90° = 2\sin 45°) = 84^p\ 51'\ 10''$,
side of equilateral triangle (=crd. $120° = 2\sin 60°) = 103^p\ 55'\ 23''$.

[b] *i.e.*, not deduced from other known chords.

$\overset{\alpha}{\mathrm{M}}\,\overline{\gamma\kappa\delta}\ \overline{\nu\epsilon}\ \overline{\mu\epsilon}$, αὐτὴ δὲ μήκει τῶν αὐτῶν $\overline{\rho\iota\delta}\ \overline{\zeta}\ \overline{\lambda\zeta}$ ἔγγιστα, καὶ ἐπὶ τῶν ἄλλων ὁμοίως.

Ὃν δὲ τρόπον ἀπὸ τούτων καὶ αἱ λοιπαὶ τῶν κατὰ μέρος δοθήσονται, δείξομεν ἐφεξῆς προεκθέμενοι λημμάτιον εὔχρηστον πάνυ πρὸς τὴν παροῦσαν πραγματείαν.

(iv.) "*Ptolemy's Theorem*"

Ibid. 36. 13–37. 18

Ἔστω γὰρ κύκλος ἐγγεγραμμένον ἔχων τετράπλευρον τυχὸν τὸ ΑΒΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ καὶ ΒΔ. δεικτέον, ὅτι τὸ ὑπὸ τῶν ΑΓ καὶ ΒΔ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ συναμφοτέροις τῷ τε ὑπὸ τῶν ΑΒ, ΔΓ καὶ τῷ ὑπὸ τῶν ΑΔ, ΒΓ.

Κείσθω γὰρ τῇ ὑπὸ τῶν ΔΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΑΒΕ. ἐὰν οὖν κοινὴν προσθῶμεν τὴν ὑπὸ ΕΒΔ,

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[a] *i.e.*, crd. $144°(=2\sin 72°)=114^p\ 7'\ 37''$. If the given chord subtends an angle 2θ at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle $(180-2\theta)$, and the theorem asserts that

$$(\text{crd. } 2\theta)^2+(\text{crd. } \overline{180-2\theta})^2=(\text{diameter})^2,$$

or $$\sin^2\theta \quad + \quad \cos^2\theta \quad =1.$$

$13024^p\ 55'\ 45''$ and the chord itself is approximately $114^p\ 7'\ 37''$, and similarly for the other chords.[a]

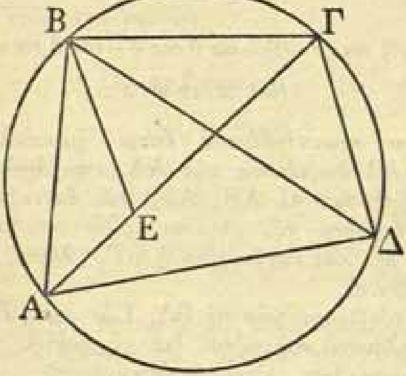
We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(iv.) "*Ptolemy's Theorem*"

Ibid. 36. 13–37. 18

Let ΑΒΓΔ be any quadrilateral inscribed in a circle, and let ΑΓ and ΒΔ be joined. It is required to prove that the rectangle contained by ΑΓ and ΒΔ is equal to the sum of the rectangles contained by ΑΒ, ΔΓ and ΑΔ, ΒΓ.

For let the angle ΑΒΕ be placed equal to the angle



ΔΒΓ. Then if we add the angle ΕΒΔ to both, the

ἔσται καὶ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΕΒΓ. ἔστιν δὲ καὶ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΒΓΕ ἴση· τὸ γὰρ αὐτὸ τμῆμα ὑποτείνουσιν· ἰσογώνιον ἄρα ἐστὶν τὸ ΑΒΔ τρίγωνον τῷ ΒΓΕ τριγώνῳ. ὥστε καὶ ἀνάλογόν ἐστιν, ὡς ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ· τὸ ἄρα ὑπὸ ΒΓ, ΑΔ ἴσον ἐστὶν τῷ ὑπὸ ΒΔ, ΓΕ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΔΒΓ γωνίᾳ, ἔστιν δὲ καὶ ἡ ὑπὸ ΒΑΕ ἴση τῇ ὑπὸ ΒΔΓ, ἰσογώνιον ἄρα ἐστὶν τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ· ἀνάλογον ἄρα ἐστίν, ὡς ἡ ΒΑ πρὸς ΑΕ, ἡ ΒΔ πρὸς ΔΓ· τὸ ἄρα ὑπὸ ΒΑ, ΔΓ ἴσον ἐστὶν τῷ ὑπὸ ΒΔ, ΑΕ. ἐδείχθη δὲ καὶ τὸ ὑπὸ ΒΓ, ΑΔ ἴσον τῷ ὑπὸ ΒΔ, ΓΕ· καὶ ὅλον ἄρα τὸ ὑπὸ ΑΓ, ΒΔ ἴσον ἐστὶν συναμφοτέροις τῷ τε ὑπὸ ΑΒ, ΔΓ καὶ τῷ ὑπὲρ ΑΔ, ΒΓ· ὅπερ ἔδει δεῖξαι.

(v.) $\sin(\theta - \phi) = \sin\theta \cos\phi - \cos\theta \sin\phi$

Ibid. 37. 19–39. 3

Τούτου προεκτεθέντος ἔστω ἡμικύκλιον τὸ ΑΒΓΔ ἐπὶ διαμέτρου τῆς ΑΔ, καὶ ἀπὸ τοῦ Α δύο διήχθωσαν αἱ ΑΒ, ΑΓ, καὶ ἔστω ἑκατέρα αὐτῶν δοθεῖσα τῷ μεγέθει, οἵων ἡ διάμετρος δοθεῖσα $\overline{\rho\kappa}$, καὶ ἐπεζεύχθω ἡ ΒΓ. λέγω, ὅτι καὶ αὕτη δέδοται.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΓΔ· δεδομέναι ἄρα εἰσὶν δηλονότι καὶ αὗται διὰ τὸ λείπειν ἐκείνων εἰς τὸ ἡμικύκλιον. ἐπεὶ οὖν ἐν κύκλῳ τετράπλευρόν ἐστιν τὸ ΑΒΓΔ, τὸ ἄρα ὑπὸ ΑΒ, ΓΔ μετὰ τοῦ

angle ΑΒΔ = the angle ΕΒΓ. But the angle ΒΔΑ = the angle ΒΓΕ [Eucl. iii. 21], for they subtend the same segment; therefore the triangle ΑΒΔ is equiangular with the triangle ΒΓΕ.

∴ ΒΓ : ΓΕ = ΒΔ : ΔΑ; [Eucl. vi. 4

∴ ΒΓ . ΑΔ = ΒΔ . ΓΕ. [Eucl. vi. 6

Again, since the angle ΑΒΕ is equal to the angle ΔΒΓ, while the angle ΒΑΕ is equal to the angle ΒΔΓ [Eucl. iii. 21], therefore the triangle ΑΒΕ is equiangular with the triangle ΒΓΔ;

∴ ΒΑ : ΑΕ = ΒΔ : ΔΓ; [Eucl. vi. 4

∴ ΒΑ . ΔΓ = ΒΔ . ΑΕ. [Eucl. vi. 6

But it was shown that

ΒΓ . ΑΔ = ΒΔ . ΓΕ;

and ∴ ΑΓ . ΒΔ = ΑΒ . ΔΓ + ΑΔ . ΒΓ;

[Eucl. ii. 1

which was to be proved.

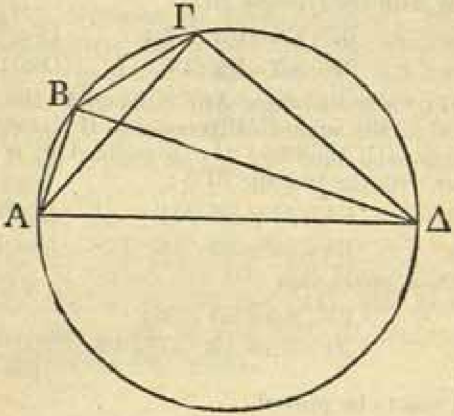
(v.) *sin* $(\theta - \phi) =$ *sin* θ *cos* $\phi -$ *cos* θ *sin* ϕ

Ibid. 37. 19–39. 3

This having first been proved, let ΑΒΓΔ be a semicircle having ΑΔ for its diameter, and from Α let the two [chords] ΑΒ, ΑΓ be drawn, and let each of them be given in length, in terms of the 120^p in the diameter, and let ΒΓ be joined. I say that this also is given.

For let ΒΔ, ΓΔ be joined; then clearly these also are given because they are the chords subtending the remainder of the semicircle. Then since ΑΒΓΔ is a quadrilateral in a circle,

ὑπὸ τῶν ΑΔ, ΒΓ ἴσον ἐστὶν τῷ ὑπὸ ΑΓ, ΒΔ. καί ἐστιν τό τε ὑπὸ τῶν ΑΓ, ΒΔ δοθὲν καὶ τὸ



ὑπὸ ΑΒ, ΓΔ· καὶ λοιπὸν ἄρα τὸ ὑπὸ ΑΔ, ΒΓ δοθέν ἐστιν. καί ἐστιν ἡ ΑΔ διάμετρος· δοθεῖσα ἄρα ἐστὶν καὶ ἡ ΒΓ εὐθεῖα.

Καὶ φανερὸν ἡμῖν γέγονεν, ὅτι, ἐὰν δοθῶσιν δύο περιφέρειαι καὶ αἱ ὑπ' αὐτὰς εὐθεῖαι, δοθεῖσα ἔσται καὶ ἡ τὴν ὑπεροχὴν τῶν δύο περιφερειῶν ὑποτείνουσα εὐθεῖα. δῆλον δέ, ὅτι διὰ τούτου τοῦ θεωρήματος ἄλλας τε οὐκ ὀλίγας εὐθείας ἐγγράψομεν ἀπὸ τῶν ἐν ταῖς καθ' αὑτὰς δεδομένων ὑπεροχῶν καὶ δὴ καὶ τὴν ὑπὸ τὰς δώδεκα μοίρας, ἐπειδήπερ ἔχομεν τήν τε ὑπὸ τὰς ξ καὶ τὴν ὑπὸ τὰς οβ.

$$AB \,.\, \Gamma\Delta + A\Delta \,.\, B\Gamma = A\Gamma \,.\, B\Delta.$$

["Ptolemy's theorem"

And $A\Gamma \,.\, B\Delta$ is given, and also $AB \,.\, \Gamma\Delta$; therefore the remaining term $A\Delta \,.\, B\Gamma$ is also given. And $A\Delta$ is the diameter; therefore the straight line $B\Gamma$ is given.[a]

And it has become clear to us that, if two arcs are given and the chords subtending them, the chord subtending the difference of the arcs will also be given. It is obvious that, by this theorem we can inscribe[b] many other chords subtending the difference between given chords, and in particular we may obtain the chord subtending 12°, since we have that subtending 60° and that subtending 72°.

[a] If $A\Gamma$ subtends an angle 2θ and AB an angle 2ϕ at the centre, the theorem asserts that

$$\text{crd. } \overline{2\theta - 2\phi} \,.\, (\text{crd. } 180^\circ) = (\text{crd. } 2\theta) \,.\, (\text{crd. } \overline{180^\circ - 2\phi}) - (\text{crd. } 2\phi) \,.\, (\text{crd. } \overline{180^\circ - 2\theta})$$

i.e., $$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

[b] Or "calculate," as we might almost translate ἐγγράφομεν: *cf. supra*, p. 414 n. *a* on ἐκ τῶν γραμμῶν.

(vi.) $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$

Ibid. 39. 4–41. 3

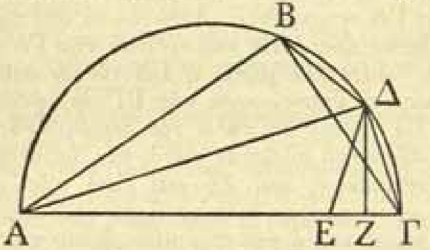
Πάλιν προκείσθω δοθείσης τινὸς εὐθείας ἐν κύκλῳ τὴν ὑπὸ τὸ ἥμισυ τῆς ὑποτεινομένης περιφερείας εὐθεῖαν εὑρεῖν. καὶ ἔστω ἡμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΓ καὶ δοθεῖσα εὐθεῖα ἡ ΓΒ, καὶ ἡ ΓΒ περιφέρεια δίχα τετμήσθω κατὰ τὸ Δ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΑΔ, ΒΔ, ΔΓ, καὶ ἀπὸ τοῦ Δ ἐπὶ τὴν ΑΓ κάθετος ἤχθω ἡ ΔΖ. λέγω, ὅτι ἡ ΖΓ ἡμίσειά ἐστι τῆς τῶν ΑΒ καὶ ΑΓ ὑπεροχῆς.

Κείσθω γὰρ τῇ ΑΒ ἴση ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΔΕ. ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΔ, δύο δὴ αἱ ΑΒ, ΑΔ δύο ταῖς ΑΕ, ΑΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ γωνία ἡ ὑπὸ ΒΑΔ γωνίᾳ τῇ ὑπὸ ΕΑΔ ἴση ἐστίν· καὶ βάσις ἄρα ἡ ΒΔ βάσει τῇ ΔΕ ἴση ἐστίν. ἀλλὰ ἡ ΒΔ τῇ ΔΓ ἴση ἐστίν· καὶ ἡ ΔΓ ἄρα τῇ ΔΕ ἴση ἐστίν. ἐπεὶ οὖν ἰσοσκελοῦς ὄντος τριγώνου τοῦ ΔΕΓ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΔΖ, ἴση ἐστὶν ἡ ΕΖ τῇ ΖΓ. ἀλλ' ἡ ΕΓ ὅλη ἡ ὑπεροχή ἐστιν τῶν ΑΒ καὶ ΑΓ εὐθειῶν· ἡ ἄρα ΖΓ ἡμίσειά ἐστιν τῆς τῶν αὐτῶν ὑπεροχῆς. ὥστε, ἐπεὶ τῆς ὑπὸ τὴν ΒΓ περιφέρειαν εὐθείας ὑποκειμένης

(vi.) $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$

Ibid. 39. 4–41. 3

Again, given any chord in a circle, let it be required to find the chord subtending half the arc subtended by the given chord. Let ABΓ be a semicircle upon the diameter AΓ and let the chord ΓB be given, and



let the arc ΓB be bisected at Δ, and let AB, AΔ, BΔ, ΔΓ be joined, and from Δ let ΔZ be drawn perpendicular to AΓ. I say that ZΓ is half of the difference between AB and AΓ.

For let AE be placed equal to AB, and let ΔE be joined. Since AB = AE and AΔ is common, [in the triangles ABΔ, AEΔ] the two [sides] AB, AΔ are equal to AE, AΔ each to each; and the angle BAΔ is equal to the angle EAΔ [Eucl. iii. 27]; and therefore the base BΔ is equal to the base ΔE [Eucl. i. 4]. But BΔ = ΔΓ; and therefore ΔΓ = ΔE. Then since the triangle ΔEΓ is isosceles and ΔZ has been drawn from the vertex perpendicular to the base, EZ = ZΓ [Eucl. i. 26]. But the whole EΓ is the difference between the chords AB and AΓ; therefore ZΓ is half of the difference. Thus, since the chord subtending the arc BΓ is given, the chord AB subtending the remainder

αὐτόθεν δέδοται καὶ ἡ λείπουσα εἰς τὸ ἡμικύκλιον ἡ ΑΒ, δοθήσεται καὶ ἡ ΖΓ ἡμίσεια οὖσα τῆς τῶν ΑΓ καὶ ΑΒ ὑπεροχῆς· ἀλλ' ἐπεὶ ἐν ὀρθογωνίῳ τῷ ΑΓΔ καθέτου ἀχθείσης τῆς ΔΖ ἰσογώνιον γίνεται τὸ ΑΔΓ ὀρθογώνιον τῷ ΔΓΖ, καί ἐστιν, ὡς ἡ ΑΓ πρὸς ΓΔ, ἡ ΓΔ πρὸς ΓΖ, τὸ ἄρα ὑπὸ τῶν ΑΓ, ΓΖ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶν τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. δοθὲν δὲ τὸ ὑπὸ τῶν ΑΓ, ΓΖ. δοθὲν ἄρα ἐστὶν καὶ τὸ ἀπὸ τῆς ΓΔ τετράγωνον. ὥστε καὶ μήκει ἡ ΓΔ εὐθεῖα δοθήσεται τὴν ἡμίσειαν ὑποτείνουσα τῆς ΒΓ περιφερείας.

Καὶ διὰ τούτου δὴ πάλιν τοῦ θεωρήματος ἄλλαι τε ληφθήσονται πλεῖσται κατὰ τὰς ἡμισείας τῶν προεκτεθειμένων, καὶ δὴ καὶ ἀπὸ τῆς τὰς ι̅β̅ μοίρας ὑποτεινούσης εὐθείας ἥ τε ὑπὸ τὰς ϛ̅ καὶ ἡ ὑπὸ τὰς γ̅ καὶ ἡ ὑπὸ τὴν μίαν ἥμισυ καὶ ἡ ὑπὸ τὸ ἥμισυ τέταρτον τῆς μιᾶς μοίρας. εὑρίσκομεν δὲ ἐκ τῶν ἐπιλογισμῶν τὴν μὲν ὑπὸ τὴν μίαν ἥμισυ μοῖραν τοιούτων α̅ λδ ι̅ε̅ ἔγγιστα, οἵων ἐστὶν ἡ διάμετρος ρ̅κ̅, τὴν δὲ ὑπὸ τὸ ∠′ δ′ τῶν αὐτῶν Ο μζ η̅.

(vii.) $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$

Ibid. 41. 4–43. 5

Πάλιν ἔστω κύκλος ὁ ΑΒΓΔ περὶ διάμετρον μὲν τὴν ΑΔ, κέντρον δὲ τὸ Ζ, καὶ ἀπὸ τοῦ Α ἀπειλήφθωσαν δύο περιφέρειαι δοθεῖσαι κατὰ τὸ ἑξῆς αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ ὑπ' αὐτὰς εὐθεῖαι καὶ αὐταὶ δεδομέναι. λέγω ὅτι, ἐὰν ἐπιζεύξωμεν τὴν ΑΓ, δοθήσεται καὶ αὐτή.

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[a] If ΒΓ subtends an angle 2θ at the centre the proposition asserts that

of the semicircle is immediately given, and ZΓ will also be given, being half of the difference between AΓ and AB. But since the perpendicular ΔZ has been drawn in the right-angled triangle AΓΔ, the right-angled triangle AΔΓ is equiangular with ΔΓZ [Eucl. vi. 8], and

$$\text{A}\Gamma : \Gamma\Delta = \Gamma\Delta : \Gamma\text{Z},$$

and therefore $\text{A}\Gamma \,.\, \Gamma\text{Z} = \Gamma\Delta^2$.

But AΓ . ΓZ is given; therefore $\Gamma\Delta^2$ is also given. Therefore the chord ΓΔ, subtending half of the arc BΓ, is also given.[a]

And again by this theorem many other chords can be obtained as the halves of known chords, and in particular from the chord subtending 12° can be obtained the chord subtending 6° and that subtending 3° and that subtending $1\frac{1}{2}°$ and that subtending $\frac{1}{2}° + \frac{1}{4}°(= \frac{3}{4}°)$. We shall find, when we come to make the calculation, that the chord subtending $1\frac{1}{2}°$ is approximately $1^p\ 34'\ 15''$ (the diameter being 120^p) and that subtending $\frac{3}{4}°$ is $0^p\ 47'\ 8''$.[b]

(vii.) *cos* $(\theta + \phi) =$ *cos* θ *cos* ϕ − *sin* θ *sin* ϕ

Ibid. 41. 4–43. 5

Again, let ABΓΔ be a circle about the diameter AΔ and with centre Z, and from A let there be cut off in succession two given arcs AB, BΓ, and let there be joined AB, BΓ, which, being the chords subtending them, are also given. I say that, if we join AΓ, it also will be given.

$$(\text{crd. }\theta)^2 = \tfrac{1}{2}(\text{crd. }180)\,.\,\{(\text{crd. }180°) - \text{crd. }\overline{180° - 2\theta}\}$$

i.e., $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$.

[b] The symbol in the Greek for O should be noted; *v.* vol. i. p. 47 n. *a*.

Διήχθω γὰρ διὰ τοῦ Β διάμετρος τοῦ κύκλου ἡ ΒΖΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΔ, ΔΓ, ΓΕ, ΔΕ· δῆλον δὴ αὐτόθεν, ὅτι διὰ μὲν τὴν ΒΓ δοθήσεται καὶ ἡ ΓΕ, διὰ δὲ τὴν ΑΒ δοθήσεται ἥ τε ΒΔ καὶ ἡ ΔΕ. καὶ διὰ τὰ αὐτὰ τοῖς ἔμπροσθεν, ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστιν τὸ ΒΓΔΕ, καὶ διηγμέναι εἰσὶν αἱ ΒΔ, ΓΕ, τὸ ὑπὸ τῶν διηγμένων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶν συναμφοτέροις τοῖς ὑπὸ τῶν ἀπεναντίον· ὥστε, ἐπεὶ δεδομένου τοῦ ὑπὸ τῶν ΒΔ, ΓΕ δέδοται καὶ τὸ ὑπὸ τῶν ΒΓ, ΔΕ, δέδοται ἄρα καὶ τὸ ὑπὸ ΒΕ, ΓΔ. δέδοται δὲ καὶ ἡ ΒΕ διάμετρος, καὶ λοιπὴ ἡ ΓΔ ἔσται δεδομένη, καὶ διὰ τοῦτο καὶ ἡ λείπουσα εἰς τὸ ἡμικύκλιον ἡ ΓΑ· ὥστε, ἐὰν δοθῶσιν δύο περιφέρειαι καὶ αἱ ὑπ᾽ αὐτὰς εὐθεῖαι, δοθήσεται καὶ ἡ συναμφοτέρας τὰς περιφερείας κατὰ σύνθεσιν ὑποτείνουσα εὐθεῖα διὰ τούτου τοῦ θεωρήματος.[a]

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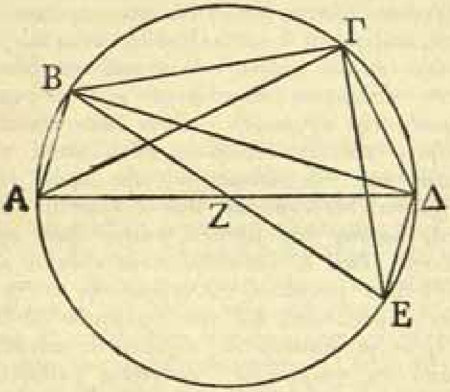
[a] If AB subtends an angle 2θ and ΒΓ an angle 2ϕ at the centre, the theorem asserts that

$$(\text{crd. } 180^\circ) \,.\, (\text{crd. } \overline{180^\circ - 2\theta - 2\phi}) = (\text{crd. } \overline{180^\circ - 2\theta}) \,.\, (\text{crd. } \overline{180^\circ - 2\phi}) - (\text{crd. } 2\theta) \,.\, (\text{crd. } 2\phi),$$

i.e., $$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi.$$

For through B let BZE, the diameter of the circle, be drawn, and let BΔ, ΔΓ, ΓE, ΔE be joined; it is



then immediately obvious that, by reason of BΓ being given, ΓE is also given, and by reason of AB being given, both BΔ and ΔE are given. And by the same reasoning as before, since BΓΔE is a quadrilateral in a circle, and BΔ, ΓE are the diagonals, the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides. And so, since BΔ . ΓE is given, while BΓ . ΔE is also given, therefore BE . ΓΔ is given. But the diameter BE is given, and [therefore] the remaining term ΓΔ will be given, and therefore the chord ΓA subtending the remainder of the semicircle [a]; accordingly, if two arcs be given, and the chords subtending them, by this theorem the chord subtending the sum of the arcs will also be given.

Φανερὸν δέ, ὅτι συντιθέντες ἀεὶ μετὰ τῶν προεκτεθειμένων πασῶν τὴν ὑπὸ α̅ ∠′ μοῖραν καὶ τὰς συναπτομένας ἐπιλογιζόμενοι πάσας ἁπλῶς ἐγγράψομεν, ὅσαι δὶς γινόμεναι τρίτον μέρος ἕξουσιν, καὶ μόναι ἔτι περιλειφθήσονται αἱ μεταξὺ τῶν ἀνὰ α̅ ∠′ μοῖραν διαστημάτων δύο καθ' ἕκαστον ἐσόμεναι, ἐπειδήπερ καθ' ἡμιμοίριον ποιούμεθα τὴν ἐγγραφήν. ὥστε, ἐὰν τὴν ὑπὸ τὸ ἡμιμοίριον εὐθεῖαν εὕρωμεν, αὕτη κατά τε τὴν σύνθεσιν καὶ τὴν ὑπεροχὴν τὴν πρὸς τὰς τὰ διαστήματα περιεχούσας καὶ δεδομένας εὐθείας καὶ τὰς λοιπὰς τὰς μεταξὺ πάσας ἡμῖν συναναπληρώσει. ἐπεὶ δὲ δοθείσης τινὸς εὐθείας ὡς τῆς ὑπὸ τὴν α̅ ∠′ μοῖραν ἡ τὸ τρίτον τῆς αὐτῆς περιφερείας ὑποτείνουσα διὰ τῶν γραμμῶν οὐ δίδοταί πως· εἰ δέ γε δυνατὸν ἦν, εἴχομεν ἂν αὐτόθεν καὶ τὴν ὑπὸ τὸ ἡμιμοίριον· πρότερον μεθοδεύσομεν τὴν ὑπὸ τὴν α̅ μοῖραν ἀπό τε τῆς ὑπὸ τὴν α̅ ∠′ μοῖραν καὶ τῆς ὑπὸ ∠′ δ′ ὑποτεθέμενοι λημμάτιον, ὅ, κἂν μὴ πρὸς τὸ καθόλου δύνηται τὰς πηλικότητας ὁρίζειν, ἐπί γε τῶν οὕτως ἐλαχίστων τὸ πρὸς τὰς ὡρισμένας ἀπαράλλακτον δύναιτ' ἂν συντηρεῖν.

(viii.) *Method of Interpolation*

Ibid. 43. 6–46. 20

Λέγω γάρ, ὅτι ἐὰν ἐν κύκλῳ διαχθῶσιν ἄνισοι δύο εὐθεῖαι, ἡ μείζων πρὸς τὴν ἐλάσσονα ἐλάσσονα λόγον ἔχει ἤπερ ἡ ἐπὶ τῆς μείζονος εὐθείας περιφέρεια πρὸς τὴν ἐπὶ τῆς ἐλάσσονος.

Ἔστω γὰρ κύκλος ὁ ΑΒΓΔ, καὶ διήχθωσαν ἐν αὐτῷ δύο εὐθεῖαι ἄνισοι ἐλάσσων μὲν ἡ ΑΒ,

It is clear that, by continually putting next to all known chords a chord subtending $1\frac{1}{2}°$ and calculating the chords joining them, we may compute in a simple manner all chords subtending multiples of $1\frac{1}{2}°$, and there will still be left only those within the $1\frac{1}{2}°$ intervals—two in each case, since we are making the diagram in half degrees. Therefore, if we find the chord subtending $\frac{1}{2}°$, this will enable us to complete, by the method of addition and subtraction with respect to the chords bounding the intervals, both the given chords and all the remaining, intervening chords. But when any chord subtending, say, $1\frac{1}{2}°$, is given, the chord subtending the third part of the same arc is not given by the [above] calculations—if it were, we should obtain immediately the chord subtending $\frac{1}{2}°$; therefore we shall first give a method for finding the chord subtending $1°$ from the chord subtending $1\frac{1}{2}°$ and that subtending $\frac{3}{4}°$, assuming a little lemma which, even though it cannot be used for calculating lengths in general, in the case of such small chords will enable us to make an approximation indistinguishable from the correct figure.

(viii.) *Method of Interpolation*

Ibid. 43. 6–46. 20

For I say that, *if two unequal chords be drawn in a circle, the greater will bear to the less a less ratio than that which the arc on the greater chord bears to the arc on the lesser.*

For let ΑΒΓΔ be a circle, and in it let there be drawn two unequal chords, of which ΑΒ is the lesser

μείζων δὲ ἡ ΒΓ. λέγω, ὅτι ἡ ΓΒ εὐθεῖα πρὸς τὴν ΒΑ εὐθεῖαν ἐλάσσονα λόγον ἔχει ἤπερ ἡ ΒΓ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τετμήσθω γὰρ ἡ ὑπὸ ΑΒΓ γωνία δίχα ὑπὸ τῆς ΒΔ, καὶ ἐπεζεύχθωσαν ἥ τε ΑΕΓ καὶ ἡ ΑΔ καὶ ἡ ΓΔ. καὶ ἐπεὶ ἡ ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΕΔ εὐθείας, ἴση μέν ἐστιν ἡ ΓΔ εὐθεῖα τῇ ΑΔ, μείζων δὲ ἡ ΓΕ τῆς ΕΑ. ἤχθω δὴ ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ. ἐπεὶ τοίνυν μείζων ἐστὶν ἡ μὲν ΑΔ τῆς ΕΔ, ἡ δὲ ΕΔ τῆς ΔΖ, ὁ ἄρα κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεῖ, ὑπερπεσεῖται δὲ τὴν ΔΖ. γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἡ ΔΖΘ. καὶ ἐπεὶ ὁ μὲν ΔΕΘ τομεὺς μείζων ἐστὶν τοῦ ΔΕΖ τριγώνου, τὸ δὲ ΔΕΑ τρίγωνον μεῖζον τοῦ ΔΕΗ τομέως, τὸ ἄρα ΔΕΖ

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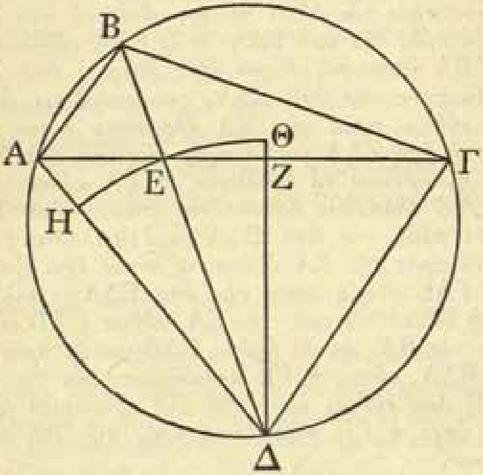
[a] Lit. "let ΔΖΘ be produced."

and ΒΓ the greater. I say that

$$\Gamma B : BA < \text{arc } B\Gamma : \text{arc } BA.$$

For let the angle ΑΒΓ be bisected by ΒΔ, and let



ΑΕΓ and ΑΔ and ΓΔ be joined. Then since the angle ΑΒΓ is bisected by the chord ΒΕΔ, the chord ΓΔ = ΑΔ [Eucl. iii. 26, 29], while ΓΕ > ΕΑ [Eucl. vi. 3]. Now let ΔΖ be drawn from Δ perpendicular to ΑΕΓ. Then since ΑΔ > ΕΔ, and ΕΔ > ΔΖ, the circle described with centre Δ and radius ΔΕ will cut ΑΔ, and will fall beyond ΔΖ. Let [the arc] ΗΕΘ be described, and let ΔΖ be produced to Θ.[a] Then since

$$\text{sector } \Delta E\Theta > \text{triangle } \Delta EZ,$$

and $$\text{triangle } \Delta EA > \text{sector } \Delta EH,$$

τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον ἐλάσσονα λόγον ἔχει ἤπερ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ. ἀλλ' ὡς μὲν τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον, οὕτως ἡ ΕΖ εὐθεῖα πρὸς τὴν ΕΑ, ὡς δὲ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ τομέα, οὕτως ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ· ἡ ἄρα ΖΕ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. καὶ συνθέντι ἄρα ἡ ΖΑ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΑ γωνία πρὸς τὴν ὑπὸ ΑΔΕ· καὶ τῶν ἡγουμένων τὰ διπλάσια, ἡ ΓΑ εὐθεῖα πρὸς τὴν ΑΕ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΓΔΑ γωνία πρὸς τὴν ὑπὸ ΕΔΑ· καὶ διελόντι ἡ ΓΕ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΓΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἀλλ' ὡς μὲν ἡ ΓΕ εὐθεῖα πρὸς τὴν ΕΑ, οὕτως ἡ ΓΒ εὐθεῖα πρὸς τὴν ΒΑ, ὡς δὲ ἡ ὑπὸ ΓΔΒ γωνία πρὸς τὴν ὑπὸ ΒΔΑ, οὕτως ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ· ἡ ΓΒ ἄρα εὐθεῖα πρὸς τὴν ΒΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τούτου δὴ οὖν ὑποκειμένου ἔστω κύκλος ὁ ΑΒΓ, καὶ διήχθωσαν ἐν αὐτῷ δύο εὐθεῖαι ἥ τε ΑΒ καὶ ἡ ΑΓ, ὑποκείσθω δὲ πρῶτον ἡ μὲν ΑΒ ὑποτείνουσα μιᾶς μοίρας ∠′ δ′, ἡ δὲ ΑΓ μοῖραν ᾱ. ἐπεὶ ἡ ΑΓ εὐθεῖα πρὸς τὴν ΒΑ εὐθεῖαν ἐλάσσονα λόγον ἔχει ἤπερ ἡ ΑΓ περιφέρεια πρὸς τὴν ΑΒ, ἡ δὲ ΑΓ περιφέρεια ἐπίτριτός ἐστιν τῆς ΑΒ, ἡ ΓΑ ἄρα εὐθεῖα τῆς ΒΑ ἐλάσσων ἐστὶν ἢ ἐπίτριτος. ἀλλὰ ἡ ΑΒ εὐθεῖα ἐδείχθη τοιούτων Ο μζ η̄, οἵων ἐστὶν ἡ διάμετρος ρκ̄· ἡ ἄρα ΓΑ

$\therefore$ triangle ΔEZ : triangle ΔEA < sector ΔEΘ : sector ΔEH.

But triangle ΔEZ : triangle ΔEA = EZ : EA, [Eucl. vi. 1

and

sector ΔEΘ : sector ΔEH = angle ZΔE : angle EΔA.

$\therefore$ ZE : EA < angle ZΔE : angle EΔA.

$\therefore$ *componendo*, ZA : EA < angle ZΔA : angle AΔE ;

and, by doubling the antecedents,

ΓA : AE < angle ΓΔA : angle EΔA ;

and *dirimendo*, ΓE : EA < angle ΓΔE : angle EΔA.

But ΓE : EA = ΓB : BA, [Eucl. vi. 3

and

angle ΓΔB : angle BΔA = arc ΓB : arc BA ; [Eucl. vi. 33

$\therefore$ ΓB : BA < arc ΓB : arc BA.[a]

On this basis, then, let ABΓ be a circle, and in it let there be drawn the two chords AB and AΓ, and let it first be supposed that AB subtends an angle of $\frac{3}{4}°$ and AΓ an angle of 1°. Then since

AΓ : BA < arc AΓ : arc AB,

while arc AΓ = $\frac{4}{3}$. arc AB,

$\therefore$ ΓA : BA < $\frac{4}{3}$.

But the chord AB was shown to be $0^p\ 47'\ 8''$ (the diameter being 120^p); therefore the chord ΓA

[a] If the chords ΓB, BA subtend angles 2θ, 2ϕ at the centre, this is equivalent to the formula,

$$\frac{\sin\theta}{\sin\phi} < \frac{\theta}{\phi},$$

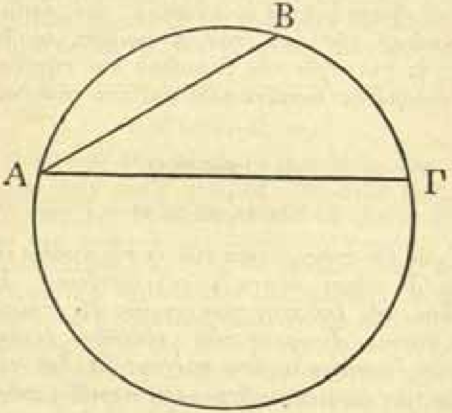
where $\theta < \phi < \frac{1}{2}\pi$.

εὐθεῖα ἐλάσσων ἐστὶν τῶν αὐτῶν α̅ β̅ ν̅· ταῦτα γὰρ ἐπίτριτά ἐστιν ἔγγιστα τῶν ο μ̅ζ̅ η̅.

Πάλιν ἐπὶ τῆς αὐτῆς καταγραφῆς ἡ μὲν ΑΒ εὐθεῖα ὑποκείσθω ὑποτείνουσα μοῖραν α̅, ἡ δὲ ΑΓ μοῖραν α̅ ∠′. κατὰ τὰ αὐτὰ δή, ἐπεὶ ἡ ΑΓ περιφέρεια τῆς ΑΒ ἐστιν ἡμιολία, ἡ ΓΑ ἄρα εὐθεῖα τῆς ΒΑ ἐλάσσων ἐστὶν ἢ ἡμιόλιος. ἀλλὰ τὴν ΑΓ ἀπεδείξαμεν τοιούτων οὖσαν α̅ λ̅δ̅ ι̅ε̅, οἵων ἐστὶν ἡ διάμετρος ρ̅κ̅· ἡ ἄρα ΑΒ εὐθεῖα μείζων ἐστὶν τῶν αὐτῶν α̅ β̅ ν̅· τούτων γὰρ ἡμιόλιά ἐστιν τὰ προκείμενα α̅ λ̅δ̅ ι̅ε̅. ὥστε, ἐπεὶ τῶν αὐτῶν ἐδείχθη καὶ μείζων καὶ ἐλάσσων ἡ τὴν μίαν μοῖραν ὑποτείνουσα εὐθεῖα, καὶ ταύτην δηλονότι ἕξομεν τοιούτων α̅ β̅ ν̅ ἔγγιστα, οἵων ἐστὶν ἡ διάμετρος ρ̅κ̅, καὶ διὰ τὰ προδεδειγμένα καὶ τὴν ὑπὸ τὸ ἡμιμοίριον, ἥτις εὑρίσκεται τῶν αὐτῶν

$<1^p\ 2'\ 50''$; for this is approximately four-thirds of $0^p\ 47'\ 8''$.

Again, with the same diagram, let the chord AB



be supposed to subtend an angle of 1°, and AΓ an angle of $1\frac{1}{2}$°. By the same reasoning,

since $\qquad$ arc $A\Gamma = \frac{3}{2}$. arc AB,

$\therefore \qquad \Gamma A : BA < \frac{3}{2}$.

But we have proved AΓ to be $1^p\ 34'\ 15''$ (the diameter being 120^p); therefore the chord AB$>1^p\ 2'\ 50''$; for $1^p\ 34'\ 15''$ is one-and-a-half times this number. Therefore, since the chord subtending an angle of 1° has been shown to be both greater and less than [approximately] the same [length], manifestly we shall find it to have approximately this identical value $1^p\ 2'\ 50''$ (the diameter being 120^p), and by what has been proved before we shall obtain the chord subtending $\frac{1}{2}$°, which is found to be approximately

ο $\overline{\lambda\alpha}$ $\overline{\kappa\epsilon}$ ἔγγιστα. καὶ συναναπληρωθήσεται τὰ λοιπά, ὡς ἔφαμεν, διαστήματα ἐκ μὲν τῆς πρὸς τὴν μίαν ἥμισυ μοῖραν λόγου ἕνεκεν ὡς ἐπὶ τοῦ πρώτου διαστήματος συνθέσεως τοῦ ἡμιμοιρίου δεικνυμένης τῆς ὑπὸ τὰς $\overline{\beta}$ μοίρας, ἐκ δὲ τῆς ὑπεροχῆς τῆς πρὸς τὰς $\overline{\gamma}$ μοίρας καὶ τῆς ὑπὸ τὰς $\overline{\beta}\,\angle'$ διδομένης· ὡσαύτως δὲ καὶ ἐπὶ τῶν λοιπῶν.

(ix.) *The Table*

Ibid. 46. 21–63. 46

Ἡ μὲν οὖν πραγματεία τῶν ἐν τῷ κύκλῳ εὐθειῶν οὕτως ἂν οἶμαι ῥᾷστα μεταχειρισθείη. ἵνα δέ, ὡς ἔφην, ἐφ' ἑκάστης τῶν χρειῶν ἐξ ἑτοίμου τὰς πηλικότητας ἔχωμεν τῶν εὐθειῶν ἐκκειμένας, κανόνια ὑποτάξομεν ἀνὰ στίχους $\overline{\mu\epsilon}$ διὰ τὸ σύμμετρον, ὧν τὰ μὲν πρῶτα μέρη περιέξει τὰς πηλικότητας τῶν περιφερειῶν καθ' ἡμιμοίριον παρηυξημένας, τὰ δὲ δεύτερα τὰς τῶν παρακειμένων ταῖς περιφερείαις εὐθειῶν πηλικότητας ὡς τῆς διαμέτρου τῶν $\overline{\rho\kappa}$ τμημάτων ὑποκειμένης, τὰ δὲ τρίτα τὸ λʹ μέρος τῆς καθ' ἕκαστον ἡμιμοίριον τῶν εὐθειῶν παραυξήσεως, ἵνα ἔχοντες καὶ τὴν τοῦ ἑνὸς ἑξηκοστοῦ μέσην ἐπιβολὴν ἀδιαφοροῦσαν πρὸς αἴσθησιν τῆς ἀκριβοῦς καὶ τῶν μεταξὺ τοῦ ἡμίσους μερῶν ἐξ ἑτοίμου τὰς ἐπιβαλλούσας πηλικότητας ἐπιλογίζεσθαι δυνώμεθα. εὐκατανόητον δ', ὅτι διὰ τῶν αὐτῶν καὶ προκειμένων θεωρημάτων, κἂν ἐν διστανμῷ γενώμεθα γραφικῆς ἁμαρτίας περί τινα τῶν ἐν τῷ κανονίῳ παρακειμένων εὐθειῶν, ῥᾳδίαν ποιησόμεθα τήν τε ἐξέτασιν καὶ τὴν

0° 31′ 25″. The remaining intervals may be completed, as we said, by means of the chord subtending $1\frac{1}{2}$°—in the case of the first interval, for example, by adding $\frac{1}{2}$° we obtain the chord subtending 2°, and from the difference between this and 3° we obtain the chord subtending $2\frac{1}{2}$°, and so on for the remainder.

(ix.) *The Table*

Ibid. 46. 21–63. 46

The theory of the chords in the circle may thus, I think, be very easily grasped. In order that, as I said, we may have the lengths of all the chords in common use immediately available, we shall draw up tables arranged in forty-five symmetrical rows.[a] The first section will contain the magnitudes of the arcs increasing by half degrees, the second will contain the lengths of the chords subtending the arcs measured in parts of which the diameter contains 120, and the third will give the thirtieth part of the increase in the chords for each half degree, in order that for every sixtieth part of a degree we may have a mean approximation differing imperceptibly from the true figure and so be able readily to calculate the lengths corresponding to the fractions between the half degrees. It should be well noted that, by these same theorems now before us, if we should suspect an error in the computation of any of the chords in the table,[b] we can easily make a test and

[a] As there are 360 half degrees in the table, the statement appears to mean that the table occupied eight pages each of 45 rows; so Manitius, *Des Claudius Ptolemäus Handbuch der Astronomie*, 1er Bd., p. 35 n. *a*.

[b] Such an error might be accumulated by using the approximations for 1° and $\frac{1}{2}$°; but, in fact, the sines in the table are generally correct to five places of decimals.

ἐπανόρθωσιν ἤτοι ἀπὸ τῆς ὑπὸ τὴν διπλασίονα τῆς ἐπιζητουμένης ἢ τῆς πρὸς ἄλλας τινὰς τῶν δεδομένων ὑπεροχῆς ἢ τῆς τὴν λείπουσαν εἰς τὸ ἡμικύκλιον περιφέρειαν ὑποτεινούσης εὐθείας. καί ἐστιν ἡ τοῦ κανονίου καταγραφὴ τοιαύτη·

ιαʹ. Κανόνιον τῶν ἐν κύκλῳ εὐθειῶν

περιφερειῶν	εὐθειῶν			ἑξηκοστῶν			
∠′	ο	λα	κε	ο	α	β	ν
α	α	β	ν	ο	α	β	ν
α ∠′	α	λδ	ιε	ο	α	β	ν
β	β	ε	μ	ο	α	β	ν
β ∠′	β	λζ	δ	ο	α	β	μη
γ	γ	η	κη	ο	α	β	μη
γ ∠′	γ	λθ	νβ	ο	α	β	μη
δ	δ	ια	ιϛ	ο	α	β	μζ
δ ∠′	δ	μβ	μ	ο	α	β	μζ
.	.	.	.	.	.	.	.
ξ	ξ	ο	ο	ο	ο	νδ	κα
.	.	.	.	.	.	.	.
ροϛ	ριθ	νε	λη	ο	ο	β	γ
ροϛ ∠′	ριθ	νϛ	λθ	ο	ο	α	μζ
ροζ	ριθ	νζ	λβ	ο	ο	α	λ
ροζ ∠′	ριθ	νη	ιη	ο	ο	α	ιδ
ροη	ριθ	νη	νε	ο	ο	ο	νζ
ροη ∠′	ριθ	νθ	κδ	ο	ο	ο	μα
ροθ	ριθ	νθ	μδ	ο	ο	ο	κε
ροθ ∠′	ριθ	νθ	νϛ	ο	ο	ο	θ
ρπ	ρκ	ο	ο	ο	ο	ο	ο

apply a correction, either from the chord subtending double of the arc which is under investigation, or from the difference with respect to any others of the given magnitudes, or from the chord subtending the remainder of the semicircular arc. And this is the diagram of the table :

11. Table of the Chords in a Circle

Arcs	Chords			Sixtieths			
½°	0ᵖ	31′	25″	0ᵖ	1′	2″	50‴
1	1	2	50	0	1	2	50
1½	1	34	15	0	1	2	50
2	2	5	40	0	1	2	50
2½	2	37	4	0	1	2	48
3	3	8	28	0	1	2	48
3½	3	39	52	0	1	2	48
4	4	11	16	0	1	2	47
4½	4	42	40	0	1	2	47
.		.	.		.		.
60	60	0	0	0	0	54	21
.		.	.		.		.
176	119	55	38	0	0	2	3
176½	119	56	39	0	0	1	47
177	119	57	32	0	0	1	30
177½	119	58	18	0	0	1	17
178	119	58	55	0	0	0	57
178½	119	59	24	0	0	0	41
179	119	59	44	0	0	0	25
179½	119	59	56	0	0	0	9
180	120	0	0	0	0	0	0

(c) Menelaus's Theorem

(i.) *Lemmas*

Ibid. 68. 14–74. 8

ιγʹ. Προλαμβανόμενα εἰς τὰς σφαιρικὰς δείξεις

Ἀκολούθου δ' ὄντος ἀποδεῖξαι καὶ τὰς κατὰ μέρος γινομένας πηλικότητας τῶν ἀπολαμβανομένων περιφερειῶν μεταξὺ τοῦ τε ἰσημερινοῦ καὶ τοῦ διὰ μέσων τῶν Ζῳδίων κύκλου τῶν γραφομένων μεγίστων κύκλων διὰ τῶν τοῦ ἰσημερινοῦ πόλων προεκθησόμεθα λημμάτια βραχέα καὶ εὔχρηστα, δι' ὧν τὰς πλείστας σχεδὸν δείξεις τῶν σφαιρικῶς θεωρουμένων, ὡς ἔνι μάλιστα, ἁπλούστερον καὶ μεθοδικώτερον ποιησόμεθα.

Εἰς δύο δὴ εὐθείας τὰς ΑΒ καὶ ΑΓ διαχθεῖσαι δύο εὐθεῖαι ἥ τε ΒΕ καὶ ἡ ΓΔ τεμνέτωσαν ἀλλήλας

(c) Menelaus's Theorem

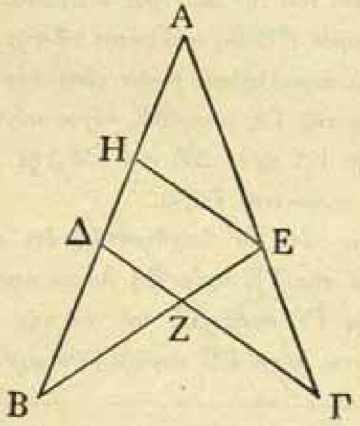
(i.) *Lemmas*

Ibid. 68. 14–74. 8

13. Preliminary matter for the spherical proofs

The next subject for investigation being to show the lengths of the arcs, intercepted between the celestial equator and the zodiac circle, of great circles drawn through the poles of the equator, we shall set out some brief and serviceable little lemmas, by means of which we shall be able to prove more simply and more systematically most of the questions investigated spherically.

Let two straight lines ΒΕ and ΓΔ be drawn so as



to meet the straight lines ΑΒ and ΑΓ and to cut one

κατὰ τὸ Ζ σημεῖον. λέγω, ὅτι ὁ τῆς ΓΑ πρὸς ΑΕ λόγος συνῆπται ἔκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ.

Ἤχθω γὰρ διὰ τοῦ Ε τῇ ΓΔ παράλληλος ἡ ΕΗ. ἐπεὶ παράλληλοί εἰσιν αἱ ΓΔ καὶ ΕΗ, ὁ τῆς ΓΑ πρὸς ΕΑ λόγος ὁ αὐτός ἐστιν τῷ τῆς ΓΔ πρὸς ΕΗ. ἔξωθεν δὲ ἡ ΖΔ· ὁ ἄρα τῆς ΓΔ πρὸς ΕΗ λόγος συγκείμενος ἔσται ἔκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ· ὥστε καὶ ὁ τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἔκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ. ἔστιν δὲ καὶ ὁ τῆς ΔΖ πρὸς ΗΕ λόγος ὁ αὐτὸς τῷ τῆς ΖΒ πρὸς ΒΕ διὰ τὸ παραλλήλους πάλιν εἶναι τὰς ΕΗ καὶ ΖΔ· ὁ ἄρα τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἔκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ· ὅπερ προέκειτο δεῖξαι.

Κατὰ τὰ αὐτὰ δὲ δειχθήσεται, ὅτι καὶ κατὰ διαίρεσιν ὁ τῆς ΓΕ πρὸς ΕΑ λόγος συνῆπται ἔκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ, διὰ τοῦ Α τῇ ΕΒ παραλλήλου ἀχθείσης καὶ

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[a] Lit. "the ratio of ΓΑ to ΑΕ is compounded of the ratio of ΓΔ to ΔΖ and ΖΒ to ΒΕ."

another at the point Z. I say that

$$\Gamma A : AE = (\Gamma\Delta : \Delta Z)(ZB : BE).^{a}$$

For through E let EH be drawn parallel to ΓΔ. Since ΓΔ and EH are parallel,

$$\Gamma A : EA = \Gamma\Delta : EH. \qquad \text{[Eucl. vi. 4}$$

But ZΔ is an external [straight line];

$\therefore$ $$\Gamma\Delta : EH = (\Gamma\Delta : \Delta Z)(\Delta Z : HE);$$

$\therefore$ $$\Gamma A : AE = (\Gamma\Delta : \Delta Z)(\Delta Z : HE).$$

But $$\Delta Z : HE = ZB : BE, \qquad \text{[Eucl. vi. 4}$$

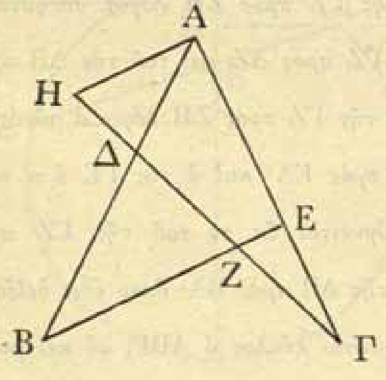
by reason of the fact that EH and ZΔ are parallels;

$\therefore$ $$\Gamma A : AE = (\Gamma\Delta : \Delta Z)(ZB : BE); \quad . \quad (1)$$

which was set to be proved.

With the same premises, it will be shown by transformation of ratios that

$$\Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA),$$



a parallel to EB being drawn through A and ΓΔH

προσεκβληθείσης ἐπ' αὐτὴν τῆς ΓΔΗ. ἐπεὶ γὰρ πάλιν παράλληλός ἐστιν ἡ ΑΗ τῇ ΕΖ, ἔστιν, ὡς ἡ ΓΕ πρὸς ΕΑ, ἡ ΓΖ πρὸς ΖΗ. ἀλλὰ τῆς ΖΔ ἔξωθεν λαμβανομένης ὁ τῆς ΓΖ πρὸς ΖΗ λόγος σύγκειται ἔκ τε τοῦ τῆς ΓΖ πρὸς ΖΔ καὶ τοῦ τῆς ΔΖ πρὸς ΖΗ· ἔστιν δὲ ὁ τῆς ΔΖ πρὸς ΖΗ λόγος ὁ αὐτὸς τῷ τῆς ΔΒ πρὸς ΒΑ διὰ τὸ εἰς παραλλήλους τὰς ΑΗ καὶ ΖΒ διῆχθαι τὰς ΒΑ καὶ ΖΗ· ὁ ἄρα τῆς ΓΖ πρὸς ΖΗ λόγος συνῆπται ἔκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ. ἀλλὰ τῷ τῆς ΓΖ πρὸς ΖΗ λόγῳ ὁ αὐτός ἐστιν ὁ τῆς ΓΕ πρὸς ΕΑ· καὶ ὁ τῆς ΓΕ ἄρα πρὸς ΕΑ λόγος σύγκειται ἔκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ· ὅπερ ἔδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Δ, καὶ εἰλήφθω ἐπὶ τὰς περιφερείας αὐτοῦ τυχόντα

being produced to it. For, again, since AH is parallel to EZ,

$$\Gamma E : EA = \Gamma Z : ZH. \qquad \text{[Eucl. vi. 2}$$

But, an external straight line $Z\Delta$ having been taken,

$$\Gamma Z : ZH = (\Gamma Z : Z\Delta)(\Delta Z : ZH) ;$$

and $$\Delta Z : ZH = \Delta B : BA,$$

by reason of BA and ZH being drawn to meet the parallels AH and ZB ;

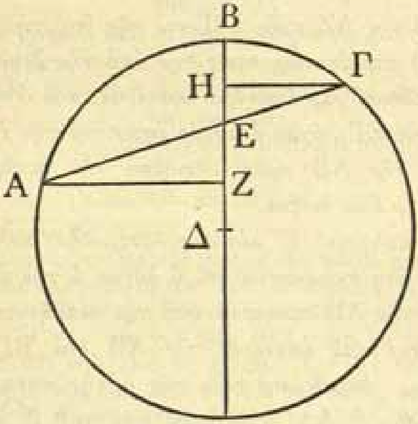
$\therefore$ $$\Gamma Z : ZH = (\Gamma Z : \Delta Z)(\Delta B : BA).$$

But $$\Gamma Z : ZH = \Gamma E : EA ; \qquad \text{[}\textit{supra}$$

and $\therefore$ $$\Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA) ; \quad . \quad (2)$$

which was to be proved.

Again, let $AB\Gamma$ be a circle with centre Δ, and let



there be taken on its circumference any three points

τρία σημεῖα τὰ Α, Β, Γ, ὥστε ἑκατέραν τῶν ΑΒ, ΒΓ περιφερειῶν ἐλάσσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ τῶν ἑξῆς δὲ λαμβανομένων περιφερειῶν τὸ ὅμοιον ὑπακουέσθω· καὶ ἐπεζεύχθωσαν αἱ ΑΓ καὶ ΔΕΒ. λέγω, ὅτι ἐστίν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, οὕτως ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΓ εὐθεῖαν.

Ἤχθωσαν γὰρ κάθετοι ἀπὸ τῶν Α καὶ Γ σημείων ἐπὶ τὴν ΔΒ ἥ τε ΑΖ καὶ ἡ ΓΗ. ἐπεὶ παράλληλός ἐστιν ἡ ΑΖ τῇ ΓΗ, καὶ διῆκται εἰς αὐτὰς εὐθεῖα ἡ ΑΕΓ, ἔστιν, ὡς ἡ ΑΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΑΕ πρὸς ΕΓ. ἀλλ' ὁ αὐτός ἐστιν λόγος ὁ τῆς ΑΖ πρὸς ΓΗ καὶ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ἡμίσεια γὰρ ἑκατέρα ἑκατέρας· καὶ ὁ τῆς ΑΕ ἄρα πρὸς ΕΓ λόγος ὁ αὐτός ἐστιν τῷ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ὅπερ ἔδει δεῖξαι.

Παρακολουθεῖ δ' αὐτόθεν, ὅτι, κἂν δοθῶσιν ἥ τε ΑΓ ὅλη περιφέρεια καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, δοθήσεται καὶ ἑκατέρα τῶν ΑΒ καὶ ΒΓ περιφερειῶν. ἐκτεθείσης γὰρ τῆς αὐτῆς καταγραφῆς ἐπεζεύχθω ἡ ΑΔ, καὶ ἤχθω ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ. ὅτι μὲν οὖν τῆς ΑΓ περι-

A, B, Γ, in such a manner that each of the arcs AB, BΓ is less than a semicircle; and upon the arcs taken in succession let there be a similar relationship; and let AΓ be joined and ΔEB. I say that

the chord subtended by double of the arc AB :
the chord subtended by double of the arc BΓ

[*i.e.*, sin AB : sin BΓ [a]] = AE : EΓ.

For let perpendiculars AZ and ΓH be drawn from the points A and Γ to ΔB. Since AZ is parallel to ΓH, and the straight line AEΓ has been drawn to meet them,

AZ : ΓH = AE : EΓ. [Eucl. vi. 4

But AZ : ΓH = the chord subtended by double of the arc AB :
the chord subtended by double of the arc BΓ,

for each term is half of the corresponding term;
and therefore

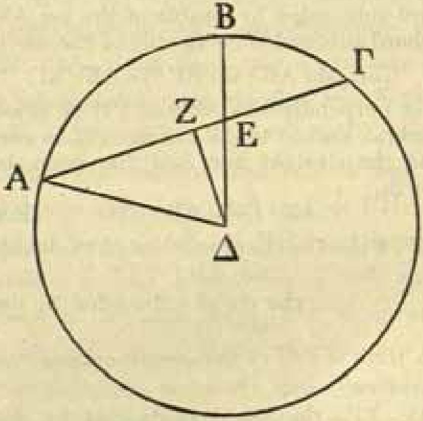
AE : EΓ = the chord subtended by double of the arc AB :
the chord subtended by double of the arc BΓ. . . . (3)
[= sin AB : sin BΓ],

which was to be proved.

It follows immediately that, if the whole arc AΓ be given, and the ratio of the chord subtended by double of the arc AB to the chord subtended by double of the arc BΓ [*i.e.* sin AB : sin BΓ], each of the arcs AB and BΓ will also be given. For let the same diagram be set out, and let AΔ be joined, and from Δ let ΔZ be drawn perpendicular to AEΓ. If the arc

[a] *v. supra*, p. 420 n. *a*.

φερείας δοθείσης ἥ τε ὑπὸ ΑΔΖ γωνία τὴν ἡμίσειαν αὐτῆς ὑποτείνουσα δεδομένη ἔσται καὶ ὅλον τὸ ΑΔΖ τρίγωνον, δῆλον· ἐπεὶ δὲ τῆς ΑΓ



εὐθείας ὅλης δεδομένης ὑπόκειται καὶ ὁ τῆς ΑΕ πρὸς ΕΓ λόγος ὁ αὐτὸς ὢν τῷ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, ἥ τε ΑΕ ἔσται δοθεῖσα καὶ λοιπὴ ἡ ΖΕ. καὶ διὰ τοῦτο καὶ τῆς ΔΖ δεδομένης δοθήσεται καὶ ἥ τε ὑπὸ ΕΔΖ γωνία τοῦ ΕΔΖ ὀρθογωνίου καὶ ὅλη ἡ ὑπὸ ΑΔΒ· ὥστε καὶ ἥ τε ΑΒ περιφέρεια δοθήσεται καὶ λοιπὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ περὶ κέντρον τὸ Δ, καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω τρία σημεῖα τὰ Α, Β, Γ, ὥστε ἑκατέραν τῶν ΑΒ, ΑΓ περιφερειῶν ἐλάσσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ

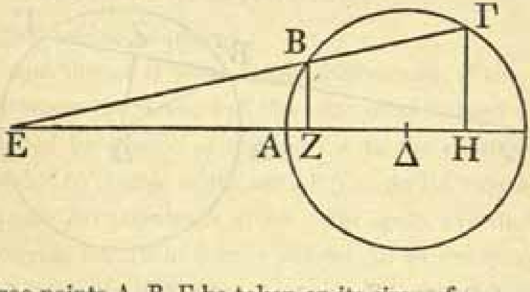
ΑΓ is given, it is then clear that the angle ΑΔΖ, subtending half the same arc, will also be given and therefore the whole triangle ΑΔΖ ; and since the whole chord ΑΓ is given, and by hypothesis

ΑΕ : ΕΓ = the chord subtended by double of the arc ΑΒ :
the chord subtended by double of the arc ΒΓ,

[*i.e.* = sin ΑΒ : sin ΒΓ],

therefore ΑΕ will be given [Eucl. *Dat.* 7], and the remainder ΖΕ. And for this reason, ΔΖ also being given, the angle ΕΔΖ will be given in the right-angled triangle ΕΔΖ, and [therefore] the whole angle ΑΔΒ ; therefore the arc ΑΒ will be given and also the remainder ΒΓ ; which was to be proved.

Again, let ΑΒΓ be a circle about centre Δ, and let

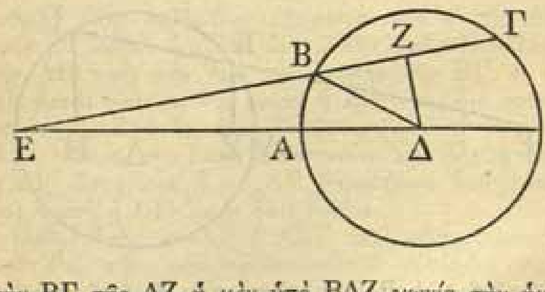


three points Α, Β, Γ be taken on its circumference so that each of the arcs ΑΒ, ΑΓ is less than a semicircle ;

τῶν ἑξῆς δὲ λαμβανομένων περιφερειῶν τὸ ὅμοιον ὑπακουέσθω· καὶ ἐπιζευχθεῖσαι ἥ τε ΔΑ καὶ ἡ ΓΒ ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Ε σημεῖον. λέγω, ὅτι ἐστίν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΑ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΑΒ, οὕτως ἡ ΓΕ εὐθεῖα πρὸς τὴν ΒΕ.

Ὁμοίως γὰρ τῷ προτέρῳ λημματίῳ, ἐὰν ἀπὸ τῶν Β καὶ Γ ἀγάγωμεν καθέτους ἐπὶ τὴν ΔΑ τήν τε ΒΖ καὶ τὴν ΓΗ, ἔσται διὰ τὸ παραλλήλους αὐτὰς εἶναι, ὡς ἡ ΓΗ πρὸς τὴν ΒΖ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΒ· ὥστε καί, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΑΒ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΒ· ὅπερ ἔδει δεῖξαι.

Καὶ ἐνταῦθα δὲ αὐτόθεν παρακολουθεῖ, διότι, κἂν ἡ ΓΒ περιφέρεια μόνη δοθῇ, καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΑΒ δοθῇ, καὶ ἡ ΑΒ περιφέρεια δοθήσεται. πάλιν γὰρ ἐπὶ τῆς ὁμοίας καταγραφῆς ἐπιζευχθείσης τῆς ΔΒ καὶ καθέτου ἀχθείσης ἐπὶ



τὴν ΒΓ τῆς ΔΖ ἡ μὲν ὑπὸ ΒΔΖ γωνία τὴν ἡμίσειαν ὑποτείνουσα τῆς ΒΓ περιφερείας ἔσται

and upon the arcs taken in succession let there be a similar relationship; and let ΔA be joined and let ΓB be produced so as to meet it at the point E. I say that

the chord subtended by double of the arc ΓA :
the chord subtended by double of the arc AB
[*i.e.*, sin ΓA : sin AB] = ΓE : BE.

For, as in the previous lemma, if from B and Γ we draw BZ and ΓH perpendicular to ΔA, then, by reason of the fact that they are parallel,

ΓH : BZ = ΓE : EB. [Eucl. vi. 4

∴ the chord subtended by double of the arc ΓA :
the chord subtended by double of the arc AB
[*i.e.*, sin ΓA : sin AB] = ΓE : EB ; . . . (4)

which was to be proved.

And thence it immediately follows why, if the arc ΓB alone be given, and the ratio of the chord subtended by double of the arc ΓA to the chord subtended by double of the arc AB [*i.e.*, sin ΓA : sin AB], the arc AB will also be given. For again, in a similar diagram let ΔB be joined and let ΔZ be drawn perpendicular to BΓ; then the angle BΔZ subtended by half the arc BΓ will be given; and therefore the

δεδομένη· καὶ ὅλον ἄρα τὸ ΒΔΖ ὀρθογώνιον. ἐπεὶ δὲ καὶ ὅ τε τῆς ΓΕ πρὸς τὴν ΕΒ λόγος δέδοται καὶ ἔτι ἡ ΓΒ εὐθεῖα, δοθήσεται καὶ ἥ τε ΕΒ καὶ ἔτι ὅλη ἡ ΕΒΖ· ὥστε καί, ἐπεὶ ἡ ΔΖ δέδοται, δοθήσεται καὶ ἥ τε ὑπὸ ΕΔΖ γωνία τοῦ αὐτοῦ ὀρθογωνίου καὶ λοιπὴ ἡ ὑπὸ ΕΔΒ. ὥστε καὶ ἡ ΑΒ περιφέρεια ἔσται δεδομένη.

(ii.) *The Theorem*

Ibid. 74. 9–76. 9

Τούτων προληφθέντων γεγράφθωσαν ἐπὶ σφαιρικῆς ἐπιφανείας μεγίστων κύκλων περιφέρειαι, ὥστε εἰς δύο τὰς ΑΒ καὶ ΑΓ δύο γραφείσας τὰς ΒΕ καὶ ΓΔ τέμνειν ἀλλήλας κατὰ τὸ Ζ σημεῖον· ἔστω δὲ ἑκάστη αὐτῶν ἐλάσσων ἡμικυκλίου· τὸ δὲ αὐτὸ καὶ ἐπὶ πασῶν τῶν καταγραφῶν ὑπακουέσθω.

Λέγω δή, ὅτι ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΕ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ λόγος συνῆπται ἔκ τε τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ.

Εἰλήφθω γὰρ τὸ κέντρον τῆς σφαίρας καὶ ἔστω τὸ Η, καὶ ἤχθωσαν ἀπὸ τοῦ Η ἐπὶ τὰς Β, Ζ, Ε τομὰς τῶν κύκλων ἥ τε ΗΒ καὶ ἡ ΗΖ καὶ ἡ ΗΕ, καὶ ἐπιζευχθεῖσα ἡ ΑΔ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ ΗΒ ἐκβληθείσῃ καὶ αὐτῇ κατὰ τὸ Θ σημεῖον, ὁμοίως δὲ ἐπιζευχθεῖσαι αἱ ΔΓ καὶ ΑΓ τεμνέτωσαν τὰς ΗΖ καὶ ΗΕ κατὰ τὸ Κ καὶ Λ

whole of the right-angled triangle BΔZ. But since the ratio ΓE : EB is given and also the chord ΓB, therefore EB will also be given and, further, the whole [straight line] EBZ; therefore, since ΔZ is given, the angle EΔZ in the same right-angled triangle will be given, and the remainder EΔB. Therefore the arc AB will be given.

(ii.) *The Theorem*

Ibid. 74. 9–76. 9

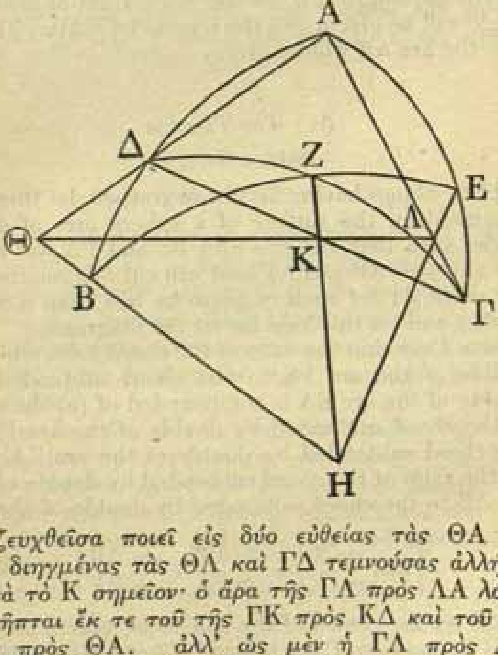
These things having first been grasped, let there be described on the surface of a sphere arcs of great circles such that the two arcs BE and ΓΔ will meet the two arcs AB and AΓ and will cut one another at the point Z; let each of them be less than a semicircle; and let this hold for all the diagrams.

Now I say that the ratio of the chord subtended by double of the arc ΓE to the chord subtended by double of the arc EA is compounded of (*a*) the ratio of the chord subtended by double of the arc ΓZ to the chord subtended by double of the arc ZΔ, and (*b*) the ratio of the chord subtended by double of the arc ΔB to the chord subtended by double of the arc BA,

$$\left[i.e., \frac{\sin \Gamma\mathrm{E}}{\sin \mathrm{EA}} = \frac{\sin \Gamma\mathrm{Z}}{\sin \mathrm{Z}\Delta} \cdot \frac{\sin \Delta\mathrm{B}}{\sin \mathrm{BA}}\right].$$

For let the centre of the sphere be taken, and let it be H, and from H let HB and HZ and HE be drawn to B, Z, E, the points of intersection of the circles, and let AΔ be joined and produced, and let it meet HB produced at the point Θ, and similarly let ΔΓ and AΓ be joined and cut HZ and HE at K and the point

σημεῖον· ἐπὶ μιᾶς δὴ γίνεται εὐθείας τὰ Θ, Κ, Λ σημεῖα διὰ τὸ ἐν δυσὶν ἅμα εἶναι ἐπιπέδοις τῷ τε τοῦ ΛΓΔ τριγώνου καὶ τῷ τοῦ ΒΖΕ κύκλου, ἥτις



ἐπιζευχθεῖσα ποιεῖ εἰς δύο εὐθείας τὰς ΘΑ καὶ ΓΑ διηγμένας τὰς ΘΛ καὶ ΓΔ τεμνούσας ἀλλήλας κατὰ τὸ Κ σημεῖον· ὁ ἄρα τῆς ΓΛ πρὸς ΛΑ λόγος συνῆπται ἔκ τε τοῦ τῆς ΓΚ πρὸς ΚΔ καὶ τοῦ τῆς ΔΘ πρὸς ΘΑ. ἀλλ' ὡς μὲν ἡ ΓΛ πρὸς ΛΑ, οὕτως ἡ ὑπὸ τὴν διπλῆν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ περιφερείας, ὡς δὲ ἡ ΓΚ πρὸς ΚΔ, οὕτως ἡ ὑπὸ τὴν διπλῆν τῆς ΓΖ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ, ὡς δὲ ἡ ΘΔ

Λ; then the points Θ, K, Λ will lie on one straight line because they lie simultaneously in two planes, that of the triangle ΑΓΔ and that of the circle BZE, and therefore we have straight lines ΘΛ and ΓΔ meeting the two straight lines ΘΑ and ΓΑ and cutting one another at the point K; therefore

$$\Gamma\Lambda : \Lambda\mathrm{A} = (\Gamma\mathrm{K} : \mathrm{K}\Delta)(\Delta\Theta : \Theta\mathrm{A}). \qquad \text{[by (2)}$$

But ΓΛ : ΛΑ = the chord subtended by double of the arc ΓΕ :

the chord subtended by double of the arc ΕΑ

[*i.e.*, sin ΓΕ : sin ΕΑ],

while ΓΚ : ΚΔ = the chord subtended by double of the arc ΓΖ :

the chord subtended by double of the arc ΖΔ [by (3)

[*i.e.*, sin ΓΖ : sin ΖΔ],

πρὸς ΘΑ, οὕτως ἡ ὑπὸ τὴν διπλῆν τῆς ΔΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ· καὶ ὁ λόγος ἄρα ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ συνῆπται ἔκ τε τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ.

Κατὰ τὰ αὐτὰ δὴ καὶ ὥσπερ ἐπὶ τῆς ἐπιπέδου καταγραφῆς τῶν εὐθειῶν δείκνυται, ὅτι καὶ ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ λόγος συνῆπται ἔκ τε τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΔ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΔΖ καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΖΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΕ· ἅπερ προέκειτο δεῖξαι.

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[a] From the Arabic version, it is known that "Menelaus's Theorem" was the first proposition in Book iii. of his *Sphaerica*, and several interesting deductions follow.

and $\Theta\Delta : \Theta A =$ the chord subtended by double of the arc ΔB :

the chord subtended by double of the arc BA [by (4)

[*i.e.*, $\sin \Delta B : \sin BA$],

and therefore the ratio of the chord subtended by double of the arc ΓE to the chord subtended by double of the arc EA is compounded of (*a*) the ratio of the chord subtended by double of the arc ΓZ to the chord subtended by double of the arc ZΔ, and (*b*) the ratio of the chord subtended by double of the arc ΔB to the chord subtended by double of the arc BA,

$$\left[\text{i.e., } \frac{\sin \Gamma E}{\sin EA} = \frac{\sin \Gamma Z}{\sin Z\Delta} \cdot \frac{\sin \Delta B}{\sin BA}\right].$$

Now with the same premises, and as in the case of the straight lines in the plane diagram [by (1)], it is shown that the ratio of the chord subtended by double of the arc ΓA to the chord subtended by double of the arc EA is compounded of (*a*) the ratio of the chord subtended by double of the arc ΓΔ to the chord subtended by double of the arc ΔZ, and (*b*) the ratio of the chord subtended by double of the arc ZB to the chord subtended by double of the chord BE,

$$\left[\text{i.e., } \frac{\sin \Gamma A}{\sin EA} = \frac{\sin \Gamma \Delta}{\sin \Delta Z} \cdot \frac{\sin ZB}{\sin BE}\right];$$

which was set to be proved.[a]

XXII. MENSURATION: HERON OF ALEXANDRIA

XXII. MENSURATION: HERON OF ALEXANDRIA

(a) Definitions

Heron, *Def.*, ed. Heiberg (Heron iv.) 14. 1-24

Καὶ τὰ μὲν πρὸ τῆς γεωμετρικῆς στοιχειώσεως τεχνολογούμενα ὑπογράφων σοι καὶ ὑποτυπούμενος, ὡς ἔχει μάλιστα συντόμως, Διονύσιε λαμπρότατε, τήν τε ἀρχὴν καὶ τὴν ὅλην σύνταξιν ποιήσομαι κατὰ τὴν τοῦ Εὐκλείδου τοῦ Στοιχειωτοῦ τῆς ἐν γεωμετρίᾳ θεωρίας διδασκαλίαν· οἶμαι γὰρ οὕτως οὐ μόνον τὰς ἐκείνου πραγματείας

[a] The problem of Heron's date is one of the most disputed questions in the history of Greek mathematics. The only details certainly known are that he lived after Apollonius, whom he quotes, and before Pappus, who cites him, say between 150 B.C. and A.D. 250. Many scraps of evidence have been thrown into the dispute, including the passage here first cited; for it is argued that the title λαμπρότατος corresponds to the Latin *clarissimus*, which was not in common use in the third century A.D. Both Heiberg (Heron, vol. v. p. ix) and Heath (*H.G.M.* ii. 306) place him, however, in the third century A.D., only a little earlier than Pappus.

The chief works of Heron are now definitively published in five volumes of the Teubner series. Perhaps the best known are his *Pneumatica* and the *Automata*, in which he shows how to use the force of compressed air, water or steam; they are of great interest in the history of physics, and have led some to describe Heron as "the father of the turbine," but

XXII. MENSURATION : HERON OF ALEXANDRIA [a]

(a) Definitions

Heron, *Definitions*, ed. Heiberg (Heron iv.) 14. 1-24

In setting out for you as briefly as possible, O most excellent Dionysius, a sketch of the technical terms premised in the elements of geometry, I shall take as my starting point, and shall base my whole arrangement upon, the teaching of Euclid, the writer of the elements of theoretical geometry ; for in this way I think I shall give you a good general understanding,

as they have no mathematical interest they cannot be noticed here. Heron also wrote a *Belopoeica* on the construction of engines of war, and a *Mechanics*, which has survived in Arabic and in a few fragments of the Greek.

In geometry, Heron's elaborate collection of *Definitions* has survived, but his *Commentary on Euclid's Elements* is known only from extracts preserved by Proclus and an-Nairīzī, the Arabic commentator. In mensuration there are extant the *Metrica*, *Geometrica*, *Stereometrica*, *Geodaesia*, *Mensurae* and *Liber Geëponicus*. The *Metrica*, discovered in a Constantinople MS. in 1896 by R. Schöne and edited by his son H. Schöne, seems to have preserved its original form more closely than the others, and will be relied on here in preference to them. Heron's *Dioptra*, describing an instrument of the nature of a theodolite and its application to surveying, is also extant and will be cited here.

For a full list of Heron's many works, v. Heath, *H.G.M.* ii. 308-310.

εὐσυνόπτους ἔσεσθαί σοι, ἀλλὰ καὶ πλείστας ἄλλας τῶν εἰς γεωμετρίαν ἀνηκόντων. ἄρξομαι τοίνυν ἀπὸ σημείου.

α′. Σημεῖόν ἐστιν, οὗ μέρος οὐθὲν ἢ πέρας ἀδιάστατον ἢ πέρας γραμμῆς, πέφυκε δὲ διανοίᾳ μόνῃ ληπτὸν εἶναι ὡσανεὶ ἀμερές τε καὶ ἀμέγεθες τυγχάνον. τοιοῦτον οὖν αὐτό φασιν εἶναι οἷον ἐν χρόνῳ τὸ ἐνεστὸς καὶ οἷον μονάδα θέσιν ἔχουσαν. ἔστι[1] μὲν οὖν τῇ οὐσίᾳ ταὐτὸν τῇ μονάδι· ἀδιαίρετα γὰρ ἄμφω καὶ ἀσώματα καὶ ἀμέριστα· τῇ δὲ ἐπιφανείᾳ καὶ τῇ σχέσει διαφέρει· ἡ μὲν γὰρ μονὰς ἀρχὴ ἀριθμοῦ, τὸ δὲ σημεῖον τῆς γεωμετρουμένης οὐσίας ἀρχή, ἀρχὴ δὲ κατὰ ἔκθεσιν, οὐχ ὡς μέρος ὂν τῆς γραμμῆς, ὡς τοῦ ἀριθμοῦ μέρος ἡ μονάς, προεπινοούμενον δὲ αὐτῆς· κινηθέντος γὰρ ἢ μᾶλλον νοηθέντος ἐν ῥύσει νοεῖται γραμμή, καὶ οὕτω σημεῖον ἀρχή ἐστι γραμμῆς, ἐπιφάνεια δὲ στερεοῦ σώματος.

Ibid. 60. 22–62. 9

ϛζ′. Σπεῖρα γίνεται, ὅταν κύκλος ἐπὶ κύκλου τὸ κέντρον ἔχων ὀρθὸς ὢν πρὸς τὸ τοῦ κύκλου ἐπίπεδον περιενεχθεὶς εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ· τὸ δὲ αὐτὸ τοῦτο καὶ κρίκος καλεῖται. διεχὴς μὲν οὖν ἐστι σπεῖρα ἡ ἔχουσα διάλειμμα, συνεχὴς δὲ ἡ καθ' ἓν σημεῖον συμπίπτουσα, ἐπαλλάττουσα δέ, καθ' ἣν ὁ περιφερόμενος κύκλος

[1] ἔστι Friedlein, ὅτι codd..

[a] The first definition is that of Euclid i. Def. 1, the third in effect that of Plato, who defined a point as ἀρχὴ γραμμῆς (Aristot. *Metaph.* 992 a 20): the second is reminiscent of Nicomachus, *Arith. Introd.* ii. 7. 1, v. vol. i. pp. 86-89.

not only of Euclid's works, but of many others pertaining to geometry. I shall begin, then, with the point.

1. A *point* is that which has no parts, or an extremity without extension, or the extremity of a line,[a] and, being both without parts and without magnitude, it can be grasped by the understanding only. It is said to have the same character as the moment in time or the unit having position.[b] It is the same as the unit in its fundamental nature, for they are both indivisible and incorporeal and without parts, but in relation to surface and position they differ ; for the unit is the beginning of number, while the point is the beginning of geometrical being—but a beginning by way of setting out only, not as a part of a line, in the way that the unit is a part of number—and is prior to geometrical being in conception ; for when a point moves, or rather is conceived in motion, a line is conceived, and in this way a point is the beginning of a line and a surface is the beginning of a solid body.

Ibid. 60. 22–62. 9

97. A *spire* is generated when a circle revolves and returns to its original position in such a manner that its centre traces a circle, the original circle remaining at right angles to the plane of this circle ; this same curve is also called a *ring*. A spire is *open* when there is a gap, *continuous* when it touches at one point, and *self-crossing* when the revolving circle cuts itself.

[a] The Pythagorean definition of a point : *v.* Proclus, *in Eucl.* i., ed. Friedlein 95. 22. Proclus's whole comment is worth reading, and among modern writers there is a full discussion in Heath, *The Thirteen Books of Euclid's Elements*, vol. i. pp. 155-158.

αὐτὸς αὑτὸν τέμνει. γίνονται δὲ καὶ τούτων τομαὶ γραμμαί τινες ἰδιάζουσαι. οἱ δὲ τετράγωνοι κρίκοι ἐκπρίσματά εἰσι κυλίνδρων· γίνονται δὲ καὶ ἄλλα τινὰ ποικίλα πρίσματα ἔκ τε σφαιρῶν καὶ ἐκ μικτῶν ἐπιφανειῶν.

(*b*) Measurement of Areas and Volumes

(i.) *Area of a Triangle Given the Sides*

Heron, *Metr.* i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

Ἔστι δὲ καθολικὴ μέθοδος ὥστε τριῶν πλευρῶν δοθεισῶν οἱουδηποτοῦν τριγώνου τὸ ἐμβαδὸν εὑρεῖν χωρὶς καθέτου· οἷον ἔστωσαν αἱ τοῦ τριγώνου πλευραὶ μονάδων ζ̅, η̅, θ̅. σύνθες τὰ ζ̅ καὶ τὰ η̅ καὶ τὰ θ̅· γίγνεται κ̅δ̅. τούτων λαβὲ τὸ ἥμισυ· γίγνεται ι̅β̅. ἄφελε τὰς ζ̅ μονάδας· λοιπαὶ ε̅. πάλιν ἄφελε ἀπὸ τῶν ι̅β̅ τὰς η̅· λοιπαὶ δ̅. καὶ ἔτι τὰς θ̅· λοιπαὶ γ̅. ποίησον τὰ ι̅β̅ ἐπὶ τὰ ε̅· γίγνονται ξ̅. ταῦτα ἐπὶ τὸν δ̅· γίγνονται σ̅μ̅· ταῦτα ἐπὶ τὸν γ̅· γίγνεται ψ̅κ̅· τούτων λαβὲ πλευρὰν καὶ ἔσται τὸ ἐμβαδὸν τοῦ τριγώνου. ἐπεὶ οὖν αἱ ψ̅κ̅ ῥητὴν τὴν πλευρὰν οὐκ ἔχουσι, ληψόμεθα μετὰ διαφόρου ἐλαχίστου τὴν πλευρὰν οὕτως· ἐπεὶ ὁ συνεγγίζων τῷ ψ̅κ̅ τετράγωνός ἐστιν ὁ ψ̅κ̅θ̅ καὶ πλευρὰν ἔχει τὸν κ̅ζ̅, μέρισον τὰς ψ̅κ̅ εἰς τὸν κ̅ζ̅· γίγνεται κ̅ϛ̅ καὶ τρίτα δύο· πρόσθες τὰς κ̅ζ̅· γίγνεται ν̅γ̅ τρίτα δύο. τούτων τὸ ἥμισυ· γίγνεται κ̅ϛ̅ ∠γ′. ἔσται ἄρα τοῦ ψ̅κ̅ ἡ πλευρὰ ἔγγιστα τὰ κ̅ϛ̅ ∠γ′. τὰ γὰρ κ̅ϛ̅ ∠γ′ ἐφ᾽ ἑαυτὰ γίγνεται ψ̅κ̅ λϛ′· ὥστε τὸ διάφορον μονάδος ἐστὶ μόριον λϛ′. ἐὰν δὲ βουλώμεθα

Certain special curves are generated by sections of these spires. But the *square rings* are prismatic sections of cylinders ; various other kinds of prismatic sections are formed from spheres and *mixed* surfaces.[a]

(*b*) Measurement of Areas and Volumes

(i.) *Area of a Triangle Given the Sides*

Heron, *Metrica* i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9. Add together 7, 8 and 9 ; the result is 24. Take half of this, which gives 12. Take away 7 ; the remainder is 5. Again, from 12 take away 8 ; the remainder is 4. And again 9 ; the remainder is 3. Multiply 12 by 5 ; the result is 60. Multiply this by 4 ; the result is 240. Multiply this by 3 ; the result is 720. Take the square root of this and it will be the area of the triangle. Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square nearest to 720 is 729, having a root 27, divide 27 into 720 ; the result is $26\frac{2}{3}$; add 27 ; the result is $53\frac{2}{3}$. Take half of this ; the result is $26\frac{1}{2} + \frac{1}{3} (= 26\frac{5}{6})$. Therefore the square root of 720 will be very nearly $26\frac{5}{6}$. For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $\frac{1}{36}$. If we wish to make the difference less than $\frac{1}{36}$,

[a] The passage should be read in conjunction with those from Proclus cited *supra*, pp. 360-365 ; note the slight difference in terminology—*self-crossing* for *interlaced*.

ἐν ἐλάσσονι μορίῳ τοῦ λϛʹ τὴν διαφορὰν γίγνεσθαι, ἀντὶ τοῦ $\overline{\psi\kappa\theta}$ τάξομεν τὰ νῦν εὑρεθέντα $\overline{\psi\kappa}$ καὶ λϛʹ, καὶ ταὐτὰ ποιήσαντες εὑρήσομεν πολλῷ ἐλάττονα ⟨τοῦ⟩[1] λϛʹ τὴν διαφορὰν γιγνομένην.

Ἡ δὲ γεωμετρικὴ τούτου ἀπόδειξίς ἐστιν ἥδε· τριγώνου δοθεισῶν τῶν πλευρῶν εὑρεῖν τὸ ἐμβαδόν. δυνατὸν μὲν οὖν ἐστιν ἀγαγόντα[ς][2] μίαν κάθετον καὶ πορισάμενον αὐτῆς τὸ μέγεθος εὑρεῖν τοῦ τριγώνου τὸ ἐμβαδόν, δέον δὲ ἔστω χωρὶς τῆς καθέτου τὸ ἐμβαδὸν πορίσασθαι.

Ἔστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ καὶ ἔστω ἑκάστη τῶν ΑΒ, ΒΓ, ΓΑ δοθεῖσα· εὑρεῖν τὸ ἐμβα-

[1] τοῦ add. Heiberg.
[2] ἀγαγόντα[ς] corr. H. Schöne.

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[a] If a non-square number A is equal to $a^2 \pm b$, Heron's method gives as a first approximation to $\sqrt{A}$,

$$a_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right),$$

and as a second approximation,

$$a_2 = \tfrac{1}{2}\left(a_1 + \frac{A}{a_1}\right).$$

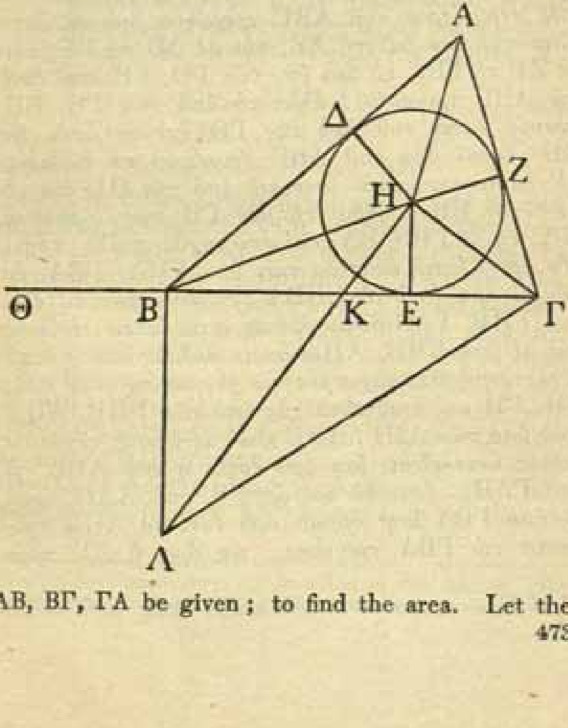
An equivalent formula is used by Rhabdas (*v.* vol. i. p. 30 n. *b*) and by a fourteenth century Calabrian monk Barlaam, who wrote in Greek and who indicated that the process could be continued indefinitely. Several modern writers have used the formula to account for Archimedes' approximations to $\sqrt{3}$ (*v.* vol. i. p. 322 n. *a*).

[b] Heron had previously shown how to do this.

instead of 729 we shall take the number now found, $720\frac{1}{36}$, and by the same method we shall find an approximation differing by much less than $\frac{1}{36}$.[a]

The geometrical proof of this is as follows: *In a triangle whose sides are given to find the area.* Now it is possible to find the area of the triangle by drawing one perpendicular and calculating its magnitude,[b] but let it be required to calculate the area without the perpendicular.

Let ΑΒΓ be the given triangle, and let each of



ΑΒ, ΒΓ, ΓΑ be given; to find the area. Let the

δόν. ἐγγεγράφθω εἰς τὸ τρίγωνον κύκλος ὁ ΔΕΖ, οὗ κέντρον ἔστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΒΗ, ΓΗ, ΔΗ, ΕΗ, ΖΗ. τὸ μὲν ἄρα ὑπὸ ΒΓ, ΕΗ διπλάσιόν ἐστι τοῦ ΒΗΓ τριγώνου, τὸ δὲ ὑπὸ ΓΑ, ΖΗ τοῦ ΑΓΗ τριγώνου, ⟨τὸ δὲ ὑπὸ ΑΒ, ΔΗ τοῦ ΑΒΗ τριγώνου⟩[1]· τὸ ἄρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου καὶ τῆς ΕΗ, τουτέστι τῆς ἐκ τοῦ κέντρου τοῦ ΔΕΖ κύκλου, διπλάσιόν ἐστι τοῦ ΑΒΓ τριγώνου. ἐκβεβλήσθω ἡ ΓΒ, καὶ τῇ ΑΔ ἴση κείσθω ἡ ΒΘ· ἡ ἄρα ΓΒΘ ἡμίσειά ἐστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου διὰ τὸ ἴσην εἶναι τὴν μὲν ΑΔ τῇ ΑΖ, τὴν δὲ ΔΒ τῇ ΒΕ, τὴν δὲ ΖΓ τῇ ΓΕ. τὸ ἄρα ὑπὸ τῶν ΓΘ, ΕΗ ἴσον ἐστὶ τῷ ΑΒΓ τριγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν ΓΘ, ΕΗ πλευρά ἐστιν τοῦ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ· ἔσται ἄρα τοῦ ΑΒΓ τριγώνου τὸ ἐμβαδὸν ἐφ' ἑαυτὸ γενόμενον ἴσον τῷ ἀπὸ τῆς ΘΓ ἐπὶ τὸ ἀπὸ τῆς ΕΗ. ἤχθω τῇ μὲν ΓΗ πρὸς ὀρθὰς ἡ ΗΛ, τῇ δὲ ΓΒ ἡ ΒΛ, καὶ ἐπεζεύχθω ἡ ΓΛ. ἐπεὶ οὖν ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΓΗΛ, ΓΒΛ, ἐν κύκλῳ ἄρα ἐστὶ τὸ ΓΗΒΛ τετράπλευρον· αἱ ἄρα ὑπὸ ΓΗΒ, ΓΛΒ δυσὶν ὀρθαῖς εἰσιν ἴσαι. εἰσὶν δὲ καὶ αἱ ὑπὸ ΓΗΒ, ΑΗΔ δυσὶν ὀρθαῖς ἴσαι διὰ τὸ δίχα τετμῆσθαι τὰς πρὸς τῷ Η γωνίας ταῖς ΑΗ, ΒΗ, ΓΗ καὶ ἴσας εἶναι τὰς ὑπὸ τῶν ΓΗΒ, ΑΗΔ ταῖς ὑπὸ τῶν ΑΗΓ, ΔΗΒ καὶ τὰς πάσας τέτρασιν ὀρθαῖς ἴσας εἶναι· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΗΔ τῇ ὑπὸ ΓΛΒ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΔΗ ὀρθῇ τῇ ὑπὸ ΓΒΛ ἴση· ὅμοιον ἄρα ἐστὶ τὸ ΑΗΔ τρίγωνον τῷ ΓΒΛ τριγώνῳ. ὡς ἄρα ἡ ΒΓ πρὸς

[1] τὸ δὲ . . . τριγώνου: these words, along with several

circle ΔEZ be inscribed in the triangle with centre H [Eucl. iv. 4], and let AH, BH, ΓH, ΔH, EH, ZH be joined. Then

$$\text{BΓ} \,.\, \text{EH} = 2 \,.\, \text{triangle BHΓ}, \quad \text{[Eucl. i. 41}$$

$$\text{ΓA} \,.\, \text{ZH} = 2 \,.\, \text{triangle AHΓ}, \quad \text{[\textit{ibid.}}$$

$$\text{AB} \,.\, \text{ΔH} = 2 \,.\, \text{triangle ABH}. \quad \text{[\textit{ibid.}}$$

Therefore the rectangle contained by the perimeter of the triangle ABΓ and EH, that is the radius of the circle ΔEZ, is double of the triangle ABΓ. Let ΓB be produced and let BΘ be placed equal to AΔ; then ΓBΘ is half of the perimeter of the triangle ABΓ because AΔ = AZ, ΔB = BE, ZΓ = ΓE [by Eucl. iii. 17]. Therefore

$$\text{ΓΘ} \,.\, \text{EH} = \text{triangle ABΓ}. \quad \text{[\textit{ibid.}}$$

But $\quad \text{ΓΘ} \,.\, \text{EH} = \sqrt{\text{ΓΘ}^2 \,.\, \text{EH}^2}$;

therefore $\quad (\text{triangle ABΓ})^2 = \text{ΘΓ}^2 \,.\, \text{EH}^2$.

Let HΛ be drawn perpendicular to ΓH and BΛ perpendicular to ΓB, and let ΓΛ be joined. Then since each of the angles ΓHΛ, ΓBΛ is right, a circle can be described about the quadrilateral ΓHBΛ [by Eucl. iii. 31]; therefore the angles ΓHB, ΓΛB are together equal to two right angles [Eucl. iii. 22]. But the angles ΓHB, AHΔ are together equal to two right angles because the angles at H are bisected by AH, BH, ΓH and the angles ΓHB, AHΔ together with AHΓ, ΔHB are equal to four right angles; therefore the angle AHΔ is equal to the angle ΓΛB. But the right angle AΔH is equal to the right angle ΓBΛ; therefore the triangle AHΔ is similar to the triangle ΓBΛ.

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other obvious corrections not specified in this edition, were rightly added to the text by a fifteenth-century scribe.

ΒΛ, ἡ ΑΔ πρὸς ΔΗ, τουτέστιν ἡ ΒΘ πρὸς ΕΗ, καὶ ἐναλλάξ, ὡς ἡ ΓΒ πρὸς ΒΘ, ἡ ΒΛ πρὸς ΕΗ, τουτέστιν ἡ ΒΚ πρὸς ΚΕ διὰ τὸ παράλληλον εἶναι τὴν ΒΛ τῇ ΕΗ, καὶ συνθέντι, ὡς ἡ ΓΘ πρὸς ΒΘ, οὕτως ἡ ΒΕ πρὸς ΕΚ· ὥστε καὶ ὡς τὸ ἀπὸ τῆς ΓΘ πρὸς τὸ ὑπὸ τῶν ΓΘ, ΘΒ, οὕτως τὸ ὑπὸ ΒΕΓ πρὸς τὸ ὑπὸ ΓΕΚ, τουτέστι πρὸς τὸ ἀπὸ ΕΗ· ἐν ὀρθογωνίῳ γὰρ ἀπὸ τῆς ὀρθῆς ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΕΗ· ὥστε τὸ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ, οὗ πλευρὰ ἦν τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου, ἴσον ἔσται τῷ ὑπὸ ΓΘΒ ἐπὶ τὸ ὑπὸ ΓΕΒ. καὶ ἔστι δοθεῖσα ἑκάστη τῶν ΓΘ, ΘΒ, ΒΕ, ΓΕ· ἡ μὲν γὰρ ΓΘ ἡμίσειά ἐστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου, ἡ δὲ ΒΘ ἡ ὑπεροχή, ᾗ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΓΒ, ἡ δὲ ΒΕ ἡ ὑπεροχή, ᾗ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΓ, ἡ δὲ ΕΓ ἡ ὑπεροχή, ᾗ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΒ, ἐπειδήπερ ἴση ἐστὶν ἡ μὲν ΕΓ τῇ ΓΖ, ἡ δὲ ΒΘ τῇ ΑΖ, ἐπεὶ καὶ τῇ ΑΔ ἐστιν ἴση. δοθὲν ἄρα καὶ τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου.

(ii.) *Volume of a Spire*

Ibid. ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

Ἔστω γάρ τις ἐν ἐπιπέδῳ εὐθεῖα ἡ ΑΒ καὶ δύο τυχόντα ἐπ' αὐτῆς σημεῖα. εἰλήφθω ὁ ΒΓΔΕ ⟨κύκλος⟩[1] ὀρθὸς ὢν πρὸς τὸ ὑποκείμενον ἐπίπεδον, ἐν ᾧ ἐστιν ἡ ΑΒ εὐθεῖα, καὶ μένοντος τοῦ Α

[1] κύκλος add. H. Schöne.

Therefore $\quad B\Gamma : B\Lambda = A\Delta : \Delta H$

$= B\Theta : EH,$

and *permutando*, $\quad \Gamma B : B\Theta = B\Lambda : EH$

$= BK : KE,$

because $B\Lambda$ is parallel to EH,

and *componendo* $\quad \Gamma\Theta : B\Theta = BE : EK$;

therefore $\quad \Gamma\Theta^2 : \Gamma\Theta \,.\, \Theta B = BE \,.\, E\Gamma : \Gamma E \,.\, EK,$

i.e. $\quad = BE \,.\, E\Gamma : EH^2,$

for in a right-angled triangle EH has been drawn from the right angle perpendicular to the base; therefore $\Gamma\Theta^2 \,.\, EH^2$, whose square root is the area of the triangle $AB\Gamma$, is equal to $(\Gamma\Theta \,.\, \Theta B)(\Gamma E \,.\, EB)$. And each of $\Gamma\Theta$, ΘB, BE, ΓE is given; for $\Gamma\Theta$ is half of the perimeter of the triangle $AB\Gamma$, while $B\Theta$ is the excess of half the perimeter over ΓB, BE is the excess of half the perimeter over $A\Gamma$, and $E\Gamma$ is the excess of half the perimeter over AB, inasmuch as $E\Gamma = \Gamma Z$, $B\Theta = A\Delta = AZ$. Therefore the area of the triangle $AB\Gamma$ is given.[a]

(ii.) *Volume of a Spire*

Ibid. ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

Let AB be any straight line in a plane and A, B any two points taken on it. Let the circle $B\Gamma\Delta E$ be taken perpendicular to the plane of the horizontal, in which lies the straight line AB, and, while the point

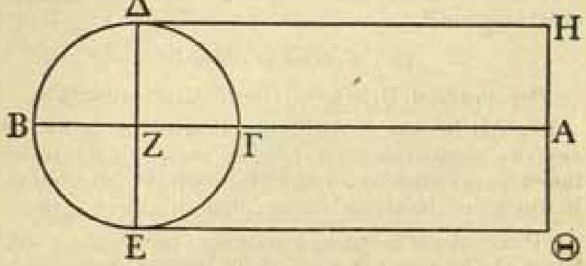
[a] If the sides of the triangle are a, b, c, and $s = \frac{1}{2}(a+b+c)$, Heron's formula may be stated in the familiar terms,

$$\text{area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Heron also proves the formula in his *Dioptra* 30, but it is now known from Arabian sources to have been discovered by Archimedes.

σημείου περιφερέσθω κατὰ τὸ ἐπίπεδον ἡ ΑΒ, ἄχρι οὗ εἰς τὸ αὐτὸ ἀποκατασταθῇ συμπεριφερομένου καὶ τοῦ ΒΓΔΕ κύκλου ὀρθοῦ διαμένοντος πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπογεννήσει ἄρα τινὰ ἐπιφάνειαν ἡ ΒΓΔΕ περιφέρεια, ἣν δὴ σπειρικὴν καλοῦσιν· κἂν μὴ ᾖ δὲ ὅλος ὁ κύκλος, ἀλλὰ τμῆμα αὐτοῦ, πάλιν ἀπογεννήσει τὸ τοῦ κύκλου τμῆμα σπειρικῆς ἐπιφανείας τμῆμα, καθάπερ εἰσὶ καὶ αἱ ταῖς κίοσιν ὑποκείμεναι σπεῖραι· τριῶν γὰρ οὐσῶν ἐπιφανειῶν ἐν τῷ καλουμένῳ ἀναγραφεῖ, ὃν δή τινες καὶ ἐμβολέα καλοῦσιν, δύο μὲν κοίλων τῶν ἄκρων, μιᾶς δὲ μέσης καὶ κυρτῆς, ἅμα περιφερόμεναι αἱ τρεῖς ἀπογεννῶσι τὸ εἶδος τῆς τοῖς κίοσιν ὑποκειμένης σπείρας.

Δέον οὖν ἔστω τὴν ἀπογεννηθεῖσαν σπεῖραν ὑπὸ τοῦ ΒΓΔΕ κύκλου μετρῆσαι. δεδόσθω ἡ μὲν ΑΒ μονάδων κ̄, ἡ δὲ ΒΓ διάμετρος μονάδων ιβ̄.

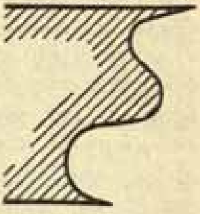


εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Ζ, καὶ ἀπὸ τῶν Α, Ζ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ ΔΖΕ, ΗΑΘ. καὶ διὰ τῶν Δ, Ε τῇ ΑΒ παράλ-

A remains stationary, let AB revolve in the plane until it concludes its motion at the place where it started, the circle ΒΓΔΕ remaining throughout perpendicular to the plane of the horizontal. Then the circumference ΒΓΔΕ will generate a certain surface, which is called *spiric* ; and if the whole circle do not revolve, but only a segment of it, the segment of the circle will again generate a segment of a spiric surface, such as are the *spirae* on which columns rest ; for as there are three surfaces in the so-called *anagrapheus*, which some call also *emboleus*, two concave (the extremes) and one (the middle) convex, when the three are moved round simultaneously they generate the form of the *spira* on which columns rest.[a]

Let it then be required to measure the spire generated by the circle ΒΓΔΕ. Let AB be given as 20, and the diameter ΒΓ as 12. Let Z be the centre of the circle, and through[b] A, Z let ΗΑΘ, ΔΖΕ be drawn perpendicular to the plane of the horizontal. And through Δ, E let ΔΗ, ΕΘ be drawn parallel to

[a] The ἀναγραφεύς or ἐμβολεύς is the pattern or templet for applying to an architectural feature, in this case an Attic-Ionic column-base. The Attic-Ionic base consists essentially of two convex mouldings, separated by a concave one. In practice, there are always narrow vertical ribbons between the convex mouldings and the concave one, but Heron ignores them. In the *templet*, there are naturally two concave surfaces separated by a convex, and the kind of figure Heron had in mind appears to be that here illustrated. I am indebted to Mr. D. S. Robertson, Regius Professor of Greek in the University of Cambridge, for help in elucidating this passage.



[b] Lit. "from."

ληλοι ἤχθωσαν αἱ ΔΗ, ΕΘ. δέδεικται δὲ Διονυσοδώρῳ ἐν τῷ Περὶ τῆς σπείρας ἐπιγραφομένῳ, ὅτι ὃν λόγον ἔχει ὁ ΒΓΔΕ κύκλος πρὸς τὸ ἥμισυ τοῦ ΔΕΗΘ παραλληλογράμμου, τοῦτον ἔχει καὶ ἡ γεννηθεῖσα σπεῖρα ὑπὸ τοῦ ΒΓΔΕ κύκλου πρὸς τὸν κύλινδρον, οὗ ἄξων μέν ἐστιν ὁ ΗΘ, ἡ δὲ ἐκ τοῦ κέντρου τῆς βάσεως ἡ ΕΘ. ἐπεὶ οὖν ἡ ΒΓ μονάδων ιβ̅ ἐστίν, ἡ ἄρα ΖΓ ἔσται μονάδων ϛ̅. ἔστι δὲ καὶ ἡ ΑΓ μονάδων η̅· ἔσται ἄρα ἡ ΑΖ μονάδων ιδ̅, τουτέστιν ἡ ΕΘ, ἥτις ἐστὶν ἐκ τοῦ κέντρου τῆς βάσεως τοῦ εἰρημένου κυλίνδρου· δοθεὶς ἄρα ἐστὶν ὁ κύκλος· ἀλλὰ καὶ ὁ ἄξων δοθείς· ἔστιν γὰρ μονάδων ιβ̅, ἐπεὶ καὶ ἡ ΔΕ. ὥστε δοθεὶς καὶ ὁ εἰρημένος κύλινδρος· καὶ ἔστι τὸ ΔΘ παραλληλόγραμμον ⟨δοθέν⟩[1]· ὥστε καὶ τὸ ἥμισυ αὐτοῦ. ἀλλὰ καὶ ὁ ΒΓΔΕ κύκλος· δοθεῖσα γὰρ ἡ ΓΒ διάμετρος. λόγος ἄρα τοῦ ΒΓΔΕ κύκλου πρὸς τὸ ΔΘ παραλληλόγραμμον δοθείς· ὥστε καὶ τῆς σπείρας πρὸς τὸν κύλινδρον λόγος ἔστι δοθείς. καὶ ἔστι δοθεὶς ὁ κύλινδρος· δοθὲν ἄρα καὶ τὸ στερεὸν τῆς σπείρας.

Συντεθήσεται δὴ ἀκολούθως τῇ ἀναλύσει οὕτως. ἄφελε ἀπὸ τῶν κ̅ τὰ ιβ̅· λοιπὰ η̅. καὶ πρόσθες τὰ κ̅· γίγνεται κη̅· καὶ μέτρησον κύλινδρον, οὗ ἡ μὲν διάμετρος τῆς βάσεώς ἐστι μονάδων κη̅, τὸ δὲ ὕψος ιβ̅· καὶ γίγνεται τὸ στερεὸν αὐτοῦ ͵ζτϙβ̅. καὶ μέτρησον κύκλον, οὗ διάμετρός ἐστι μονάδων ιβ̅· γίγνεται τὸ ἐμβαδὸν αὐτοῦ, καθὼς ἐμάθομεν, ρια̅ γ̅ ζʹ· καὶ λαβὲ τῶν κη̅ τὸ ἥμισυ· γίγνεται ιδ̅. ἐπὶ τὸ ἥμισυ τῶν ιβ̅· γίγνεται πδ̅· καὶ πολλα-

[1] δοθέν add. H. Schöne.

AB. Now it is proved by Dionysodorus[a] in the book which he wrote *On the Spire* that the circle ΒΓΔΕ bears to half of the parallelogram ΔΕΗΘ the same ratio as the spire generated by the circle ΒΓΔΕ bears to the cylinder having ΗΘ for its axis and ΕΘ for the radius of its base. Now, since ΒΓ is 12, ΖΓ will be 6. But ΑΓ is 8 ; therefore ΑΖ will be 14, and likewise ΕΘ, which is the radius of the base of the aforesaid cylinder. Therefore the circle is given ; but the axis is also given ; for it is 12, since this is the length of ΔΕ. Therefore the aforesaid cylinder is also given ; and the parallelogram ΔΘ is given, so that its half is also given. But the circle ΒΓΔΕ is also given ; for the diameter ΓΒ is given. Therefore the ratio of the circle ΒΓΔΕ to the parallelogram is given ; and so the ratio of the spire to the cylinder is given. And the cylinder is given ; therefore the volume of the spire is also given.

Following the analysis, the synthesis may thus be done. Take 12 from 20 ; the remainder is 8. And add 20 ; the result is 28. Let the measure be taken of the cylinder having for the diameter of its base 28 and for height 12 ; the resulting volume is 7392. Now let the area be found of a circle having a diameter 12 ; the resulting area, as we learnt, is 113⅐. Take the half of 28 ; the result is 14. Multiply it by the half of 12 ; the result is 84. Now multiply

[a] For Dionysodorus *v. supra*, p. 162 n. *a* and p. 364 n. *a*. If $\Delta E = H\Theta = 2r$ and $E\Theta = a$, then the volume of the spire bears to the volume of the cylinder the ratio $2\pi a \,.\, \pi r^2 : 2r \,.\, \pi a^2$ or $\pi r : a$, which, as Dionysodorus points out, is identical with the ratio of the circle to half the parallelogram, that is, $\pi r^2 : ra$ or $\pi r : a$.

πλασιάσας τὰ $\overline{,ζτϛβ}$ ἐπὶ τὰ $\overline{ριγ}$ ζ′· καὶ τὰ γενόμενα παράβαλε παρὰ τὸν $\overline{πδ}$· γίγνεται $\overline{,θλνϛ}\ \frac{ζ}{δ}$. τοσούτου ἔσται τὸ στερεὸν τῆς σπείρας.

(iii.) *Division of a Circle*

Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 13–174. 2

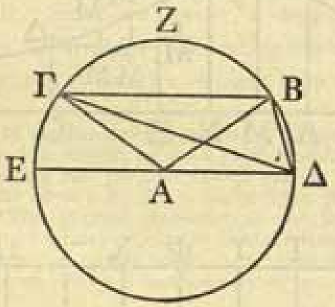
Τὸν δοθέντα κύκλον διελεῖν εἰς τρία ἴσα δυσὶν εὐθείαις. τὸ μὲν οὖν πρόβλημα ὅτι οὐ ῥητόν ἐστι, δῆλον, τῆς εὐχρηστίας δὲ ἕνεκεν διελοῦμεν αὐτὸν ὡς ἔγγιστα οὕτω. ἔστω ὁ δοθεὶς κύκλος, οὗ κέντρον τὸ Α, καὶ ἐνηρμόσθω εἰς αὐτὸν τρίγωνον ἰσόπλευρον, οὗ πλευρὰ ἡ ΒΓ, καὶ παράλληλος αὐτῇ ἤχθω ἡ ΔΑΕ καὶ ἐπεζεύχθωσαν αἱ ΒΔ, ΔΓ. λέγω, ὅτι τὸ ΔΒΓ τμῆμα τρίτον ἔγγιστά ἐστι μέρος τοῦ ὅλου κύκλου. ἐπεζεύχθωσαν γὰρ αἱ ΒΑ, ΑΓ. ὁ ἄρα ΑΒΓΖΒ τομεὺς τρίτον ἐστὶ μέρος τοῦ ὅλου κύκλου. καὶ ἔστιν ἴσον τὸ ΑΒΓ τρίγωνον τῷ ΒΓΔ τριγώνῳ· τὸ ἄρα ΒΔΓΖ σχῆμα τρίτον μέρος ἐστὶ τοῦ ὅλου κύκλου, ᾧ δὴ μεῖζόν ἐστιν αὐτοῦ τὸ ΔΒΓ τμῆμα ἀνεπαισθήτου ὄντος ὡς πρὸς τὸν ὅλον κύκλον. ὁμοίως δὲ καὶ ἑτέραν

7392 by $113\frac{1}{7}$ and divide the product by 84; the result is $9956\frac{1}{7}$. This will be the volume of the spire.

(iii.) *Division of a Circle*

Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 13–174. 2

To divide a given circle into three equal parts by two straight lines. It is clear that this problem is not rational, and for practical convenience we shall make the division as closely as possible in this way. Let the given circle have A for its centre, and let there be inserted in it an equilateral triangle with side BΓ, and let ΔAE be drawn parallel to it, and let BΔ, ΔΓ



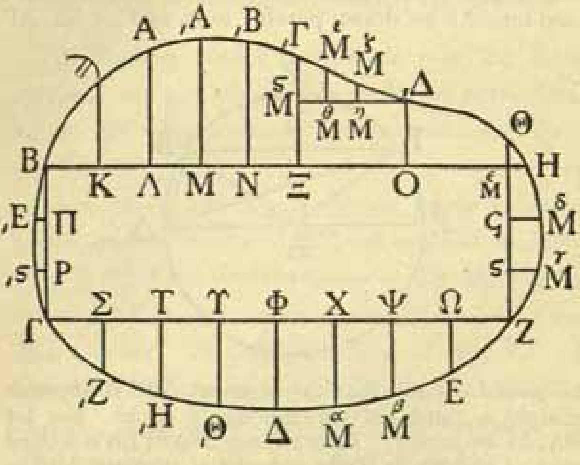
be joined. I say that the segment ΔBΓ is approximately a third part of the whole circle. For let BA, AΓ be joined. Then the sector ABΓZB is a third part of the whole circle. And the triangle ABΓ is equal to the triangle BΓΔ [Eucl. i. 37]; therefore the figure BΔΓZ is a third part of the whole circle, and the excess of the segment ΔBΓ over it is negligible in comparison with the whole circle. Similarly, if we

πλευρὰν ἰσοπλεύρου τριγώνου ἐγγράψαντες ἀφελοῦμεν ἕτερον τρίτον μέρος· ὥστε καὶ τὸ καταλειπόμενον τρίτον μέρος ἔσται [μέρος]¹ τοῦ ὅλου κύκλου.

(iv.) *Measurement of an Irregular Area*

Heron, *Diopt.* 23, ed. H. Schöne (Heron iii.) 260. 18–264. 15

Τὸ δοθὲν χωρίον μετρῆσαι διὰ διόπτρας. ἔστω τὸ δοθὲν χωρίον περιεχόμενον ὑπὸ γραμμῆς



ἀτάκτου τῆς ΑΒΓΔΕΖΗΘ. ἐπεὶ οὖν ἐμάθομεν διὰ τῆς κατασκευασθείσης διόπτρας διάγειν πάσῃ τῇ δοθείσῃ εὐθείᾳ ⟨ἑτέραν⟩² πρὸς ὀρθάς, ἔλαβόν τι σημεῖον ἐπὶ τῆς περιεχούσης τὸ χωρίον γραμμῆς

inscribe another side of the equilateral triangle, we may take away another third part; and therefore the remainder will also be a third part of the whole circle.[a]

(iv.) *Measurement of an Irregular Area*

Heron, *Dioptra*[b] 23, ed. H. Schöne (Heron iii.) 260. 18–264. 15

To measure a given area by means of the dioptra. Let the given area be bounded by the irregular line ΑΒΓΔΕΖΗΘ. Since we learnt to draw, by setting the dioptra, a straight line perpendicular to any other straight line, I took any point B on the line en-

[a] Euclid, in his book *On Divisions of Figures* which has partly survived in Arabic, solved a similar problem—*to draw in a given circle two parallel chords cutting off a certain fraction of the circle*; Euclid actually takes the fraction as one-third. The general character of the third book of Heron's *Metrics* is very similar to Euclid's treatise.

It is in the course of this book (iii. 20) that Heron extracts the cube root of 100 by a method already noted (vol. i. pp. 60-63).

[b] The dioptra was an instrument fulfilling the same purposes as the modern theodolite. An elaborate description of the instrument prefaces Heron's treatise on the subject, and it was obviously a fine piece of craftsmanship, much superior to the "parallactic" instrument with which Ptolemy had to work—another piece of evidence against an early date for Heron.

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[1] *μέρος* om. H. Schöne.
[2] *ἑτέραν* add. H. Schöne.

τὸ Β, καὶ ἤγαγον εὐθεῖαν τυχοῦσαν διὰ τῆς διόπτρας τὴν ΒΗ, καὶ ταύτῃ πρὸς ὀρθὰς τὴν ΒΓ, ⟨καὶ ταύτῃ⟩[1] ἑτέραν πρὸς ὀρθὰς τὴν ΓΖ, καὶ ὁμοίως τῇ ΓΖ πρὸς ὀρθὰς τὴν ΖΘ. καὶ ἔλαβον ἐπὶ τῶν ἀχθεισῶν εὐθειῶν συνεχῆ σημεῖα, ἐπὶ μὲν τῆς ΒΗ τὰ Κ, Λ, Μ, Ν, Ξ, Ο· ἐπὶ δὲ τῆς ΒΓ τὰ Π, Ρ· ἐπὶ δὲ τῆς ΓΖ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω· ἐπὶ δὲ τῆς ΖΘ τὰ ϛ, ϙ. καὶ ἀπὸ τῶν ληφθέντων σημείων ταῖς εὐθείαις, ἐφ' ὧν ἐστι τὰ σημεῖα, πρὸς ὀρθὰς ἤγαγον τὰς Κϡ, ΛΑ, Μ͵Α, Ν͵Β, Ξ͵Γ, Ο͵Δ, Π͵Ε, Ρ͵ϛ, Σ͵Ζ, Τ͵Η, Υ͵Θ, ΦΔ, Χ$\overset{\alpha}{\mathrm{M}}$, Ψ$\overset{\beta}{\mathrm{M}}$, ΩΕ, ϛ$\overset{\gamma}{\mathrm{M}}$, ϙ$\overset{\varepsilon}{\mathrm{M}}$ οὕτως ὥστε [τὰς ἐπὶ][2] τὰ πέρατα τῶν ἀχθεισῶν πρὸς ὀρθὰς [ἐπιζευγνυμένας][3] ἀπολαμβάνειν γραμμὰς ἀπὸ τῆς περιεχούσης τὸ χωρίον γραμμῆς σύνεγγυς εὐθείας· καὶ τούτων γενηθέντων ἔσται δυνατὸν τὸ χωρίον μετρεῖν. τὸ μὲν γὰρ ΒΓΖ$\overset{\alpha}{\mathrm{M}}$ παραλληλόγραμμον ὀρθογώνιόν ἐστιν· ἔπειτα τὰς πλευρὰς ἁλύσει ἢ σχοινίῳ βεβασανισμένῳ, τουτέστιν μήτ' ἐκτείνεσθαι μήτε συστέλλεσθαι δυναμένῳ, μετρήσαντες ἕξομεν τὸ ἐμβαδὸν τοῦ παραλληλογράμμου. τὰ δ' ἐκτὸς τούτου τρίγωνα ὀρθογώνια καὶ τραπέζια ὁμοίως μετρήσομεν, ἔχοντες τὰς πλευρὰς αὐτῶν· ἔσται γὰρ τρίγωνα μὲν ὀρθογώνια τὰ ΒΚϡ, ΒΠ͵Ε, ΓΡ͵ϛ, ΓΣ͵Ζ, ΖΩΕ, Ζϛ$\overset{\gamma}{\mathrm{M}}$, ΘΗ$\overset{\alpha}{\mathrm{M}}$· τὰ δὲ λοιπὰ τραπέζια ὀρθογώνια. τὰ μὲν οὖν τρίγωνα μετρεῖται τῶν περὶ τὴν ὀρθὴν γωνίαν πολλαπλασιαζομένων ἐπ' ἄλληλα· καὶ τοῦ γενομένου τὸ ἥμισυ. τὰ δὲ τραπέζια· συναμφοτέρων τῶν παραλλήλων τὸ ἥμισυ ἐπὶ τὴν ἐπ' αὐτὰς κάθετον οὖσαν, οἷον

closing the area, and by means of the dioptra drew any straight line BH, and drew BΓ perpendicular to it, and drew another straight line ΓZ perpendicular to this last, and similarly drew ZΘ perpendicular to ΓZ. And on the straight lines so drawn I took a series of points—on BH taking K, Λ, M, N, Ξ, O, on BΓ taking Π, P, on ΓZ taking Σ, T, Y, Φ, X, Ψ, Ω, and on ZΘ taking ϛ, ϙ. And from the points so taken on the straight lines designated by the letters, I drew the perpendiculars Kϡ, ΛA, M͵A, N͵B, Ξ͵Γ, O͵Δ, Π͵E, P͵ϛ, Σ͵Z, T͵H, Y͵Θ, ΦΔ, X$\overset{\alpha}{\mathrm{M}}$, Ψ$\overset{\beta}{\mathrm{M}}$, ΩE, ϛ$\overset{\gamma}{\mathrm{M}}$, ϙ$\overset{\delta}{\mathrm{M}}$ in such a manner that the extremities of the perpendiculars cut off from the line enclosing the area approximately straight lines. When this is done it will be possible to measure the area. For the parallelogram BΓZṀ is right-angled ; so that if we measure the sides by a chain or measuring-rod, which has been carefully tested so that it can neither expand nor contract, we shall obtain the area of the parallelogram. We may similarly measure the right-angled triangles and trapezia outside this by taking their sides ; for BKϡ, BΠ͵E, ΓPϛ, ΓΣ͵Z, ZΩE, Zϛ$\overset{\gamma}{\mathrm{M}}$, ΘHṀ are right-angled triangles, and the remaining figures are right-angled trapezia. The triangles are measured by multiplying together the sides about the right angle and taking half the product. As for the trapezia—take half of the sum of the two parallel sides and multiply it by the perpendicular upon

[1] *καὶ ταύτῃ* add. H. Schöne.
[2] *τὰς ἐπὶ* om. H. Schöne.
[3] *ἐπιζευγνυμένας* om. H. Schöne.

τῶν ΚΛ, ΑΛ τὸ ἥμισυ ἐπὶ τὴν ΚΛ· καὶ τῶν λοιπῶν δὲ ὁμοίως. ἔσται ἄρα μεμετρημένον ὅλον τὸ χωρίον διά τε τοῦ μέσου παραλληλογράμμου καὶ τῶν ἐκτὸς αὐτοῦ τριγώνων καὶ τραπεζίων. ἐὰν δὲ τύχῃ ποτὲ μεταξὺ αὐτῶν τῶν ἀχθεισῶν πρὸς ὀρθὰς ταῖς τοῦ παραλληλογράμμου πλευραῖς καμπύλη γραμμὴ μὴ συνεγγίζουσα εὐθείᾳ (οἷον μεταξὺ τῶν Ξ,Γ, Ο,Δ γραμμὴ ἡ ,Γ,Δ), ἀλλὰ περιφερεῖ, μετρήσομεν οὕτως· ἀγαγόντες ⟨τῇ⟩[1] Ο,Δ πρὸς ὀρθὰς τὴν ,Δ$\overset{\varsigma}{\mathrm{M}}$, καὶ ἐπ' αὐτῆς λαβόντες σημεῖα συνεχῆ τὰ $\overset{\theta}{\mathrm{M}}$, $\overset{\eta}{\mathrm{M}}$, καὶ ἀπ' αὐτῶν πρὸς ὀρθὰς ἀγαγόντες τῇ $\overset{\varsigma}{\mathrm{M}}$,Δ τὰς $\overset{\theta}{\mathrm{M}}\overset{\iota}{\mathrm{M}}$, $\overset{\eta}{\mathrm{M}}\overset{\zeta}{\mathrm{M}}$, ὥστε τὰς μεταξὺ τῶν ἀχθεισῶν σύνεγγυς εὐθείας εἶναι, πάλιν μετρήσομεν τό τε $\overset{\varsigma}{\mathrm{M}}$ΞΟ,Δ παραλληλόγραμμον καὶ τὸ $\overset{\eta}{\mathrm{M}}\overset{\zeta}{\mathrm{M}}$,Δ τρίγωνον, καὶ τὸ ,Γ$\overset{\varsigma}{\mathrm{M}}\overset{\theta}{\mathrm{M}}\overset{\iota}{\mathrm{M}}$ τραπέζιον, καὶ ἔτι τὸ ἕτερον τραπέζιον, καὶ ἕξομεν τὸ περιεχόμενον χωρίον ὑπό τε τῆς ,Γ$\overset{\iota}{\mathrm{M}}\overset{\zeta}{\mathrm{M}}$,Δ γραμμῆς καὶ τῶν ,ΓΞ, ⟨ΞΟ,⟩[2] Ο,Δ εὐθειῶν μεμετρημένον.

(c) Mechanics

Heron, *Diopt.* 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

Τῇ δοθείσῃ δυνάμει τὸ δοθὲν βάρος κινῆσαι

[1] τῇ add. H. Schöne. [2] ΞΟ add. H. Schöne.

[a] Heron's *Mechanics* in three books has survived in Arabic, but has obviously undergone changes in form. It begins with the problem of arranging toothed wheels so as to move

them, as, for example, half of Κϡ, ΑΛ by ΚΛ ; and similarly for the remainder. Then the whole area will have been measured by means of the parallelogram in the middle and the triangles and trapezia outside it. If perchance the curved line between the perpendiculars drawn to the sides of the parallelogram should not approximate to a straight line (as, for example, the curve ͵Γ͵Δ between Ξ͵Γ, Ο͵Δ), but to an arc, we may measure it thus : Draw ͵Δ$\overset{\varsigma}{\mathrm{M}}$ perpendicular to Ο͵Δ, and on it take a series of points $\overset{\theta}{\mathrm{M}}$, $\overset{\eta}{\mathrm{M}}$, and from them draw $\overset{\theta}{\mathrm{M}}\overset{\iota}{\mathrm{M}}$, $\overset{\eta}{\mathrm{M}}\overset{\varsigma}{\mathrm{M}}$ perpendicular to $\overset{\varsigma}{\mathrm{M}}$͵Δ, so that the portions between the straight lines so drawn approximate to straight lines, and again we can measure the parallelogram $\overset{\varsigma}{\mathrm{M}}$ΞΟ͵Δ and the triangle $\overset{\eta}{\mathrm{M}}\overset{\varsigma}{\mathrm{M}}$͵Δ, and the trapezium ͵Γ$\overset{\varsigma}{\mathrm{M}}\overset{\theta}{\mathrm{M}}\overset{\iota}{\mathrm{M}}$, and also the other trapezium, and so we shall obtain the area bounded by the line ͵Γ$\overset{\iota}{\mathrm{M}}\overset{\varsigma}{\mathrm{M}}$͵Δ and the straight lines ͵ΓΞ, ΞΟ, Ο͵Δ.

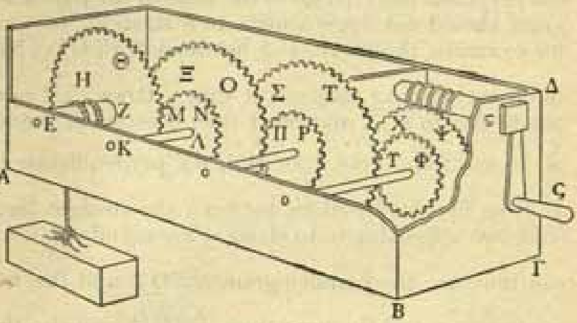
(c) Mechanics [a]

Heron, *Dioptra* 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

With a given force to move a given weight by th

a given weight by a given force. This account is the same as that given in the passage here reproduced from the *Dioptra*, and it is obviously the same as the account found by Pappus (viii. 19, ed. Hultsch 1060. 1–1068. 23) in a work of Heron's (now lost) entitled Βαρουλκός ("weight-lifter")—though Pappus himself took the ratio of force to weight as 4 : 160 and the ratio of successive diameters as 2 : 1. It is suggested by Heath (*H.G.M.* ii. 346-347) that the chapter from the

διὰ τυμπάνων ὀδοντωτῶν παραθέσεως. κατεσκευάσθω πῆγμα καθάπερ γλωσσόκομον· εἰς τοὺς μακροὺς καὶ παραλλήλους τοίχους διακείσθωσαν ἄξονες παράλληλοι ἑαυτοῖς, ἐν διαστήμασι κείμενοι



ὥστε τὰ συμφυῆ αὐτοῖς ὀδοντωτὰ τύμπανα παρακεῖσθαι καὶ συμπεπλέχθαι ἀλλήλοις, καθὰ μέλλομεν δηλοῦν. ἔστω τὸ εἰρημένον γλωσσόκομον τὸ ΑΒΓΔ, ἐν ᾧ ἄξων ἔστω διακείμενος, ὡς εἴρηται, καὶ δυνάμενος εὐλύτως στρέφεσθαι, ὁ ΕΖ. τούτῳ δὲ συμφυὲς ἔστω τύμπανον ὠδοντωμένον τὸ ΗΘ ἔχον τὴν διάμετρον, εἰ τύχοι, πενταπλασίονα ⟨τῆς⟩[1] τοῦ ΕΖ ἄξονος διαμέτρου. καὶ ἵνα ἐπὶ παραδείγματος τὴν κατασκευὴν ποιησώμεθα, ἔστω τὸ μὲν ἀγόμενον βάρος ταλάντων χιλίων, ἡ δὲ κινοῦσα δύναμις ἔστω ταλάντων ε̄, τουτέστιν ὁ κινῶν ἄνθρωπος ἢ παιδάριον, ὥστε δύνασθαι καθ' ἑαυτὸν ἄνευ μηχανῆς ἕλκειν τάλαντα ε̄. οὐκοῦν ἐὰν τὰ ἐκ τοῦ φορτίου ἐκδεδεμένα ὅπλα διά τινος ⟨ὀπῆς

juxtaposition of toothed wheels.[a] Let a framework be prepared like a chest ; and in the long, parallel walls let there lie axles parallel one to another, resting at such intervals that the toothed wheels fitting on to them will be adjacent and will engage one with the other, as we shall explain. Let ΑΒΓΔ be the aforesaid chest, and let ΕΖ be an axle lying in it, as stated above, and able to revolve freely. Fitting on to this axle let there be a toothed wheel ΗΘ whose diameter, say, is five times the diameter of the axle ΕΖ. In order that the construction may serve as an illustration, let the weight to be raised be 1000 talents, and let the moving force be 5 talents, that is, let the man or slave who moves it be able by himself, without mechanical aid, to lift 5 talents. Then if the rope holding the load passes through some aperture in

Βαρουλκός was substituted for the original opening of the *Mechanics*, which had become lost.

Other problems dealt with in the *Mechanics* are the paradox of motion known as Aristotle's wheel, the parallelogram of velocities, motion on an inclined plane, centres of gravity, the five mechanical powers, and the construction of engines. Edited with a German translation by L. Nix and W. Schmidt, it is published as vol. ii. in the Teubner Heron.

[a] Perhaps "rollers."

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[1] τῆς add. Vincentius.

οὔσης⟩[1] ἐν τῷ ΑΒ τοίχῳ ἐπειληθῇ περὶ τὸν ΕΖ ἄξονα ⟨. . . . .⟩[2] κατειλούμενα τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος[3]· ἵνα δὲ κινηθῇ τὸ ΗΘ τύμπανον, ⟨δεῖ δυνά⟩μει[4] ὑπάρχειν πλέον ταλάντων διακοσίων, διὰ τὸ τὴν διάμετρον τοῦ τυμπάνου τῆς διαμέτρου τοῦ ἄξονος, ὡς ὑπεθέμεθα, πενταπλῆν ⟨εἶναι⟩[5]· ταῦτα γὰρ ἀπεδείχθη ἐν ταῖς τῶν ε̄ δυνάμεων ἀποδείξεσιν. ἀλλ' ⟨. . . . . .⟩[6] ἔχομεν τὶ τὴν δύναμιν ταλάντων διακοσίων, ἀλλὰ πέντε. γεγονέτω οὖν ἕτερος ἄξων ⟨παράλληλος⟩[7] διακείμενος τῷ ΕΖ, ὁ ΚΛ, ἔχων συμφυὲς τύμπανον ὠδοντωμένον τὸ ΜΝ. ὀδοντῶδες δὲ καὶ τὸ ΗΘ τύμπανον, ὥστε ἐναρμόζειν ταῖς ὀδοντώσεσι τοῦ ΜΝ τυμπάνου. τῷ δὲ αὐτῷ ἄξονι τῷ ΚΛ συμφυὲς τύμπανον τὸ ΞΟ, ἔχον ὁμοίως τὴν διάμετρον πενταπλασίονα τῆς τοῦ ΜΝ τυμπάνου διαμέτρου. διὰ δὴ τοῦτο δεήσει τὸν βουλόμενον κινεῖν διὰ τοῦ ΞΟ τυμπάνου τὸ βάρος ἔχειν δύναμιν ταλάντων μ̄, ἐπειδήπερ τῶν σ̄ ταλάντων τὸ πέμπτον ἐστὶ τάλαντα μ̄. πάλιν οὖν παρακείσθω ⟨τῷ ΞΟ τυμπάνῳ ὠδοντωμένῳ⟩[8] τύμπανον ὀδοντωθὲν ἕτερον ⟨τὸ ΠΡ, καὶ ἔστω τῷ⟩[9] τυμπάνῳ ὠδοντωμένῳ τῷ ΠΡ συμφυὲς ἕτερον τύμπανον τὸ ΣΤ[10] ἔχον ὁμοίως πενταπλῆν τὴν διάμετρον τῆς ΠΡ τυμπάνου διαμέτρου· ἡ δὲ ἀ⟨νάλογος ἔσται δύναμις⟩[11] τοῦ ΣΤ τυμπάνου ἡ ἔχουσα τὸ βάρος ταλάντων η̄·

[1] ὀπῆς οὔσης add. Hultsch et H. Schöne.

[2] After ἄξονα there is a lacuna of five letters.

[3] τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος H. Schöne, τὰ ἐκ τοῦ φορτίου ἐπλακων εν τισι το βαρος cod.

[4] δεῖ δυνάμει—"septem litteris madore absumptis, supplevi dubitanter," H. Schöne.

[5] εἶναι add. H. Schöne.

the wall AB and is coiled round the axle EZ, the rope holding the load will move the weight as it winds up. In order that the wheel HΘ may be moved, a force of more than 200 talents is necessary, owing to the diameter of the wheel being, as postulated, five times the diameter of the axle ; for this was shown in the proofs of the five mechanical powers.[a] We have [not, however . . .] a force of 200 talents, but only of 5. Therefore let there be another axle KΛ, lying parallel to EZ, and having the toothed wheel MN fitting on to it. Now let the teeth of the wheel HΘ be such as to engage with the teeth of the wheel MN. On the same axle KΛ let there be fitted the wheel ΞO, whose diameter is likewise five times the diameter of the wheel MN. Now, in consequence, anyone wishing to move the weight by means of the wheel ΞO will need a force of 40 talents, since the fifth part of 200 talents is 40 talents. Again, then, let another toothed wheel ΠP lie alongside the toothed wheel ΞO, and let there be fitted to the toothed wheel ΠP another toothed wheel ΣT whose diameter is likewise five times the diameter of the wheel ΠP ; then the force needing to be applied to the wheel ΣT will be 8 talents ; but the force actually available

[a] The wheel and axle, the lever, the pulley, the wedge and the screw, which are dealt with in Book ii. of Heron's *Mechanics*.

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[6] After ἀλλ' is a special sign and a lacuna of 22 letters.
[7] παράλληλος add. H. Schöne.
[8] τῷ ΞΟ τυμπάνῳ ὠδοντωμένῳ add. H. Schöne.
[9] τὸ ΠΡ, καὶ ἔστω τῷ add. H. Schöne.
[10] τύμπανον τὸ ΣΤ, so I read in place of the *συμφυὴς* in Schöne's text.
[11] ἀνάλογος ἔσται δύναμις—so H. Schöne completes the lacuna.

ἀλλ' ἡ ὑπάρχουσα ἡμῖν δύναμις δέδοται ταλάντων ε̄. ὁμοίως ἕτερον παρακείσθω τύμπανον ὠδοντωμένον τὸ ΥΦ τῷ ΣΤ ὀδοντωθέντι· τοῦδε τοῦ ΥΦ τυμπάνου τῷ ἄξονι συμφυὲς ἔστω τύμπανον τὸ ΧΨ ὠδοντωμένον, οὗ ἡ διάμετρος πρὸς τὴν τοῦ ΥΦ τυμπάνου διάμετρον λόγον ἐχέτω, ὃν τὰ ὀκτὼ τάλαντα πρὸς τὰ τῆς δοθείσης δυνάμεως τάλαντα ε̄.

Καὶ τούτων παρασκευασθέντων, ἐὰν ἐπινοήσωμεν τὸ ΑΒΓΔ ⟨γλωσσόκομον⟩[1] μετέωρον κείμενον, καὶ ἐκ μὲν τοῦ ΕΖ ἄξονος τὸ βάρος ἐξάψωμεν, ἐκ δὲ τοῦ ΧΨ τυμπάνου τὴν ἕλκουσαν δύναμιν, οὐδοπότερον αὐτῶν κατενεχθήσεται, εὐλύτως στρεφομένων τῶν ἀξόνων, καὶ τῆς τῶν τυμπάνων παραθέσεως καλῶς ἁρμοζούσης, ἀλλ' ὥσπερ ζυγοῦ τινος ἰσορροπήσει ἡ δύναμις τῷ βάρει. ἐὰν δὲ ἑνὶ αὐτῶν προσθῶμεν ὀλίγον ἕτερον βάρος, καταρρέψει καὶ ἐνεχθήσεται ἐφ' ὃ προσετέθη βάρος, ὥστε ἐὰν ἓν τῶν ε̄ ταλάντων δυνάμει ⟨. . . . . .⟩[2] εἰ τύχοι μναϊαῖον προστεθῇ βάρος, κατακρατήσει καὶ ἐπισπάσεται τὸ βάρος. ἀντὶ δὲ[3] τῆς προσθέσεως τούτῳ παρακείσθω κοχλίας ἔχων τὴν ἕλικα ἁρμοστὴν τοῖς ὀδοῦσι τοῦ τυμπάνου, στρεφόμενος εὐλύτως περὶ τόρμους ἐνόντας ἐν τρήμασι στρογγύλοις, ὧν ὁ μὲν ἕτερος ὑπερεχέτω εἰς τὸ ἐκτὸς μέρος τοῦ γλωσσοκόμου κατὰ τὸν ΓΔ ⟨τοῖχον τὸν παρακείμενον⟩[4] τῷ κοχλίᾳ· ἡ ἄρα ὑπεροχὴ τετραγωνισθεῖσα λαβέτω χειρολάβην τὴν ϛϛ, δι' ἧς ἐπιλαμβανόμενός τις καὶ ἐπιστρέφων ἐπιστρέψει τὸν κοχλίαν καὶ τὸ ΧΨ τύμπανον, ὥστε καὶ τὸ ΥΦ συμφυὲς αὐτῷ. διὰ δὲ τοῦτο καὶ τὸ παρακείμενον τὸ ΣΤ ἐπιστραφήσεται, καὶ τὸ συμφυὲς αὐτῷ τὸ ΠΡ, καὶ τὸ τούτῳ παρακείμενον τὸ ΞΟ,

to us is 5 talents. Let there be placed another toothed wheel ΥΦ engaging with the toothed wheel ΣΤ; and fitting on to the axle of the wheel ΥΦ let there be a toothed wheel ΧΨ, whose diameter bears to the diameter of the wheel ΥΦ the same ratio as 8 talents bears to the given force 5 talents.

When this construction is done, if we imagine the chest ΑΒΓΔ as lying above the ground, with the weight hanging from the axle ΕΖ and the force raising it applied to the wheel ΧΨ, neither of them will descend, provided the axles revolve freely and the juxtaposition of the wheels is accurate, but as in a beam the force will balance the weight. But if to one of them we add another small weight, the one to which the weight was added will tend to sink down and will descend, so that if, say, a mina is added to one of the 5 talents in the force it will overcome and draw the weight. But instead of this addition to the force, let there be a screw having a spiral which engages the teeth of the wheel, and let it revolve freely about pins in round holes, of which one projects beyond the chest through the wall ΓΔ adjacent to the screw; and then let the projecting piece be made square and be given a handle ϚϚ. Anyone who takes this handle and turns, will turn the screw and the wheel ΧΨ, and therefore the wheel ΥΦ joined to it. Similarly the adjacent wheel ΣΤ will revolve, and ΠΡ joined to it, and then the adjacent wheel ΞΟ, and then ΜΝ fitting

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[1] γλωσσόκομον add. H. Schöne.
[2] After δυνάμει is a lacuna of seven letters.
[3] In Schöne's text δὲ is printed after τούτῳ.
[4] τοῖχον τὸν παρακείμενον add. H. Schöne.

καὶ τὸ τούτῳ συμφυὲς τὸ ΜΝ, καὶ τὸ τούτῳ παρακείμενον τὸ ΗΘ, ὥστε καὶ ὁ τούτῳ συμφυὴς ἄξων ὁ ΕΖ, περὶ ὃν ἐπειλούμενα τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος. ὅτι γὰρ κινήσει, πρόδηλον ἐκ τοῦ προστεθῆναι ἑτέρᾳ δυνάμει ⟨τὴν⟩[1] τῆς χειρολάβης, ἥτις περιγράφει κύκλον τῆς τοῦ κοχλίου περιμέτρου μείζονα· ἀπεδείχθη γὰρ ὅτι οἱ μείζονες κύκλοι τῶν ἐλασσόνων κατακρατοῦσιν, ὅταν περὶ τὸ αὐτὸ κέντρον κυλίωνται.

(d) Optics: Equality of Angles of Incidence and Reflection

Damian. *Opt.* 14, ed. R. Schöne 20. 12-18

Ἀπέδειξε γὰρ ὁ μηχανικὸς Ἥρων ἐν τοῖς αὐτοῦ Κατοπτρικοῖς, ὅτι αἱ πρὸς ἴσας γωνίας κλώμεναι εὐθεῖαι ἐλάχισταί εἰσι πασῶν[2] τῶν ἀπὸ τῆς αὐτῆς καὶ ὁμοιομεροῦς γραμμῆς πρὸς τὰ αὐτὰ κλωμένων [πρὸς ἀνίσους γωνίας].[3] τοῦτο δὲ ἀποδείξας φησὶν ὅτι εἰ μὴ μέλλοι ἡ φύσις μάτην περιάγειν τὴν ἡμετέραν ὄψιν, πρὸς ἴσας αὐτὴν ἀνακλάσει γωνίας.

Olympiod. *In Meteor.* iii. 2 (Aristot. 371 b 18), ed. Stüve 212. 5–213. 21

Ἐπειδὴ γὰρ τοῦτο ὡμολογημένον ἐστὶ παρὰ πᾶσιν, ὅτι οὐδὲν μάτην ἐργάζεται ἡ φύσις οὐδὲ ματαιοπονεῖ, ἐὰν μὴ δώσωμεν πρὸς ἴσας γωνίας γίνεσθαι τὴν ἀνάκλασιν, πρὸς ἀνίσους ματαιοπονεῖ

[1] τὴν add. H. Schöne.
[2] πασῶν G. Schmidt, τῶν μέσων codd.
[3] πρὸς ἀνίσους γωνίας om. R. Schöne.

on to this last, and then the adjacent wheel HΘ, and so finally the axle EZ fitting on to it; and the rope, winding round the axle, will move the weight. That it will move the weight is obvious because there has been added to the one force that moving the handle which describes a circle greater than that of the screw; for it has been proved that greater circles prevail over lesser when they revolve about the same centre.

(*d*) Optics: Equality of Angles of Incidence and Reflection

Damianus,[a] *On the Hypotheses in Optics* 14, ed. R. Schöne 20. 12-18

For the mechanician Heron showed in his *Catoptrica* that of all [mutually] inclined straight lines drawn from the same homogenous straight line [surface] to the same [points], those are the least which are so inclined as to make equal angles. In his proof he says that if Nature did not wish to lead our sight in vain, she would incline it so as to make equal angles.

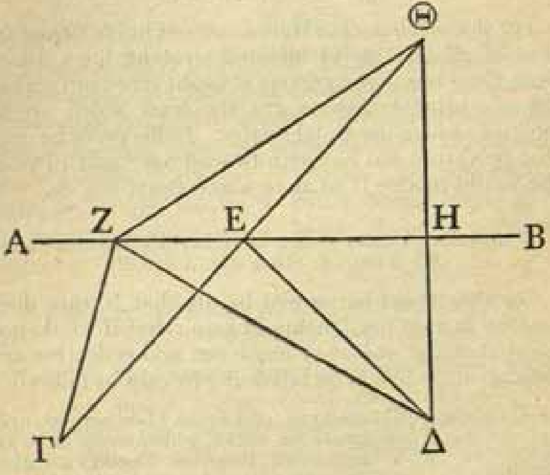
Olympiodorus, *Commentary on Aristotle's Meteora* iii. 2 (371 b 18), ed. Stüve 212. 5-213. 21

For this would be agreed by all, that Nature does nothing in vain nor labours in vain; but if we do not grant that the angles of incidence and reflection are equal, Nature would be labouring in vain by following

[a] Damianus, or Heliodorus, of Larissa (date unknown) is the author of a small work on optics, which seems to be an abridgement of a large work based on Euclid's treatise. The full title given in some MSS.—*Δαμιανοῦ φιλοσόφου τοῦ Ἡλιοδώρου Λαρισσαίου Περὶ ὀπτικῶν ὑποθέσεων βιβλία β* leaves uncertain which was his real name.

ἡ φύσις, καὶ ἀντὶ τοῦ διὰ βραχείας περιόδου φθάσαι τὸ ὁρώμενον τὴν ὄψιν, διὰ μακρᾶς περιόδου τοῦτο φανήσεται καταλαμβάνουσα.[1] εὑρεθήσονται γὰρ αἱ τὰς ἀνίσους γωνίας περιέχουσαι εὐθεῖαι, αἵτινες ἀπὸ τῆς ὄψεως [περιέχουσαι][2] φέρονται[3] πρὸς τὸ κάτοπτρον κἀκεῖθεν πρὸς τὸ ὁρώμενον, μείζονες οὖσαι τῶν τὰς ἴσας γωνίας περιεχουσῶν εὐθειῶν. καὶ ὅτι τοῦτο ἀληθές, δῆλον ἐντεῦθεν.

Ὑποκείσθω γὰρ τὸ κάτοπτρον εὐθεῖά τις ἡ ΑΒ, καὶ ἔστω τὸ μὲν ὁρῶν Γ, τὸ δ' ὁρώμενον τὸ Δ, τὸ δὲ Ε σημεῖον τοῦ κατόπτρου, ἐν ᾧ προσπίπτουσα ἡ ὄψις ἀνακλᾶται πρὸς τὸ ὁρώμενον, ἔστω,



καὶ ἐπεζεύχθω ἡ ΓΕ, ΕΔ. λέγω ὅτι ἡ ὑπὸ ΑΕΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΔΕΒ.

unequal angles, and instead of the eye apprehending the visible object by the shortest route it would do so by a longer. For straight lines so drawn from the eye to the mirror and thence to the visible object as to make unequal angles will be found to be greater than straight lines so drawn as to make equal angles. That this is true, is here made clear.

For let the straight line AB be supposed to be the mirror, and let Γ be the observer, Δ the visible object, and let E be a point on the mirror, falling on which the sight is bent towards the visible object, and let ΓE, EΔ be joined. I say that the angle AEΓ is equal to the angle ΔEB.[a]

[a] Different figures are given in different MSS., with corresponding small variants in the text. With G. Schmidt, I have reproduced the figure in the Aldine edition.

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[1] καταλαμβάνουσα om. Ideler.
[2] περιέχουσαι om. R. Schöne, περιέχουσι Ideler, Stüve.
[3] φέρονται R. Schöne, φερομένας codd.

Εἰ γὰρ μὴ ἔστιν ἴση, ἔστω ἕτερον σημεῖον τοῦ κατόπτρου, ἐν ᾧ προσπίπτουσα ἡ ὄψις πρὸς ἀνίσους γωνίας ἀνακλᾶται, τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΓΖ, ΖΔ. δῆλον ὅτι ἡ ὑπὸ ΓΖΑ γωνία μείζων ἐστὶ τῆς ὑπὸ ΔΖΕ γωνίας. λέγω ὅτι αἱ ΓΖ, ΖΔ εὐθεῖαι, αἵτινες τὰς ἀνίσους γωνίας περιέχουσιν ὑποκειμένης τῆς ΑΒ εὐθείας, μείζονές εἰσι τῶν ΓΕ, ΕΔ εὐθειῶν, αἵτινες τὰς ἴσας γωνίας περιέχουσι μετὰ τῆς ΑΒ. ἤχθω γὰρ κάθετος ἀπὸ τοῦ Δ ἐπὶ τὴν ΑΒ κατὰ τὸ Η σημεῖον καὶ ἐκβεβλήσθω ἐπ' εὐθείας ὡς ἐπὶ τὸ Θ. φανερὸν δὴ ὅτι αἱ πρὸς τῷ Η γωνίαι ἴσαι εἰσίν· ὀρθαὶ γάρ εἰσι. καὶ ἔστω ἡ ΔΗ τῇ ΗΘ ἴση, καὶ ἐπεζεύχθω ἡ ΘΖ καὶ ἡ ΘΕ. αὕτη μὲν ἡ κατασκευή. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΗ τῇ ΗΘ, ἀλλὰ καὶ ἡ ὑπὸ ΔΗΕ γωνία τῇ ὑπὸ ΘΗΕ γωνίᾳ ἴση ἐστί, κοινὴ δὲ πλευρὰ τῶν δύο τριγώνων ἡ ΗΕ, [καὶ βάσις ἡ ΘΕ βάσει τῇ ΕΔ ἴση ἐστί, καὶ][1] τὸ ΗΘΕ τρίγωνον τῷ ΔΗΕ τριγώνῳ ἴσον ἐστί, καὶ ⟨αἱ⟩[2] λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις εἰσὶν ἴσαι, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἡ ΘΕ τῇ ΕΔ. πάλιν ἐπειδὴ τῇ ΗΘ ἴση ἐστὶν ἡ ΗΔ καὶ γωνία ἡ ὑπὸ ΔΗΖ γωνίᾳ τῇ ὑπὸ ΘΗΖ ἴση ἐστί, κοινὴ δὲ ἡ ΗΖ τῶν δύο τριγώνων τῶν ΔΗΖ καὶ ΘΗΖ, [καὶ βάσις ἄρα ἡ ΘΖ βάσει τῇ ΖΔ ἴση ἐστί, καὶ][3] τὸ ΖΗΔ τρίγωνον τῷ ΘΗΖ τριγώνῳ ἴσον ἐστίν. ἴση ἄρα ἐστὶν ἡ ΘΖ τῇ ΖΔ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΘΕ τῇ ΕΔ, κοινὴ προσκείσθω ἡ ΕΓ. δύο ἄρα αἱ ΓΕ, ΕΔ δυσὶ ταῖς ΓΕ, ΕΘ ἴσαι εἰσίν. ὅλη ἄρα ἡ ΓΘ δυσὶ ταῖς ΓΕ, ΕΔ ἴση ἐστί. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ

For if it be not equal, let there be another point Z, on the mirror, falling on which the sight makes unequal angles, and let ΓZ, ZΔ be joined. It is clear that the angle ΓZA is greater than the angle ΔZE. I say that the sum of the straight lines ΓZ, ZΔ which make unequal angles with the base line AB, is greater than the sum of the straight lines ΓE, EΔ, which make equal angles with AB. For let a perpendicular be drawn from Δ to AB at the point H and let it be produced in a straight line to Θ. Then it is obvious that the angles at H are equal; for they are right angles. And let ΔH = HΘ, and let ΘZ and ΘE be joined. This is the construction. Then since ΔH = HΘ, and the angle ΔHE is equal to the angle ΘHE, while HE is a common side of the two triangles, the triangle HΘE is equal to the triangle ΔHE, and the remaining angles, subtended by the equal sides are severally equal one to the other [Eucl. i. 4]. Therefore ΘE = EΔ. Again, since HΔ = HΘ and angle ΔHZ = angle ΘHZ, while HZ is common to the two triangles ΔHZ and ΘHZ, the triangle ZHΔ is equal to the triangle ΘHZ [*ibid.*]. Therefore ΘZ = ZΔ. And since ΘE = EΔ, let EΓ be added to both. Then the sum of the two straight lines ΓE, EΔ is equal to the sum of the two straight lines ΓE, EΘ. Therefore the whole ΓΘ is equal to the sum of the two straight lines ΓE, EΔ. And since in any triangle the sum of two sides is always greater than

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[1] *καὶ . . . καὶ*. These words are out of place here and superfluous.

[2] *αἱ* add. Schmidt. But possibly *καὶ . . . ὑποτείνουσιν*, being superfluous, should be omitted.

[3] *καὶ . . . καὶ*. These words are out of place here and superfluous.

τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, τριγώνου ἄρα τοῦ ΘΖΓ αἱ δύο πλευραὶ αἱ ΘΖ, ΖΓ μιᾶς τῆς ΓΘ μείζονές εἰσιν. ἀλλ' ἡ ΓΘ ἴση ἐστὶ ταῖς ΓΕ, ΕΔ· αἱ ΘΖ, ΖΓ ἄρα μείζονές εἰσι τῶν ΓΕ, ΕΔ. ἀλλ' ἡ ΘΖ τῇ ΖΔ ἐστὶν ἴση· αἱ ΖΓ, ΖΔ ἄρα τῶν ΓΕ, ΕΔ μείζονές εἰσι. καί εἰσιν αἱ ΓΖ, ΖΔ αἱ τὰς ἀνίσους γωνίας περιέχουσαι· αἱ ἄρα τὰς ἀνίσους γωνίας περιέχουσαι μείζονές εἰσι τῶν τὰς ἴσας γωνίας περιεχουσῶν· ὅπερ ἔδει δεῖξαι.

(e) Quadratic Equations

Heron, *Geom.* 21. 9-10, ed. Heiberg (Heron iv.) 380. 15-31

Δοθέντων συναμφοτέρων τῶν ἀριθμῶν ἤγουν τῆς διαμέτρου, τῆς περιμέτρου καὶ τοῦ ἐμβαδοῦ τοῦ κύκλου ἐν ἀριθμῷ ἑνὶ διαστεῖλαι καὶ εὑρεῖν ἕκαστον ἀριθμόν. ποίει οὕτως· ἔστω ὁ δοθεὶς ἀριθμὸς μονάδες $\overline{\sigma\iota\beta}$. ταῦτα ἀεὶ ἐπὶ τὰ $\overline{\rho\nu\delta}$· γίνονται μυριάδες $\bar{\gamma}$ καὶ $\overline{,\beta\chi\mu\eta}$. τούτοις προστίθει καθολικῶς $\overline{\omega\mu\alpha}$· γίνονται μυριάδες τρεῖς καὶ ,γυπθ· ὧν πλευρὰ τετράγωνος γίνεται $\overline{\rho\pi\gamma}$· ἀπὸ τούτων κούφισον $\overline{\kappa\theta}$· λοιπὰ $\overline{\rho\nu\delta}$· ὧν μέρος ια´ γίνεται $\overline{\iota\delta}$· τοσούτου ἡ διάμετρος τοῦ κύκλου. ἐὰν δὲ θέλῃς καὶ τὴν περιφέρειαν εὑρεῖν, ὕφειλον τὰ $\overline{\kappa\theta}$ ἀπὸ τῶν $\overline{\rho\pi\gamma}$· λοιπὰ $\overline{\rho\nu\delta}$· ταῦτα ποίησον δίς· γίνονται $\overline{\tau\eta}$· τούτων λαβὲ μέρος ζ´· γίνονται $\overline{\mu\delta}$· τοσούτου ἡ

ᵃ The proof here given appears to have been taken by Olympiodorus from Heron's *Catoptrica*, and it is substantially identical with the proof in *De Speculis* 4. This work was formerly attributed to Ptolemy, but the discovery of Ptolemy's *Optics* in Arabic has encouraged the belief, now

the remaining side, in whatever way these may be taken [Eucl. i. 20], therefore in the triangle ΘΖΓ the sum of the two sides ΘΖ, ΖΓ is greater than the one side ΓΘ. But

$$\Gamma\Theta = \Gamma\mathrm{E} + \mathrm{E}\Delta ;$$

$\therefore$ $$\Theta\mathrm{Z} + \mathrm{Z}\Gamma > \Gamma\mathrm{E} + \mathrm{E}\Delta.$$

But $$\Theta\mathrm{Z} = \mathrm{Z}\Delta ;$$

$\therefore$ $$\mathrm{Z}\Gamma + \mathrm{Z}\Delta > \Gamma\mathrm{E} + \mathrm{E}\Delta.$$

And ΓΖ, ΖΔ make unequal angles; therefore the sum of straight lines making unequal angles is greater than the sum of straight lines making equal angles; which was to be proved.[a]

(e) Quadratic Equations

Heron, *Geometrica* 21. 9-10, ed. Heiberg (Heron iv.) 380. 15-31

Given the sum of the diameter, perimeter and area of a circle, to find each of them separately. It is done thus: Let the given sum be 212. Multiply this by 154; the result is 32648. To this add 841, making 33489, whose square root is 183. From this take away 29, leaving 154, whose eleventh part is 14; this will be the diameter of the circle. If you wish to find the circumference, take 29 from 183, leaving 154; double this, making 308, and take the seventh part, which is 44; this will be the perimeter. *To*

usually held, that it is a translation of Heron's *Catoptrica*. The translation, made by William of Moerbeke in 1269, can be shown by internal evidence to have been made from the Greek original and not from an Arabic translation. It is published in the Teubner edition of Heron's works, vol. ii. part i.

περίμετρος. τὸ δὲ ἐμβαδὸν εὑρεῖν. ποίει οὕτως· τὰ ι̅δ̅ τῆς διαμέτρου ἐπὶ τὰ μ̅δ̅ τῆς περιμέτρου· γίνονται χ̅ι̅ϛ̅· τούτων λαβὲ μέρος τέταρτον· γίνονται ρνδ· τοσοῦτον τὸ ἐμβαδὸν τοῦ κύκλου. ὁμοῦ τῶν τριῶν ἀριθμῶν μονάδες σ̅ι̅β̅.

(*f*) Indeterminate Analysis

Heron, *Geom.* 24. 1, ed. Heiberg
(Heron iv.) 414. 28–415. 10

Εὑρεῖν δύο χωρία τετράγωνα, ὅπως τὸ τοῦ πρώτου ἐμβαδὸν τοῦ τοῦ δευτέρου ἐμβαδοῦ ἔσται τριπλάσιον. ποιῶ οὕτως· τὰ γ̅ κύβισον· γίνονται

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[a] If d is the diameter of the circle, then the given relation is that

$$d+\frac{22}{7}d+\frac{11}{14}d^2=212,$$

i.e.

$$\frac{11}{14}d^2+\frac{29}{7}d=212.$$

To solve this quadratic equation, we should divide by $\frac{11}{14}$ so as to make the first term a square; Heron makes the first term a square by multiplying by the lowest requisite factor, in this case 154, obtaining the equation

$$11^2d^2+2\,.\,29\,.\,11d=154\,.\,212.$$

By adding 841 he completes the square on the left-hand side

$$\begin{aligned}(11d+29)^2&=154\,.\,212+841\\&=32648+841\\&=33489.\end{aligned}$$

$$\begin{aligned}\therefore\quad 11d+29&=183,\\ \therefore\quad 11d&=154,\\ \text{and}\quad d&=14.\end{aligned}$$

The same equation is again solved in *Geom.* 24. 46 and a similar one in *Geom.* 24. 47. Another quadratic equation is solved in *Geom.* 24. 3 and the result of yet another is given in *Metr.* iii. 4.

find the area. It is done thus: Multiply the diameter, 14, by the perimeter, 44, making 616; take the fourth part of this, which is 154; this will be the area of the circle. The sum of the three numbers is 212.[a]

(*f*) Indeterminate Analysis [b]

Heron, *Geometrica* 24. 1, ed. Heiberg (Heron iv.) 414. 28–415. 10

To find two rectangles such that the area of the first is three times the area of the second.[c] I proceed thus:

[a] The Constantinople ms. in which Heron's *Metrica* was found in 1896 contains also a number of interesting problems in indeterminate analysis; and two were already extant in Heron's *Geëponicus*. The problems, thirteen in all, are now published by Heiberg in Heron iv. 414. 28–426. 29.

[c] It appears also to be a condition that the perimeter of the second should be three times the perimeter of the first. If we substitute any factor n for 3 the general problem becomes: To solve the equations

$$u+v=n(x+y) \quad . \quad . \quad . \quad (1)$$
$$xy=n \, . \, uv \quad . \quad . \quad . \quad (2)$$

The solution given is equivalent to

$$x=2n^3-1, \qquad y=2n^3$$
$$u=n(4n^3-2), \qquad v=n.$$

Zeuthen (*Bibliotheca mathematica*, viii. (1907–1908), pp. 118–134) solves the problem thus: Let us start with the hypothesis that $v=n$. It follows from (1) that u is a multiple of n, say nz. We have then

$$x+y=1+z,$$

while by (2) $\qquad xy=n^3z,$

whence $\qquad xy=n^3(x+y)-n^3$

or $\qquad (x-n^3)(y-n^3)=n^3(n^3-1).$

An obvious solution of this equation is

$$x-n^3=n^3-1, \; y-n^3=n^3,$$

which gives $z=4n^3-2$, whence $u=n(4n^3-2)$. The other values follow.

κ̅ζ̅· ταῦτα δίς· γίνονται ν̅δ̅. νῦν ἆρον μονάδα α̅· λοιπὸν γίνονται ν̅γ̅. ἔστω οὖν ἡ μὲν μία πλευρὰ ποδῶν ν̅γ̅, ἡ δὲ ἑτέρα πλευρὰ ποδῶν ν̅δ̅. καὶ τοῦ ἄλλου χωρίου οὕτως· θὲς ὁμοῦ τὰ ν̅γ̅ καὶ τὰ ν̅δ̅· γίνονται πόδες ρ̅ζ̅· ταῦτα ποίει ἐπὶ τὰ γ̅ . . . λοιπὸν γίνονται πόδες τ̅ι̅η̅. ἔστω οὖν ἡ τοῦ προτέρου πλευρὰ ποδῶν τ̅ι̅η̅, ἡ δὲ ἑτέρα πλευρὰ ποδῶν γ̅· τὰ δὲ ἐμβαδὰ τοῦ ἑνὸς γίνεται ποδῶν ϡ̅ν̅δ̅ καὶ τοῦ ἄλλου ποδῶν ͵β̅ω̅ξ̅β̅.

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15–424. 5

Τριγώνου ὀρθογωνίου τὸ ἐμβαδὸν μετὰ τῆς περιμέτρου ποδῶν σ̅π̅· ἀποδιαστεῖλαι τὰς πλευρὰς καὶ εὑρεῖν τὸ ἐμβαδόν. ποιῶ οὕτως· ἀεὶ ζήτει τοὺς ἀπαρτίζοντας ἀριθμούς· ἀπαρτίζει δὲ τὸν σ̅π̅ ὁ δὶς τὸν ρ̅μ̅, ὁ δʹ τὸν ο̅, ὁ εʹ τὸν ν̅ϛ̅, ὁ ζʹ τὸν μ̅, ὁ ηʹ τὸν λ̅ε̅, ὁ ιʹ τὸν κ̅η̅, ὁ ιδʹ τὸν κ̅. ἐσκεψάμην, ὅτι ὁ η̅ καὶ λ̅ε̅ ποιήσουσι τὸ δοθὲν ἐπίταγμα. τῶν σ̅π̅ τὸ ηʹ· γίνονται πόδες λ̅ε̅. διὰ παντὸς λάμβανε δυάδα τῶν η̅· λοιπὸν μένουσιν ϛ̅ πόδες. τὰ οὖν λ̅ε̅ καὶ τὰ ϛ̅ ὁμοῦ γίνονται πόδες μ̅α̅. ταῦτα ποίει ἐφʼ ἑαυτά· γίνονται πόδες ͵α̅χ̅π̅α̅. τὰ λ̅ε̅ ἐπὶ τὰ ϛ̅· γίνονται πόδες σ̅ι̅· ταῦτα ποίει ἀεὶ ἐπὶ τὰ η̅· γίνονται πόδες ͵α̅χ̅π̅. ταῦτα ἆρον ἀπὸ τῶν ͵α̅χ̅π̅α̅· λοιπὸν μένει α̅· ὧν πλευρὰ τετραγωνικὴ γίνεται α̅. ἄρτι θὲς τὰ μ̅α̅ καὶ ἆρον μονάδα α̅· λοιπὸν μ̅· ὧν ∠ʹ γίνεται κ̅· τοῦτό ἐστιν ἡ κάθετος, ποδῶν κ̅. καὶ θὲς πάλιν τὰ μ̅α̅ καὶ πρόσθες α̅· γίνονται πόδες μβ· ὧν ∠ʹ γίνεται πόδες κ̅α̅· ἔστω ἡ βάσις ποδῶν κ̅α̅. καὶ θὲς τὰ λ̅ε̅ καὶ ἆρον τὰ ϛ̅· λοιπὸν μένουσι

Take the cube of 3, making 27; double this, making 54. Now take away 1, leaving 53. Then let one side be 53 feet and the other 54 feet. As for the other rectangle, [I proceed] thus: Add together 53 and 54, making 107 feet: multiply this by 3, [making 321; take away 3], leaving 318. Then let one side be 318 feet and the other 3 feet. The area of the one will be 954 feet and of the other 2862 feet.[a]

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15–424. 5

In a right-angled triangle the sum of the area and the perimeter is 280 feet; *to separate the sides and find the area.* I proceed thus: Always look for the factors; now 280 can be factorized into 2 . 140, 4 . 70, 5 . 56, 7 . 40, 8 . 35, 10 . 28, 14 . 20. By inspection, we find 8 and 35 fulfil the requirements. For take one-eighth of 280, getting 35 feet. Take 2 from 8, leaving 6 feet. Then 35 and 6 together make 41 feet. Multiply this by itself, making 1681 feet. Now multiply 35 by 6, getting 210 feet. Multiply this by 8, getting 1680 feet. Take this away from the 1681, leaving 1, whose square root is 1. Now take the 41 and subtract 1, leaving 40, of which the half is 20; this is the perpendicular, 20 feet. And again take 41 and add 1, getting 42 feet, of which the half is 21; and let this be the base, 21 feet. And take 35 and subtract 6, leaving 29 feet. Now multiply

[a] The term "feet," πόδες, is used by Heron indiscriminately of lineal feet, square feet and the sum of numbers of lineal and square feet.

πόδες $\overline{\kappa\theta}$. ἄρτι θὲς τὴν κάθετον ἐπὶ τὴν βάσιν· ὧν ∠′ γίνεται πόδες $\overline{\sigma\iota}$· καὶ αἱ τρεῖς πλευραὶ περιμετρούμεναι ἔχουσι πόδας $\bar{o}$· ὁμοῦ σύνθες μετὰ τοῦ ἐμβαδοῦ· γίνονται πόδες $\overline{\sigma\pi}$.

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[a] Heath (*H.G.M.* ii. 446-447) shows how this solution can be generalized. Let a, b be the sides of the triangle containing the right angle, c the hypotenuse, S the area of the triangle, r the radius of the inscribed circle; and let

$$s=\tfrac{1}{2}(a+b+c).$$

Then

$$S=rs=\tfrac{1}{2}ab,\ r+s=a+b,\ c=s-r.$$

Solving the first two equations, we have

$$\left.\begin{matrix}a\\b\end{matrix}\right\}=\tfrac{1}{2}[r+s\mp\sqrt{\{(r+s)^2-8rs\}}],$$

and this formula is actually used in the problem. The

the perpendicular and the base together, [getting 420], of which the half is 210 feet; and the three sides comprising the perimeter amount to 70 feet; add them to the area, getting 280 feet.[a]

method is to take the sum of the area and the perimeter $S+2s$, separated into its two obvious factors $s(r+2)$, to put $s(r+2)=A$ (the given number), and then to separate A into suitable factors to which s and $r+2$ may be equated. They must obviously be such that sr, the area, is divisible by 6.

In the given problem $A=280$, and the suitable factors are $r+2=8$, $s=35$, because r is then equal to 6 and rs is a multiple of 6. Then

$$a=\tfrac{1}{2}[6+35-\sqrt{\{(6+35)^2-8\,.\,6\,.\,35\}}]=\tfrac{1}{2}(41-1)=20,$$
$$b=\tfrac{1}{2}(41+1)=21,$$
$$c=35-6\quad=29.$$

This problem is followed by three more of the same type.

XXIII. ALGEBRA : DIOPHANTUS

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(a) General

Anthol. Palat. xiv. 126, *The Greek Anthology*, ed. Paton (L.C.L.) v. 92-93

Οὗτός τοι Διόφαντον ἔχει τάφος· ἆ μέγα θαῦμα·
  καὶ τάφος ἐκ τέχνης μέτρα βίοιο λέγει.
ἕκτην κουρίζειν βιότου θεὸς ὤπασε μοίρην·
  δωδεκάτην δ' ἐπιθείς, μῆλα πόρεν χνοάειν·
τῇ δ' ἄφ' ἐφ' ἑβδομάτῃ τὸ γαμήλιον ἥψατο φέγγος,
  ἐκ δὲ γάμων πέμπτῳ παῖδ' ἐπένευσεν ἔτει.
αἰαῖ, τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς
  τοῦδε καὶ ἡ κρυερὸς μέτρον ἑλὼν βιότου.
πένθος δ' αὖ πισύρεσσι παρηγορέων ἐνιαυτοῖς
  τῇδε πόσου σοφίῃ τέρμ' ἐπέρησε βίου.

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[a] There are in the *Anthology* 46 epigrams which are algebraical problems. Most of them (xiv. 116-146) were collected by Metrodorus, a grammarian who lived about A.D. 500, but their origin is obviously much earlier and many belong to a type described by Plato and the scholiast to the Charmides (*v.* vol. i. pp. 16, 20).

Problems in indeterminate analysis solved before the time of Diophantus include the Pythagorean and Platonic methods of finding numbers representing the sides of right-angled triangles (*v.* vol. i. pp. 90-95), the methods (also Pythagorean) of finding "side- and diameter-numbers" (vol. i. pp. 132-139), Archimedes' *Cattle Problem* (*v. supra*, pp. 202-205) and Heron's problems (*v. supra*, pp. 504-509).

XXIII. ALGEBRA: DIOPHANTUS

(a) General

Palatine Anthology [a] xiv. 126, *The Greek Anthology*, ed. Paton (L.C.L.) v. 92-93

This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.[b]

[a] Diophantus's surviving works and ancillary material are admirably edited by Tannery in two volumes of the Teubner series (Leipzig, 1895). There is a French translation by Paul Ver Eecke, *Diophante d'Alexandre* (Bruges, 1926). The history of Greek algebra as a whole is well treated by G. F. Nesselmann, *Die Algebra der Griechen*, and by T. L. Heath, *Diophantus of Alexandria: A Study in the History of Greek Algebra*, 2nd ed. 1910.

[b] If x was his age at death, then

$$\tfrac{1}{6}x+\tfrac{1}{12}x+\tfrac{1}{7}x+5+\tfrac{1}{2}x+4=x,$$

whence $$x=84.$$

Theon Alex. *in Ptol. Math. Syn. Comm.* i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 453. 4-6

Καθ' ἃ καὶ Διόφαντός φησι· "τῆς γὰρ μονάδος ἀμεταθέτου οὔσης καὶ ἑστώσης πάντοτε, τὸ πολλαπλασιαζόμενον εἶδος ἐπ' αὐτὴν αὐτὸ τὸ εἶδος ἔσται."

Dioph. *De polyg. num.* [5], Dioph. ed. Tannery i. 470. 27–472. 4

Καὶ ἀπεδείχθη τὸ παρὰ Ὑψικλεῖ ἐν ὅρῳ λεγόμενον, ὅτι, "ἐὰν ὦσιν ἀριθμοὶ ἀπὸ μονάδος ἐν ἴσῃ ὑπεροχῇ ὁποσοιοῦν, μονάδος μενούσης τῆς ὑπεροχῆς, ὁ σύμπας ἐστὶν ⟨τρίγωνος, δυάδος δέ⟩,[1] τετράγωνος, τριάδος δέ, πεντάγωνος· λέγεται δὲ τὸ πλῆθος τῶν γωνιῶν κατὰ τὸν δυάδι μείζονα τῆς ὑπεροχῆς, πλευραὶ δὲ αὐτῶν τὸ πλῆθος τῶν ἐκτεθέντων σὺν τῇ μονάδι."

Mich. Psell. *Epist.*, Dioph. ed. Tannery ii. 38. 22–39. 1

Περὶ δὲ τῆς Αἰγυπτιακῆς μεθόδου ταύτης Διόφαντος μὲν διέλαβεν ἀκριβέστερον, ὁ δὲ λογιώτατος Ἀνατόλιος τὰ συνεκτικώτατα μέρη τῆς κατ'

[1] τρίγωνος, δυάδος δέ add. Bachet.

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[a] *Cf.* Dioph. ed. Tannery i. 8. 13-15. The word εἶδος, as will be seen in due course, is regularly used by Diophantus for a *term* of an equation.

Theon of Alexandria, *Commentary on Ptolemy's Syntaxis* i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 453. 4-6

As Diophantus says: "The unit being without dimensions and everywhere the same, a term that is multiplied by it will remain the same term." [a]

Diophantus, *On Polygonal Numbers* [5], Dioph. ed. Tannery i. 470. 27–472. 4

There has also been proved what was stated by Hypsicles in a definition, namely, that "if there be as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square number; when 3, a pentagonal number [; and so on]. The number of angles is called after the number which exceeds the common difference by 2, and the sides after the number of terms including 1." [b]

Michael Psellus,[c] *A Letter*, Dioph. ed. Tannery ii. 38. 22–39. 1

Diophantus dealt more accurately with this Egyptian method, but the most learned Anatolius collected the most essential parts of the theory as stated by

[b] *i.e.*, the nth a-gonal number (1 being the first) is $\frac{1}{2}n\{2+(n-1)(a-2)\}$; *v.* vol. i. p. 98 n. *a*.

[c] Michael Psellus, "first of philosophers" in a barren age, flourished in the latter part of the eleventh century A.D. There has survived a book purporting to be by Psellus on arithmetic, music, geometry and astronomy, but it is clearly not all his own work. In the geometrical section it is observed that the most favoured method of finding the area of a circle is to take the mean between the inscribed and circumscribed squares, which would give $\pi = \sqrt{8} = 2{\cdot}8284271$.

ἐκεῖνον ἐπιστήμης ἀπολεξάμενος ἑτέρως[1] Διοφάντῳ συνοπτικώτατα προσεφώνησε.

Dioph. *Arith.* i., Praef., Dioph. ed. Tannery i. 14. 25–16. 7

Νῦν δ' ἐπὶ τὰς προτάσεις χωρήσωμεν ὁδόν, πλείστην ἔχοντες τὴν ἐπ' αὐτοῖς τοῖς εἴδεσι συνηθροισμένην ὕλην. πλείστων δ' ὄντων τῷ ἀριθμῷ καὶ μεγίστων τῷ ὄγκῳ, καὶ διὰ τοῦτο βραδέως βεβαιουμένων ὑπὸ τῶν παραλαμβανόντων αὐτὰ καὶ ὄντων ἐν αὐτοῖς δυσμνημονεύτων, ἐδοκίμασα τὰ ἐν αὐτοῖς ἐπιδεχόμενα διαιρεῖν, καὶ μάλιστα τὰ ἐν ἀρχῇ ἔχοντα στοιχειώδως ἀπὸ ἁπλουστέρων ἐπὶ σκολιώτερα διελεῖν ὡς προσῆκεν. οὕτως γὰρ εὐόδευτα γενήσεται τοῖς ἀρχομένοις, καὶ ἡ ἀγωγὴ αὐτῶν μνημονευθήσεται, τῆς πραγματείας αὐτῶν ἐν τρισκαίδεκα βιβλίοις γεγενημένης.

Ibid. v. 3, Dioph. ed. Tannery i. 316. 6

Ἔχομεν ἐν τοῖς Πορίσμασιν.

[1] ἑτέρως Tannery, ἑτέρῳ codd.

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[a] The two passages cited before this one allow us to infer that Diophantus must have lived between Hypsicles and Theon, say 150 B.C. to A.D. 350. Before Tannery edited Michael Psellus's letter, there was no further evidence, but it is reasonable to infer from this letter that Diophantus was a contemporary of Anatolius, bishop of Laodicea about A.D. 280 (*v.* vol. i. pp. 2-3). For references by Plato and a scholiast to the Egyptian methods of reckoning, *v.* vol. i. pp. 16, 20.

[b] Of these thirteen books in the *Arithmetica*, only six

him in a different way and in the most concise form, and dedicated his work to Diophantus.[a]

Diophantus, *Arithmetica* i., Preface, Dioph. ed. Tannery i. 14. 25–16. 7

Now let us tread the path to the propositions themselves, which contain a great mass of material compressed into the several species. As they are both numerous and very complex to express, they are only slowly grasped by those into whose hands they are put, and include things hard to remember; for this reason I have tried to divide them up according to their subject-matter, and especially to place, as is fitting, the elementary propositions at the beginning in order that passage may be made from the simpler to the more complex. For thus the way will be made easy for beginners and what they learn will be fixed in their memory; the treatise is divided into thirteen books.[b]

Ibid. v. 3, Dioph. ed. Tannery i. 316. 6

We have it in the Porisms.[c]

have survived. Tannery suggests that the commentary on it written by Hypatia, daughter of Theon of Alexandria, extended only to these first six books, and that consequently little notice was taken of the remaining seven. There would be a parallel in Eutocius's commentaries on Apollonius's *Conics*. Nesselmann argues that the lost books came in the middle, but Tannery (Dioph. ii. xix-xxi) gives strong reasons for thinking it is the last and most difficult books which have been lost.

[c] Whether this collection of propositions in the Theory of Numbers, several times referred to in the *Arithmetica*, formed a separate treatise from, or was included in, that work is disputed; Hultsch and Heath take the former view, in my opinion judiciously, but Tannery takes the latter.

(b) Notation

Ibid. i., Praef., Dioph. ed. Tannery i. 2. 3–6. 21

Τὴν εὕρεσιν τῶν ἐν τοῖς ἀριθμοῖς προβλημάτων, τιμιώτατέ μοι Διονύσιε, γινώσκων σε σπουδαίως ἔχοντα μαθεῖν, [ὀργανῶσαι τὴν μέθοδον][1] ἐπειράθην, ἀρξάμενος ἀφ' ὧν συνέστηκε τὰ πράγματα θεμελίων, ὑποστῆσαι τὴν ἐν τοῖς ἀριθμοῖς φύσιν τε καὶ δύναμιν.

Ἴσως μὲν οὖν δοκεῖ τὸ πρᾶγμα δυσχερέστερον, ἐπειδὴ μήπω γνώριμόν ἐστιν, δυσέλπιστοι γὰρ εἰς κατόρθωσίν εἰσιν αἱ τῶν ἀρχομένων ψυχαί, ὅμως δ' εὐκατάληπτόν σοι γενήσεται, διά τε τὴν σὴν προθυμίαν καὶ τὴν ἐμὴν ἀπόδειξιν· ταχεῖα γὰρ εἰς μάθησιν ἐπιθυμία προσλαβοῦσα διδαχήν.

Ἀλλὰ καὶ πρὸς τοῖσδε γινώσκοντί σοι πάντας τοὺς ἀριθμοὺς συγκειμένους ἐκ μονάδων πλήθους τινός, φανερὸν καθέστηκεν εἰς ἄπειρον ἔχειν τὴν ὕπαρξιν. τυγχανόντων δὴ οὖν ἐν τούτοις

ὧν μὲν τετραγώνων, οἵ εἰσιν ἐξ ἀριθμοῦ τινος ἐφ' ἑαυτὸν πολυπλασιασθέντος· οὗτος δὲ ὁ ἀριθμὸς καλεῖται πλευρὰ τοῦ τετραγώνου·

ὧν δὲ κύβων, οἵ εἰσιν ἐκ τετραγώνων ἐπὶ τὰς αὐτῶν πλευρὰς πολυπλασιασθέντων,

ὧν δὲ δυναμοδυναμέων, οἵ εἰσιν ἐκ τετραγώνων ἐφ' ἑαυτοὺς πολυπλασιασθέντων,

ὧν δὲ δυναμοκύβων, οἵ εἰσιν ἐκ τετραγώνων ἐπὶ

[1] ὀργανῶσαι τὴν μέθοδον om. Tannery, following the most ancient ms.

(b) Notation [a]

Ibid. i., Preface, Dioph. ed. Tannery i. 2. 3–6. 21

Knowing that you are anxious, my most esteemed Dionysius, to learn how to solve problems in numbers, I have tried, beginning from the foundations on which the subject is built, to set forth the nature and power in numbers.

Perhaps the subject will appear to you rather difficult, as it is not yet common knowledge, and the minds of beginners are apt to be discouraged by mistakes ; but it will be easy for you to grasp, with your enthusiasm and my teaching ; for keenness backed by teaching is a swift road to knowledge.

As you know, in addition to these things, that all numbers are made up of some multitude of units, it is clear that their formation has no limit. Among them are—

squares, which are formed when any number is multiplied by itself ; the number itself is called the *side of the square* [b] ;

cubes, which are formed when squares are multiplied by their sides,

square-squares, which are formed when squares are multiplied by themselves ;

square-cubes, which are formed when squares are

[a] This subject is admirably treated, with two original contributions, by Heath, *Diophantus of Alexandria*, 2nd ed., pp. 34–53. Diophantus's method of representing large numbers and fractions has already been discussed (vol. i. pp. 44–45). Among other abbreviations used by Diophantus are □$^{\text{ος}}$, declined throughout its cases, for *τετράγωνος* ; and ἴσ. (apparently ισ in the archetype) for the sign =, connecting two sides of an equation.

[b] Or " square root."

τοὺς ἀπὸ τῆς αὐτῆς αὐτοῖς πλευρᾶς κύβους πολυπλασιασθέντων,

ὧν δὲ κυβοκύβων, οἵ εἰσιν ἐκ κύβων ἐφ' ἑαυτοὺς πολυπλασιασθέντων,

ἔκ τε τῆς τούτων ἤτοι συνθέσεως ἢ ὑπεροχῆς ἢ πολυπλασιασμοῦ ἢ λόγου τοῦ πρὸς ἀλλήλους ἢ καὶ ἑκάστων πρὸς τὰς ἰδίας πλευρὰς συμβαίνει πλέκεσθαι πλεῖστα προβλήματα ἀριθμητικά· λύεται δὲ βαδίζοντός σου τὴν ὑποδειχθησομένην ὁδόν.

Ἐδοκιμάσθη οὖν ἕκαστος τούτων τῶν ἀριθμῶν συντομωτέραν ἐπωνυμίαν κτησάμενος στοιχεῖον τῆς ἀριθμητικῆς θεωρίας εἶναι· καλεῖται οὖν ὁ μὲν τετράγωνος δύναμις καὶ ἔστιν αὐτῆς σημεῖον τὸ Δ ἐπίσημον ἔχον Υ, Δ^{Υ} δύναμις·

ὁ δὲ κύβος καὶ ἔστιν αὐτοῦ σημεῖον Κ ἐπίσημον ἔχον Υ, K^{Υ} κύβος·

ὁ δὲ ἐκ τετραγώνου ἐφ' ἑαυτὸν πολυπλασιασθέντος δυναμοδύναμις καὶ ἔστιν αὐτοῦ σημεῖον δέλτα δύο ἐπίσημον ἔχοντα Υ, $\Delta^{\Upsilon}\Delta$ δυναμοδύναμις·

ὁ δὲ ἐκ τετραγώνου ἐπὶ τὸν ἀπὸ τῆς αὐτῆς αὐτῷ πλευρᾶς κύβου πολυπλασιασθέντος δυναμόκυβος καὶ ἔστιν αὐτοῦ σημεῖον τὰ ΔΚ ἐπίσημον ἔχοντα Υ, ΔK^{Υ} δυναμόκυβος·

ὁ δὲ ἐκ κύβου ἑαυτὸν πολυπλασιάσαντος κυβόκυβος καὶ ἔστιν αὐτοῦ σημεῖον δύο κάππα ἐπίσημον ἔχοντα Υ, $K^{\Upsilon}K$ κυβόκυβος.

multiplied by the cubes formed from the same side;

cube-cubes, which are formed when cubes are multiplied by themselves;

and it is from the addition, subtraction, or multiplication of these numbers or from the ratio which they bear one to another or to their own sides that most arithmetical problems are formed; you will be able to solve them if you follow the method shown below.

Now each of these numbers, which have been given abbreviated names, is recognized as an element in arithmetical science; the *square* [of the unknown quantity][a] is called *dynamis* and its sign is Δ with the index Υ, that is Δ^{Υ};

the cube is called *cubus* and has for its sign K with the index Υ, that is K^{Υ};

the square multiplied by itself is called *dynamodynamis* and its sign is two deltas with the index Υ, that is $\Delta^{\Upsilon}\Delta$;

the square multiplied by the cube formed from the same root is called *dynamocubus* and its sign is ΔK with the index Υ, that is ΔK^{Υ};

the cube multiplied by itself is called *cubocubus* and its sign is two kappas with the index Υ, $K^{\Upsilon}K$.

[a] It is not here stated in so many words, but becomes obvious as the argument proceeds that δύναμις and its abbreviation are restricted to the square of the *unknown* quantity: the square of a determinate number is τετράγωνος. There is only one term, κύβος, for the cube both of a determinate and of the unknown quantity. The higher terms, when written in full as δυναμοδύναμις, δυναμόκυβος and κυβόκυβος, are used respectively for the fourth, fifth and sixth powers both of determinate quantities and of the unknown, but their abbreviations, and that for κύβος, are used to denote powers of the unknown only.

Ὁ δὲ μηδὲν τούτων τῶν ἰδιωμάτων κτησάμενος, ἔχων δὲ ἐν ἑαυτῷ πλῆθος μονάδων ἀόριστον, ἀριθμὸς καλεῖται καὶ ἔστιν αὐτοῦ σημεῖον τὸ ς.*

Ἔστι δὲ καὶ ἕτερον σημεῖον τὸ ἀμετάθετον τῶν ὡρισμένων, ἡ μονάς, καὶ ἔστιν αὐτῆς σημεῖον τὸ Μ ἐπίσημον ἔχον τὸ Ο, $\overset{\circ}{\mathrm{M}}$.

Ὥσπερ δὲ τῶν ἀριθμῶν τὰ ὁμώνυμα μόρια παρομοίως καλεῖται τοῖς ἀριθμοῖς, τοῦ μὲν τρία τὸ τρίτον, τοῦ δὲ τέσσαρα τὸ τέταρτον, οὕτως καὶ τῶν νῦν ἐπονομασθέντων ἀριθμῶν τὰ ὁμώνυμα μόρια κληθήσεται παρομοίως τοῖς ἀριθμοῖς·

τοῦ μὲν ἀριθμοῦ	τὸ ἀριθμοστόν,
τῆς δὲ δυνάμεως	τὸ δυναμοστόν,
τοῦ δὲ κύβου	τὸ κυβοστόν,
τῆς δὲ δυναμοδυνάμεως	τὸ δυναμοδυναμοστόν,
τοῦ δὲ δυναμοκύβου	τὸ δυναμοκυβοστόν,
τοῦ δὲ κυβοκύβου	τὸ κυβοκυβοστόν·

ἕξει δὲ ἕκαστον αὐτῶν ἐπὶ τὸ τοῦ ὁμωνύμου ἀριθμοῦ σημεῖον γραμμὴν Χ διαστέλλουσαν τὸ εἶδος.

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* I am entirely convinced by Heath's argument, based on the Bodleian MS. of Diophantus and general considerations, that this symbol is really the first two letters of ἀριθμός; this suggestion brings the symbol into line with Diophantus's abbreviations for δύναμις, κύβος, and so on. It may be declined throughout its cases, *e.g.*, ς$^{\hat{\omega}\nu}$ for the genitive plural, *infra* p. 552, line 5.

Diophantus has only one symbol for an unknown quantity, but his problems often lead to subsidiary equations involving other unknowns. He shows great ingenuity in isolating these subsidiary unknowns. In the translation I shall use

The number which has none of these characteristics, but merely has in it an undetermined multitude of units, is called *arithmos*, and its sign is ς [x].[a]

There is also another sign denoting the invariable element in determinate numbers, the unit, and its sign is M with the index O, that is $\overset{\circ}{M}$.

As in the case of numbers the corresponding fractions are called after the numbers, a *third* being called after 3 and a *fourth* after 4, so the functions named above will have reciprocals called after them:

arithmos [x]	*arithmoston* $\left[\frac{1}{x}\right]$,
dynamis [x^2]	*dynamoston* $\left[\frac{1}{x^2}\right]$,
cubus [x^3]	*cuboston* $\left[\frac{1}{x^3}\right]$,
dynamodynamis [x^4]	*dynamodynamoston* $\left[\frac{1}{x^4}\right]$,
dynamocubus [x^5]	*dynamocuboston* $\left[\frac{1}{x^5}\right]$,
cubocubus [x^6]	*cubocuboston* $\left[\frac{1}{x^6}\right]$.

And each of these will have the same sign as the corresponding process, but with the mark ✕ to distinguish its nature.[b]

different letters for the different unknowns as they occur, for example, x, z, m.

Diophantus does not admit negative or zero values of the unknown, but positive fractional values are admitted.

[b] So the symbol is printed by Tannery, but there are many variants in the mss.

Ibid. i., Praef., Dioph. ed. Tannery i. 12. 19-21

Λεῖψις ἐπὶ λεῖψιν πολλαπλασιασθεῖσα ποιεῖ ὕπαρξιν, λεῖψις δὲ ἐπὶ ὕπαρξιν ποιεῖ λεῖψιν, καὶ τῆς λείψεως σημεῖον Ψ ἐλλιπὲς κάτω νεῦον, ⋔.

(c) Determinate Equations

(i.) *Pure Determinate Equations*

Ibid. i., Praef., Dioph. ed. Tannery i. 14. 11-20

Μετὰ δὲ ταῦτα ἐὰν ἀπὸ προβλήματός τινος γένηται εἴδη τινὰ ἴσα εἴδεσι τοῖς αὐτοῖς, μὴ ὁμοπληθῆ δέ, ἀπὸ ἑκατέρων τῶν μερῶν δεήσει ἀφαιρεῖν τὰ ὅμοια ἀπὸ τῶν ὁμοίων, ἕως ἂν ἓν εἶδος ἑνὶ εἴδει ἴσον γένηται. ἐὰν δέ πως ἐν ὁποτέρῳ ἐνυπάρχῃ ἢ ἐν ἀμφοτέροις ἐν ἐλλείψεσί τινα εἴδη, δεήσει προσθεῖναι τὰ λείποντα εἴδη ἐν ἀμφοτέροις τοῖς μέρεσιν, ἕως ἂν ἑκατέρων τῶν μερῶν τὰ εἴδη ἐνυπάρχοντα γένηται, καὶ πάλιν ἀφελεῖν τὰ ὅμοια ἀπὸ τῶν ὁμοίων, ἕως ἂν ἑκατέρῳ τῶν μερῶν ἓν εἶδος καταλειφθῇ.

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[a] Lit. "a deficiency multiplied by a deficiency makes a forthcoming."

[b] The sign has nothing to do with Ψ, but I see no reason why Diophantus should not have described it by means of Ψ.

Ibid. i., Preface, Dioph. ed. Tannery i. 12. 19-21

A *minus* multiplied by a *minus* makes a *plus*,[a] a *minus* multiplied by a *plus* makes a *minus*, and the sign of a *minus* is a truncated Ψ turned upside down, that is ⋔.[b]

(c) Determinate Equations

(i.) *Pure*[c] *Determinate Equations*

Ibid. i., Preface, Dioph. ed. Tannery i. 14. 11-20

Next, if there result from a problem an equation in which certain terms are equal to terms of the same species, but with different coefficients, it will be necessary to subtract like from like on both sides until one term is found equal to one term. If perchance there be on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides become positive, and again to subtract like from like until on each side one term only is left.[d]

and cannot agree with Heath (*H.G.M.* ii. 459) that "the description is evidently interpolated." But Heath seems right in his conjecture, first made in 1885, that the sign ⋔ is a compendium for the root of the verb λείπειν, and is, in fact, a Λ with an I placed in the middle. When the sign is resolved in the manuscripts into a word, the dative λείψει is generally used, but there is no conclusive proof that Diophantus himself used this non-classical form.

[c] A *pure* equation is one containing only one power of the unknown, whatever its degree; a *mixed* equation contains more than one power of the unknown.

[d] In modern notation, Diophantus manipulates the equation until it is of the form $Ax^n = B$; as he recognizes only one value of x satisfying this equation, it is then considered solved.

(ii.) *Quadratic Equations*

Ibid. iv. 39, Dioph. ed. Tannery i. 298. 7–306. 8

Εὑρεῖν τρεῖς ἀριθμοὺς ὅπως ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου πρὸς τὴν ὑπεροχὴν τοῦ μέσου καὶ τοῦ ἐλάσσονος λόγον ἔχῃ δεδομένον, ἔτι δὲ καὶ σὺν δύο λαμβανόμενοι, ποιῶσι τετράγωνον.

Ἐπιτετάχθω δὴ τὴν ὑπεροχὴν τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου εἶναι $\gamma^{\pi\lambda}$.

Ἐπεὶ δὲ συναμφότερος ὁ μέσος καὶ ὁ ἐλάσσων ποιεῖ $\square^{ον}$, ποιείτω M̊ δ̅. ὁ ἄρα μέσος μείζων ἐστὶ δυάδος· ἔστω ς α̅ M̊ β̅. ὁ ἄρα ἐλάχιστος ἔσται M̊ β̅ ⋔ ς α̅.

Καὶ ἐπειδὴ ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου $\gamma^{\pi\lambda}$· ⟨ἐστί⟩,[1] καὶ ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου ς β̅, ἡ ἄρα ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου ἔσται ς ς̅, καὶ ὁ μείζων ἄρα ἔσται ς ζ̅ M̊ β̅.

Λοιπόν ἐστι δύο ἐπιτάγματα, τό τε συναμφότερον ⟨τὸν μείζονα καὶ τὸν ἐλάχιστον ποιεῖν $\square^{ον}$, καὶ τὸ τὸν μείζονα⟩[2] καὶ τὸν μέσον ποιεῖν $\square^{ον}$. καὶ γίνεταί μοι διπλῆ ἡ ἰσότης·

ς η̅ M̊ δ̅ ἴσ. $\square^{ῳ}$, καὶ ς ς̅ M̊ δ̅ ἴσ. $\square^{ῳ}$.

καὶ διὰ τὸ τὰς M̊ εἶναι τετραγωνικάς, εὐχερής ἐστιν ἡ ἴσωσις.

[1] ἐστί add. Bachet.

[2] τὸν μείζονα . . . τὸν μείζονα add. Tannery.

(ii.) *Quadratic Equations* [a]

Ibid. iv. 39, Dioph. ed. Tannery i. 298. 7–306. 8

To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.

Let it be laid down that the difference of the greatest and the middle has to the difference of the middle and the least the ratio $3:1$.

Since the sum of the middle term and the least makes a square, let it be 4. Then the middle term >2. Let it be $x+2$. Then the least term $=2-x$.

And since the difference of the greatest and the middle has to the difference of the middle and the least the ratio $3:1$, and the difference of the middle and the least is $2x$, therefore the difference of the greatest and the middle is $6x$, and therefore the greatest will be $7x+2$.

There remain two conditions, that the sum of the greatest and the least make a square and the sum of the greatest and the middle make a square. And I am left with the double equation [b]

$$8x+4=\text{a square},$$
$$6x+4=\text{a square}.$$

And as the units are squares, the equation is convenient to solve.

[a] The quadratic equation takes up only a small part of this problem, but the whole problem will give an excellent illustration of Diophantus's methods, and especially of his ingenuity in passing from one unknown to another. The geometrical solution of quadratic equations by the application of areas is treated in vol. i. pp. 192-215, and Heron's algebraical formula for solving quadratics, *supra*, pp. 502-505.

[b] For double equations, *v. infra* p. 543 n. *b*.

Πλάσσω ἀριθμοὺς δύο ἵνα ὁ ὑπ' αὐτῶν ᾖ ϛ β̄, καθὼς ἴσμεν διπλῆν ἰσότητα· ἔστω οὖν ϛ ∠' καὶ M̊ δ̄· καὶ γίνεται ὁ ϛ M̊ ρι̅β̅. ἐλθὼν ἐπὶ τὰς ὑποστάσεις, οὐ δύναμαι ἀφελεῖν ἀπὸ M̊ β̄ τὸν ϛ ᾱ τουτέστι τὰς M̊ ρι̅β̅· θέλω οὖν τὸν ϛ εὑρεθῆναι ἐλάττονα M̊ β̄, ὥστε καὶ ϛ ϛ̄ M̊ δ̄ ἐλάσσονες ἔσονται M̊ ι̅ϛ̅. ἐὰν γὰρ ἡ δυὰς ἐπὶ ϛ ϛ̄ γένηται καὶ προσλάβῃ M̊ δ̄, ποιεῖ M̊ ι̅ϛ̅.

Ἐπεὶ οὖν ζητῶ ϛ η̄ M̊ δ̄ ἴσ. □ῳ καὶ ϛ ϛ̄ M̊ δ̄ ἴσ. □ῳ, ἀλλὰ καὶ ὁ ἀπὸ τῆς δυάδος, τουτέστι M̊ δ̄, □ός ἐστι, γεγόνασι τρεῖς □οι, ϛ η̄ M̊ δ̄, καὶ ϛ ϛ̄ M̊ δ̄, καὶ M̊ δ̄, καὶ ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γον μέρος ἐστίν. ἀπῆκται οὖν μοι εἰς τὸ εὑρεῖν ⟨τρεῖς⟩[1] τετραγώνους, ὅπως ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γον μέρος ᾖ, ἔτι δὲ ὁ μὲν ἐλάχιστος ᾖ M̊ δ̄, ὁ δὲ μέσος ἐλάσσων M̊ ι̅ϛ̅.

[1] τρεῖς add. Bachet.

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[a] If we put

$$8x+4=(p+q)^2,$$
$$6x+4=(p-q)^2,$$

on subtracting, $2x=4pq$.

Substituting $2p=\frac{1}{2}x$, $2q=4$ (*i.e.*, $p=\frac{1}{4}x$, $q=2$) in the first equation we get

$$8x+4=(\tfrac{1}{4}x+2)^2,$$

or $112x=x^2$,

whence $x=112$.

I form two numbers whose product is $2x$, according to what we know about a double equation; let them be $\frac{1}{2}x$ and 4; and therefore $x=112$.[a] But, returning to the conditions, I cannot subtract x, that is 112, from 2; I desire, then, that x be found <2, so that $6x+4<16$. For $2 \cdot 6+4=16$.

Then since I seek to make $8x+4=$ a square, and $6x+4=$ a square, while $2 \cdot 2=4$ is a square, there are three squares, $8x+4$, $6x+4$, and 4, and the difference of the greatest and the middle is one-third[b] of the difference of the middle and least. My problem therefore resolves itself into finding three squares such that the difference of the greatest and the middle is one-third of the difference of the middle and least, and further such that the least $=4$ and the middle <16.

[a] This method of solving such equations is explicitly given by Diophantus in ii. 11, Dioph. ed. Tannery i. 96. 8-14: ἔσται ἄρα ὁ μὲν ϛ ᾱ Μ̊ β̄, ὁ δὲ ϛ ᾱ Μ̊ γ̄, ἴσ. □· καὶ τοῦτο τὸ εἶδος καλεῖται διπλοϊσότης· ἰσοῦται δὲ τὸν τρόπον τοῦτον. ἰδὼν τὴν ὑπεροχήν, ζήτει δύο ἀριθμοὺς ἵνα τὸ ὑπ' αὐτῶν ποιῇ τὴν ὑπεροχήν· εἰσὶ δὲ Μ̊ δ̄ καὶ Μ̊ᵒⁱ δˣ. τούτων ἤτοι τῆς ὑπεροχῆς τὸ ∠' ἐφ' ἑαυτὸ ἴσον ἐστὶ τῷ ἐλάσσονι, ἢ τῆς συνθέσεως τὸ ∠' ἐφ' ἑαυτὸ ἴσον τῷ μείζονι—"The equations will then be $x+2=$ a square, $x+3=$ a square: and this species is called a double equation. It is solved in this manner: observe the difference, and seek two [suitable] numbers whose product is equal to the difference; they are 4 and $\frac{1}{4}$. Then, either the square of half the difference of these numbers is equated to the lesser, or the square of half the sum to the greater."

[b] The ratio of the differences in this subordinate problem has, of course, nothing to do with the ratio of the differences in the main problem: the fact that they are reciprocals may lead the casual reader to suspect an error.

Τετάχθω ὁ μὲν ἐλάχιστος Μ̊ δ̄, ἡ δὲ τοῦ μέσου π^{λ} ⋔ ᾱ Μ̊ β̄· αὐτὸς ἄρα ἔσται ὁ $\square^{\text{ος}}$, Δ^{Y} ᾱ ⋔ δ̄ Μ̊ δ̄.

Ἐπεὶ οὖν ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου $\gamma^{\text{ον}}$ μέρος ἐστίν, καὶ ἔστιν ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου Δ^{Y} ᾱ ⋔ δ̄, ὥστε ἡ ὑπεροχὴ τοῦ μεγίστου καὶ τοῦ μέσου ἔσται Δ^{Y} γ^{χ} ⋔ ᾱ γ^{χ}· καὶ ἔστιν ὁ μέσος Δ^{Y} ᾱ ⋔ δ̄ Μ̊ δ̄· ὁ ἄρα μέγιστος ἔσται Δ^{Y} ᾱ γ^{χ} ⋔ ε̄ γ^{χ} Μ̊ δ̄ ἴσ. $\square^{\text{ῳ}}$· πάντα $\theta^{\text{κις}}$· Δ^{Y} ἄρα $\overline{\iota\beta}$ ⋔ $\overline{\mu\eta}$ Μ̊ $\overline{\lambda\varsigma}$ ἴσ. $\square^{\text{ῳ}}$ καὶ τὸ $\delta^{\text{ον}}$ αὐτῶν· Δ^{Y} γ̄ ⋔ $\overline{\iota\beta}$ Μ̊ θ̄ ἴσ. $\square^{\text{ῳ}}$.

Ἔτι δὲ θέλω τὸν μέσον τετράγωνον ἐλάσσονα εἶναι Μ̊ $\overline{\iota\varsigma}$, καὶ τὴν π^{λ} δηλαδὴ ἐλάσσονος Μ̊ δ̄. ἡ δὲ πλευρὰ τοῦ μέσου ἐστὶν ⋔ ᾱ Μ̊ β̄· ἐλάττονές εἰσι Μ̊ δ̄. καὶ κοινῶν ἀφαιρεθεισῶν τῶν β̄ Μ̊, ὁ ⋔ ἔσται ἐλάσσονος Μ̊ β̄.

Γέγονεν οὖν μοι Δ^{Y} γ̄ ⋔ $\overline{\iota\beta}$ Μ̊ θ̄ ἴσ. ποιῆσαι $\square^{\text{ῳ}}$. πλάσσω $\square^{\text{ον}}$ τινα ἀπὸ Μ̊ γ̄ λειπουσῶν ⋔ τινας· καὶ γίνεται ὁ ⋔ ἔκ τινος ἀριθμοῦ $\varsigma^{\text{κις}}$ γενομένου καὶ προσλαβόντος τὸν $\overline{\iota\beta}$, τουτέστι τῆς ἰσώσεως τῆς ⋔ $\overline{\iota\beta}$, καὶ μερισθέντος εἰς τὴν ὑπεροχὴν ᾗ ὑπερέχει ὁ ἀπὸ τοῦ ἀριθμοῦ $\square^{\text{ος}}$ τῶν Δ^{Y} τῶν ἐν τῇ ἰσώσει γ̄. ἀπῆκται οὖν μοι εἰς τὸ εὑρεῖν τινα ἀριθμόν, ὃς $\varsigma^{\text{κις}}$ γενόμενος καὶ προσλαβὼν Μ̊ $\overline{\iota\beta}$ καὶ μεριζόμενος εἰς τὴν ὑπεροχὴν ᾗ ὑπερέχει ὁ ἀπὸ τοῦ αὐτοῦ $\square^{\text{ος}}$ τριάδος, ποιεῖ τὴν παραβολὴν ἐλάσσονος Μ̊ β̄.

Let the least be taken as 4, and the side of the middle as $z+2$; then the square is z^2+4z+4.

Then since the difference of the greatest and the middle is one-third of the difference of the middle and the least, and the difference of the middle and the least is z^2+4z, so that the difference of the greatest and the least is $\frac{1}{3}z^2+1\frac{1}{3}z$, while the middle term is z^2+4z+4, therefore the greatest term $=1\frac{1}{3}z^2+5\frac{1}{3}z+4=$ a square. Multiply throughout by 9 :

$$12z^2+48z+36=\text{a square};$$

and take the fourth part :

$$3z^2+12z+9=\text{a square}.$$

Further, I desire that the middle square <16, whence clearly its side <4. But the side of the middle square is $z+2$, and so $z+2<4$. Take away 2 from each side, and $z<2$.

My equation is now

$$3z^2+12z+9=\text{a square}.$$
$$=(mz-3)^2, \text{ say.}^a$$

Then $$z=\frac{6m+12}{m^2-3},$$

and the equation to which my problem is now resolved is

$$\frac{6m+12}{m^2-3}<2,$$

i.e., $$<\frac{2}{1}.$$

[a] As a literal translation of the Greek at this point would be intolerably prolix, I have made free use of modern notation.

Ἔστω ὁ ζητούμενος ϛ ᾱ· οὗτος $\varsigma^{\kappa\iota\varsigma}$ γενόμενος καὶ προσλαβὼν $\overset{\circ}{M}$ ιβ, ποιεῖ ϛ ϛ̄ $\overset{\circ}{M}$ ιβ̄· ὁ δὲ ἀπ' αὐτοῦ $\square^{\text{ος}}$, ⋔ $\overset{\circ}{M}$ γ̄, ποιεῖ Δ^{Y} ᾱ ⋔ $\overset{\circ}{M}$ γ̄. θέλω οὖν ϛ ϛ̄ $\overset{\circ}{M}$ ιβ̄ μερίζεσθαι εἰς Δ^{Y} ᾱ ⋔ $\overset{\circ}{M}$ γ̄ καὶ ποιεῖν τὴν παραβολὴν ἐλάσσονος $\overset{\circ}{M}$ β̄. ἀλλὰ καὶ ὁ β̄ μεριζόμενος εἰς $\overset{\circ}{M}$ ᾱ, ποιεῖ τὴν παραβολὴν β̄· ὥστε ϛ ϛ̄ $\overset{\circ}{M}$ ιβ̄ πρὸς Δ^{Y} ᾱ ⋔ $\overset{\circ}{M}$ γ̄ ἐλάσσονα λόγον ἔχουσιν ἤπερ β̄ πρὸς ᾱ.

Καὶ χωρίον χωρίῳ ἄνισον· ὁ ἄρα ὑπὸ ϛ ϛ̄ $\overset{\circ}{M}$ ιβ̄ καὶ $\overset{\circ}{M}$ ᾱ ἐλάσσων ἐστὶν τοῦ ὑπὸ δυάδος καὶ Δ^{Y} ᾱ ⋔ $\overset{\circ}{M}$ γ̄, τουτέστιν ϛ ϛ̄ $\overset{\circ}{M}$ ιβ̄ ἐλάσσονές εἰσιν Δ^{Y} β̄ ⋔ $\overset{\circ}{M}$ ϛ̄. καὶ κοιναὶ προσκείσθωσαν αἱ $\overset{\circ}{M}$ ϛ̄. ϛ ϛ̄ $\overset{\circ}{M}$ ιη̄ ἐλάσσονες Δ^{Y} β̄.

Ὅταν δὲ τοιαύτην ἴσωσιν ἰσώσωμεν, ποιοῦμεν τῶν ϛ τὸ ∠' ἐφ' ἑαυτό, γίνεται θ̄, καὶ τὰς Δ^{Y} β̄ ἐπὶ τὰς $\overset{\circ}{M}$ ιη̄, γίνονται λϛ̄· πρόσθες τοῖς θ̄, γίνονται με̄, ὧν π^{λ}· οὐκ ἔλαττόν ἐστι $\overset{\circ}{M}$ ζ̄·[a] πρόσθες τὸ ἡμίσευμα τῶν ϛ· ⟨γίνεται οὐκ ἔλαττον $\overset{\circ}{M}$ ῑ· καὶ μέρισον εἰς τὰς Δ^{Y}·⟩[1] γίνεται οὐκ ἔλαττον $\overset{\circ}{M}$ ε̄.

Γέγονεν οὖν μοι Δ^{Y} γ̄ ϛ ιβ̄ $\overset{\circ}{M}$ θ̄ ἴσ. $\square^{\text{ῳ}}$ τῷ ἀπὸ π^{λ} $\overset{\circ}{M}$ γ̄ ⋔ ϛ ε̄, καὶ γίνεται ὁ ϛ $\overset{\circ}{M}\,\frac{\kappa\beta}{\mu\beta}$ τουτέστιν $\frac{\iota\alpha}{\kappa\alpha}$.

Τέταχα δὲ τὴν τοῦ μέσου $\square^{\text{ου}}$ π^{λ} ϛ ᾱ $\overset{\circ}{M}$ β̄·

[1] γίνεται . . . τὰς Δ^{Y} add. Tannery.

[a] This is not strictly true. But since $\sqrt{45}$ lies between 6 and 7, no smaller integral value than 7 will satisfy the conditions of the problem.

The inequality will be preserved when the term are cross-multiplied,

i.e., $$(6m+12) . 1 < 2 . (m^2-3);$$

i.e., $$6m+12 < 2m^2-6.$$

By adding 6 to both sides,

$$6m+18 < 2m^2.$$

When we solve such an equation, we multiply half the coefficient of x [or m] into itself—getting 9; then multiply the coefficient of x^2 into the units —$2 . 18 = 36$; add this last number to the 9—getting 45; take the square root—which is $\not< 7$ [a]; add half the coefficient of x—making a number $\not< 10$; and divide the result by the coefficient of x^2—getting a number $\not< 5$.[b]

My equation is therefore

$$3x^2+12x+9 = \text{a square on side } (3-5x),$$

and $$x = \frac{42}{22} = \frac{21}{11}.$$

I have made the side of the middle square to be

[a] This shows that Diophantus had a perfectly general formula for solving the equation

$$ax^2 = bx + c,$$

namely $$x = \frac{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + ac}}{a}.$$

From vi. 6 it becomes clear that he had a similar general formula for solving

$$ax^2 + bx = c,$$

and from v. 10 and vi. 22 it may be inferred that he had a general solution for

$$ax^2 + c = bx.$$

ἔσται ἡ τοῦ □ου πλ. $\overset{\circ}{M}\,\substack{\iota\alpha\\ \mu\gamma}$. αὐτὸς δὲ ὁ □ος $\overset{\circ}{M}\,\substack{\rho\alpha\kappa\\ ,\alpha\omega\mu\theta}$·

Ἔρχομαι οὖν ἐπὶ τὸ ἐξ ἀρχῆς καὶ τάσσω $\overset{\circ}{M}\,\substack{\rho\kappa\alpha\\ ,\alpha\omega\mu\theta}$, ὄντα □ον, ἴσ. τοῖς ϛ ϛ̄ Μ̊ δ̄· καὶ πάντα εἰς $\overline{\rho\kappa\alpha}$· καὶ γίνεται ὁ ϛ $\substack{\psi\kappa\varsigma\\ ,\alpha\tau\xi\epsilon}$, καὶ ἔστιν ἐλάσσων δυάδος.

Ἐπὶ τὰς ὑποστάσεις τοῦ προβλήματος τοῦ ἐξ ἀρχῆς· ὑπέστημεν δὴ τὸν μὲν μέσον ϛ ᾱ Μ̊ β̄, τὸν δὲ ἐλάχιστον Μ̊ β̄ ⋔ ϛ ᾱ, τὸν δὲ μέγιστον ϛ ζ̄ Μ̊ β̄. ἔσται ὁ μὲν μέγιστος $\overline{\alpha\,.\,,\alpha\zeta}$, ὁ δὲ βος $\overline{,\beta\omega\iota\zeta}$, ὁ δὲ ἐλάχιστος ὁ γος $\overline{\pi\zeta}$. καὶ ἐπεὶ τὸ μόριον, ἔστι τὸ $\overline{\psi\kappa\varsigma}$ων, οὐκ ἔστιν □ος, ϛον δέ ἐστιν αὐτοῦ, ἐὰν λάβωμεν $\overline{\rho\kappa\alpha}$, ὅ ἐστι □ος, πάντων οὖν τὸ ϛον, καὶ ὁμοίως ἔσται ὁ μὲν αος ρκαων $\overline{,\alpha\omega\lambda\delta\angle'}$, ὁ δὲ βος $\overline{\upsilon\xi\theta\angle'}$, ὁ δὲ γος $\overline{\iota\delta\angle'}$.

Καὶ ἐὰν ἐν ὁλοκλήροις θέλῃς ἵνα μὴ τὸ ∠′ ἐπιτρέχῃ, εἰς δον ἔμβαλε. καὶ ἔσται ὁ αος $\substack{\upsilon\pi\delta\\ ,\zeta\tau\lambda\eta}$, ὁ δὲ βος $\substack{\upsilon\pi\delta\\ ,\alpha\omega o\eta}$, ὁ δὲ γος $\substack{\upsilon\pi\delta\\ \nu\eta}$. καὶ ἡ ἀπόδειξις φανερά.

$x+2$; therefore the side will be $\frac{43}{11}$ and the square itself $\frac{1849}{121}$.

I return now to the original problem and make $\frac{1849}{121}$, which is a square, $=6x+4$. Multiplying by 121 throughout, I get $x=\frac{1365}{726}$, which is <2.

In the conditions of the original problem we made the middle term $=x+2$, the least $=2-x$, and the greatest $7x+2$.
Therefore

$$\text{the greatest} = \frac{11007}{726},$$
$$\text{the middle} = \frac{2817}{726},$$
$$\text{the least} = \frac{87}{726}.$$

Since the denominator, 726, is not a square, but its sixth part is, if we take 121, which is a square, and divide throughout by 6, then similarly the numbers are

$$\frac{1834\frac{1}{2}}{121}, \quad \frac{469\frac{1}{2}}{121}, \quad \frac{14\frac{1}{2}}{121}.$$

And if you prefer to use integers only, avoiding the $\frac{1}{2}$, multiply throughout by 4. Then the numbers will be

$$\frac{7338}{484}, \quad \frac{1878}{484}, \quad \frac{58}{484}.$$

And the proof is obvious.

(iii.) *Simultaneous Equations Leading to a Quadratic*

Ibid. i. 28, Dioph. ed. Tannery i. 62. 20–64. 10

Εὑρεῖν δύο ἀριθμοὺς ὅπως καὶ ἡ σύνθεσις αὐτῶν καὶ ἡ σύνθεσις τῶν ἀπ' αὐτῶν τετραγώνων ποιῇ δοθέντας ἀριθμούς.

Δεῖ δὴ τοὺς δὶς ἀπ' αὐτῶν τετραγώνους τοῦ ἀπὸ συναμφοτέρου αὐτῶν τετραγώνου ὑπερέχειν τετραγώνῳ. ἔστι δὲ καὶ τοῦτο πλασματικόν.

Ἐπιτετάχθω δὴ τὴν μὲν σύνθεσιν αὐτῶν ποιεῖν $\overset{\circ}{\mathrm{M}}\,\bar{\kappa}$, τὴν δὲ σύνθεσιν τῶν ἀπ' αὐτῶν τετραγώνων ποιεῖν $\overset{\circ}{\mathrm{M}}\,\overline{\sigma\eta}$.

Τετάχθω δὴ ἡ ὑπεροχὴ αὐτῶν ϛ $\bar{\beta}$. καὶ ἔστω ὁ μείζων ϛ $\bar{\alpha}$ καὶ $\overset{\circ}{\mathrm{M}}\,\bar{\iota}$, τῶν ἡμίσεων πάλιν τοῦ συνθέματος, ὁ δὲ ἐλάσσων $\overset{\circ}{\mathrm{M}}\,\bar{\iota}$ ⋔ ϛ $\bar{\alpha}$. καὶ μένει πάλιν τὸ μὲν σύνθεμα αὐτῶν $\overset{\circ}{\mathrm{M}}\,\bar{\kappa}$, ἡ δὲ ὑπεροχὴ ϛ $\bar{\beta}$.

Λοιπόν ἐστι καὶ τὸ σύνθεμα τῶν ἀπ' αὐτῶν τετραγώνων ποιεῖν $\overset{\circ}{\mathrm{M}}\,\overline{\sigma\eta}$· ἀλλὰ τὸ σύνθεμα τῶν ἀπ' αὐτῶν τετραγώνων ποιεῖ $\Delta^{\Upsilon}\,\bar{\beta}\,\overset{\circ}{\mathrm{M}}\,\bar{\sigma}$. ταῦτα ἴσα $\overset{\circ}{\mathrm{M}}\,\overline{\sigma\eta}$, καὶ γίνεται ὁ ϛ $\overset{\circ}{\mathrm{M}}\,\bar{\beta}$.

Ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν μείζων $\overset{\circ}{\mathrm{M}}\,\overline{\iota\beta}$, ὁ δὲ ἐλάσσων $\overset{\circ}{\mathrm{M}}\,\bar{\eta}$. καὶ ποιοῦσι τὰ τῆς προτάσεως.

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[a] In general terms, Diophantus's problem is to solve the simultaneous equations

$$\xi + \eta = 2a$$
$$\xi^2 + \eta^2 = A.$$

He says, in effect, let $\xi - \eta = 2x$;

then $\xi = a + x,\ \eta = a - x,$

(iii.) *Simultaneous Equations Leading to a Quadratic*

Ibid. i. 28, Dioph. ed. Tannery i. 62. 20–64. 10

To find two numbers such that their sum and the sum of their squares are given numbers.[a]

It is a necessary condition that double the sum of their squares exceed the square of their sum by a square. This is of the nature of a formula.[b]

Let it be required to make their sum 20 and the sum of their squares 208.

Let their difference be $2x$, and let the greater $=x+10$ (again adding half the sum) and the lesser $=10-x$.

Then again their sum is 20 and their difference $2x$.

It remains to make the sum of their squares 208. But the sum of their squares is $2x^2+200$.

Therefore $\qquad 2x^2+200=208,$

and $\qquad x=2.$

To return to the hypotheses—the greater $=12$ and the lesser $=8$. And these satisfy the conditions of the problem.

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and $\qquad (a+x)^2+(a-x)^2=A.$

i.e., $\qquad 2(a^2+x^2)=A.$

A procedure equivalent to the solution of the pair of simultaneous equations $\xi+\eta=2a$, $\xi\eta=A$, is given in i. 27, and a procedure equivalent to the solution of $\xi-\eta=2a$, $\xi\eta=A$, in i. 30.

[b] In other words, $2(\xi^2+\eta^2)-(\xi+\eta)^2=$ a square; it is, in fact, $(\xi-\eta)^2$. I have followed Heath in translating ἔστι δὲ καὶ τοῦτο πλασματικόν as "this is of the nature of a formula." Tannery evades the difficulty by translating "est et hoc formativum," but Bachet came nearer the mark with his "effictum aliunde." The meaning of πλασματικόν should be "easy to form a mould," *i.e.* the formula is easy to discover.

(iv.) *Cubic Equation*

Ibid. vi. 17, Dioph. ed. Tannery i. 432. 19–434. 22

Εὑρεῖν τρίγωνον ὀρθογώνιον ὅπως ὁ ἐν τῷ ἐμβαδῷ αὐτοῦ, προσλαβὼν τὸν ἐν τῇ ὑποτεινούσῃ, ποιῇ τετράγωνον, ὁ δὲ ἐν τῇ περιμέτρῳ αὐτοῦ ᾖ κύβος.

Τετάχθω ὁ ἐν τῷ ἐμβαδῷ αὐτοῦ ς ᾱ, ὁ δὲ ἐν τῇ ὑποτεινούσῃ αὐτοῦ M̊ τινῶν τετραγωνικῶν ⋔ ς ᾱ, ἔστω M̊ ι̅ϛ̅ ⋔ ς ᾱ.

Ἀλλ' ἐπεὶ ὑπεθέμεθα τὸν ἐν τῷ ἐμβαδῷ αὐτοῦ εἶναι ς ᾱ, ὁ ἄρα ὑπὸ τῶν περὶ τὴν ὀρθὴν αὐτοῦ γίνεται ς β̄. ἀλλὰ ς β̄ περιέχονται ὑπὸ ς ᾱ καὶ M̊ β̄· ἐὰν οὖν τάξωμεν μίαν τῶν ὀρθῶν M̊ β̄, ἔσται ἡ ἑτέρα ς ᾱ.

Καὶ γίνεται ἡ περίμετρος M̊ ι̅η̅ καὶ οὐκ ἔστι κύβος· ὁ δὲ ι̅η̅ γέγονεν ἔκ τινος $\square^{ου}$ καὶ M̊ β̄· δεήσει ἄρα εὑρεῖν $\square^{ον}$ τινα, ὅς, προσλαβὼν M̊ β̄, ποιεῖ κύβον, ὥστε κύβον $\square^{ου}$ ὑπερέχειν M̊ β̄.

Τετάχθω οὖν ἡ μὲν τοῦ $\square^{ου}$ πλ. ς ᾱ M̊ ᾱ, ἡ δὲ τοῦ κύβου ς ᾱ ⋔ M̊ ᾱ. γίνεται ὁ μὲν $\square^{ος}$, Δ^{Y} ᾱ ς β̄ M̊ ᾱ, ὁ δὲ κύβος, K^{Y} ᾱ ς γ̄ ⋔ Δ^{Y} γ̄ M̊ ᾱ. θέλω οὖν τὸν κύβον τὸν $\square^{ου}$ ὑπερέχειν δυάδι· ὁ ἄρα $\square^{ος}$ μετὰ δυάδος, τουτέστιν Δ^{Y} ᾱ ς β̄ M̊ γ̄, ἔστιν ἴσος K^{Y} ᾱ ς γ̄ ⋔ Δ^{Y} γ̄ M̊ ᾱ, ὅθεν ὁ ς εὑρίσκεται M̊ δ̄.

Ἔσται οὖν ἡ μὲν τοῦ $\square^{ου}$ πλ. M̊ ε̄, ἡ δὲ τοῦ

(iv.) *Cubic Equation* [a]

Ibid. vi. 17, Dioph. ed. Tannery i. 432. 19–434. 22

To find a right-angled triangle such that its area, added to one of the perpendiculars, makes a square, while its perimeter is a cube.

Let its area $=x$, and let its hypotenuse be some square number *minus* x, say $16-x$.

But since we supposed the area $=x$, therefore the product of the sides about the right angle $=2x$. But $2x$ can be factorized into x and 2; if, then, we make one of the sides about the right angle $=2$, the other $=x$.

The perimeter then becomes 18, which is not a cube; but 18 is made up of a square [16] $+2$. It shall be required, therefore, to find a square number which, when 2 is added, shall make a cube, so that the cube shall exceed the square by 2.

Let the side of the square $=m+1$ and that of the cube $m-1$. Then the square $=m^2+2m+1$ and the cube $=m^3+3m-3m^2-1$. Now I want the cube to exceed the square by 2. Therefore, by adding 2 to the square,

$$m^2+2m+3=m^3+3m-3m^2-1,$$

whence $$m=4.$$

Therefore the side of the square $=5$ and that of

[a] This is the only example of a cubic equation solved by Diophantus. For Archimedes' geometrical solution of a cubic equation, *v. supra*, pp. 126-163.

κύβου Μ̊ γ̄. αὐτοὶ ἄρα ὁ μὲν □ος Μ̊ κε̄, ὁ δὲ κύβος Μ̊ κζ̄.

Μεθυφίσταμαι οὖν τὸ ὀρθογώνιον, καὶ τάξας αὐτοῦ τὸ ἐμβαδὸν ς ᾱ, τάσσω τὴν ὑποτείνουσαν Μ̊ κε̄ ⋔ ς ᾱ· μένει δὲ καὶ ἡ βάσις Μ̊ β̄, ἡ δὲ κάθετος ς ᾱ.

Λοιπόν ἐστιν τὸν ἀπὸ τῆς ὑποτεινούσης ἴσον εἶναι τοῖς ἀπὸ τῶν περὶ τὴν ὀρθήν· γίνεται δὲ Δ^Υ ᾱ Μ̊ χκε̄ ⋔ ς ν̄· ἔσται ἴση Δ^Υ ᾱ Μ̊ δ̄. ὅθεν ὁ ς Μ̊ ν/χκα.

Ἐπὶ τὰς ὑποστάσεις καὶ μένει.

(d) Indeterminate Equations[a]

(i.) *Indeterminate Equations of the Second Degree*

(α) *Single Equations*

Ibid. ii. 20, Dioph. ed. Tannery i. 114. 11-22

Εὑρεῖν δύο ἀριθμοὺς ὅπως ὁ ἀπὸ τοῦ ἑκατέρου αὐτῶν τετράγωνος, προσλαβὼν τὸν λοιπόν, ποιῇ τετράγωνον.

Τετάχθω ὁ αος ς ᾱ, ὁ δὲ βος Μ̊ ᾱ ς β̄, ἵνα ὁ ἀπὸ τοῦ αου □ος, προσλαβὼν τὸν βον, ποιῇ □ον. λοιπόν ἐστι καὶ τὸν ἀπὸ τοῦ βου □ον, προσλαβόντα τὸν αον, ποιεῖν □ον· ἀλλ᾽ ὁ ἀπὸ τοῦ βου □ος, προσλαβὼν τὸν αον, ποιεῖ Δ^Υ δ̄ ς ε̄ Μ̊ ᾱ· ταῦτα ἴσα □ῳ.

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[a] Diophantus makes no mention of indeterminate equations of the first degree, presumably because he admits

the cube $=3$; and hence the square is 25 and the cube 27.

I now transform the right-angled [triangle], and, assuming its area to be x, I make the hypotenuse $= 25 - x$; the base remains $= 2$ and the perpendicular $= x$.

The condition is still left that the square on the hypotenuse is equal to the sum of the squares on the sides about the right angle;

i.e., $$x^2 + 625 - 50x = x^2 + 4,$$

whence $$x = \frac{621}{50}.$$

This satisfies the conditions.

(*d*) Indeterminate Equations [a]

(i.) *Indeterminate Equations of the Second Degree*

(α) *Single Equations*

Ibid. ii. 20, Dioph. ed. Tannery i. 114. 11-22

To find two numbers such that the square of either, added to the other, shall make a square.

Let the first be x, and the second $2x+1$, in order that the square on the first, added to the second, may make a square. There remains to be satisfied the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is $4x^2+5x+1$; and therefore this must be a square.

rational *fractional* solutions, and the whole point of solving an indeterminate equation of the first degree is to get a solution in integers.

Πλάσσω τὸν □ον ἀπὸ ς β̄ ⋔ Μ̊ β̄· αὐτὸς ἄρα ἔσται Δ^Υ δ̄ Μ̊ δ̄ ⋔ ς η̄· καὶ γίνεται ὁ ς $\frac{\iota\gamma}{\gamma}$.

Ἔσται ὁ μὲν αος $\frac{\iota\gamma}{\gamma}$, ὁ δὲ βος $\frac{\iota\gamma}{\iota\theta}$, καὶ ποιοῦσι τὸ πρόβλημα.

(β) *Double Equations*

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

Δοθέντα ἀριθμὸν διελεῖν εἰς τρεῖς ἀριθμοὺς ὅπως ὁ ὑπὸ τοῦ πρώτου καὶ τοῦ δευτέρου, ἐάν τε προσλάβῃ τὸν τρίτον, ἐάν τε λείψῃ, ποιῇ τετράγωνον. Ἔστω ὁ δοθεὶς ὁ ς̄.

Τετάχθω ὁ γος ς ᾱ, καὶ ὁ βος Μ̊ ἐλασσόνων τοῦ ς̄· ἔστω Μ̊ β̄· ὁ ἄρα αος ἔσται Μ̊ δ̄ ⋔ ς ᾱ· καὶ λοιπά ἐστι δύο ἐπιτάγματα, τὸν ὑπὸ αου καὶ βου, ἐάν τε προσλάβῃ τὸν γον, ἐάν τε λείψῃ, ποιεῖν □ον. καὶ γίνεται διπλῆ ἡ ἰσότης· Μ̊ η̄ ⋔ ς ᾱ ἴσ. □ῳ· καὶ Μ̊ η̄ ⋔ ς γ̄ ἴσ. □ῳ· καὶ οὐ ῥητόν

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[a] The problem, in its most general terms, is to solve the equation

$$Ax^2 + Bx + C = y^2.$$

Diophantus does not give a general solution, but takes a number of special cases. In this case A is a square number ($=a^2$, say), and in the equation

$$a^2x^2 + Bx + C = y^2$$

he apparently puts $y^2 = (ax - m)^2$,

where m is some integer,

whence $x = \dfrac{m^2 - C}{2am + B}$.

I form the square from $2x-2$; it will be $4x^2+4-8x$; and $x=\frac{3}{13}$.

The first number will be $\frac{3}{13}$, the second $\frac{19}{13}$, and they satisfy the conditions of the problem.[a]

(β) *Double Equations* [b]

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

To divide a given number into three parts such that the product of the first and second ± the third shall make a square.

Let the given number be 6.

Let the third part be x, and the second part any number <6, say 2; then the first part $=4-x$; and the two remaining conditions are that the product of the first and second ± the third = a square. There results the double equation

$$8-x\ = \text{a square},$$
$$8-3x = \text{a square}.$$

And this does not give a rational result since the ratio

[b] Diophantus's term for a double equation is διπλοϊσότης, διπλῆ ἰσότης or διπλῆ ἴσωσις. It always means with him that two different functions of the unknown have to be made simultaneously equal to two squares. The general equations are therefore

$$A_1x^2+B_1x+C_1=u_1^2,$$
$$A_2x^2+B_2x+C_2=u_2^2.$$

Diophantus solves several examples in which the terms in x^2 are missing, and also several forms of the general equation.

ἐστι διὰ τὸ μὴ εἶναι τοὺς ϛ πρὸς ἀλλήλους λόγον ἔχοντας ὃν □ος ἀριθμὸς πρὸς □ον ἀριθμόν.

Ἀλλὰ ὁ ϛ ὁ ᾱ μονάδι ἐλάσσων τοῦ β̄, οἱ δὲ ϛ γ̄ ὁμοίως μείζονες Μ̊· τοῦ β̄. ἀπῆκται οὖν μοι εἰς τὸ εὑρεῖν ἀριθμόν τινα, ὡς τὸν β̄, ἵνα ὁ Μ̊· αὐτοῦ μείζων, πρὸς τὸν Μ̊· ⟨αὐτοῦ ἐλάσσονα, λόγον ἔχῃ ὃν □ος ἀριθμὸς πρὸς⟩[1] □ον ἀριθμόν.

Ἔστω ἡ ζητούμενος ϛ ᾱ, καὶ ὁ Μ̊· ᾱ αὐτοῦ μείζων ἔσται ϛ ᾱ Μ̊ ᾱ, ὁ δὲ Μ̊· αὐτοῦ ἐλάσσων ϛ ᾱ ⋔ Μ̊ ᾱ. θέλομεν οὖν αὐτοὺς πρὸς ἀλλήλους λόγον ἔχειν ὃν □ος ἀριθμὸς πρὸς □ον ἀριθμόν. ἔστω ὃν δ̄ πρὸς ᾱ· ὥστε ϛ ᾱ ⋔ Μ̊ ᾱ ἐπὶ Μ̊ δ̄ γίνονται ϛ δ̄ ⋔ Μ̊ δ̄· καὶ ϛ ᾱ Μ̊ ᾱ ἐπὶ τὴν Μ̊ ᾱ ⟨γίνονται ϛ ᾱ Μ̊ ᾱ⟩.[2] καί εἰσιν οὗτοι οἱ ἐκκείμενοι ἀριθμοὶ λόγον ἔχοντες πρὸς ἀλλήλους ὃν ἔχει □ος ἀριθμὸς πρὸς □ον ἀριθμόν· νῦν ϛ δ̄ ⋔ Μ̊ δ̄ ἴσ. ϛ ᾱ Μ̊ ᾱ, καὶ γίνεται ὁ ϛ $\mathring{M}^{\gamma}_{\epsilon}$.

Τάσσω οὖν τὸν βον $\mathring{M}^{\gamma}_{\epsilon}$· ὁ γὰρ γος ἐστὶν ϛ ᾱ· ὁ ἄρα αος ἔσται $\mathring{M}^{\gamma}_{\iota\gamma}$ ⋔ ϛ ᾱ.

Λοιπὸν δεῖ εἶναι τὸ ἐπίταγμα, ἔστω τὸν ὑπὸ αου καὶ βου, προσλαβόντα τὸν γον, ποιεῖν □ον, καὶ λείψαντα τὸν γον, ποιεῖν □ον· ἀλλ' ὁ ὑπὸ αου καὶ βου, προσλαβὼν τὸν γον, ποιεῖ $\mathring{M}^{\theta}_{\xi\epsilon}$ ⋔ ϛ ω′ ἴσ. □ῳ· ⋔ δὲ τοῦ γου, ποιεῖ $\mathring{M}^{\theta}_{\xi\epsilon}$ ⋔ ϛ β̄ ω′ ἴσ. □ῳ. καὶ

of the coefficients of x is not the ratio of a square to a square.

But the coefficient 1 of x is $2-1$ and the coefficient 3 of x likewise is $2+1$; therefore my problem resolves itself into finding a number to take the place of 2 such that (the number $+1$) bears to (the number -1) the same ratio as a square to a square.

Let the number sought be y; then (the number $+1$) $=y+1$, and (the number -1) $=y-1$. We require these to have the ratio of a square to a square, say $4:1$. Now $(y-1) \cdot 4=4y-4$ and $(y+1) \cdot 1=y+1$. And these are the numbers having the ratio of a square to a square. Now I put

$$4y-4=y+1,$$

giving

$$y=\frac{5}{3}.$$

Therefore I make the second part $\frac{5}{3}$, for the third $=x$; and therefore the first $=\frac{13}{3}-x$.

There remains the condition, that the product of the first and second $\pm$ the third $=$ a square. But the product of the first and second $+$ the third $=$

$$\frac{65}{9}-\frac{2}{3}x=\text{a square},$$

and the product of the first and second $-$ the third $=$

$$\frac{65}{9}-2\tfrac{2}{3}x=\text{a square}.$$

[1] αὐτοῦ . . . πρὸς add. Bachet.

[2] γίνονται ϛ ᾱ $\overset{\circ}{\mathrm{M}}$ ᾱ add. Tannery.

πάντα ἐπὶ τὸν $\bar{\theta}$, καὶ γίνονται $\overset{\circ}{M}\,\overline{\xi\epsilon}\ \pitchfork\ \varsigma\,\overline{\varsigma}$ ἴσ. $\square^{\omega}$, καὶ $\overset{\circ}{M}\,\overline{\xi\epsilon}\ \pitchfork\ \varsigma\,\overline{\kappa\delta}$ ἴσ. $\square^{\omega}$. καὶ ἐξισῶ, τοὺς ς τῆς μείζονος ἰσότητος ποιήσας δ$^{\kappa\iota\varsigma}$, καὶ ἔστι

$\overset{\circ}{M}\,\overline{\sigma\xi}\ \pitchfork\ \varsigma\,\overline{\kappa\delta}$ ἴσ. $\square^{\omega}$ καὶ $\overset{\circ}{M}\,\overline{\xi\epsilon}\ \pitchfork\ \varsigma\,\overline{\kappa\delta}$ ἴσ. $\square^{\omega}$.

Νῦν τούτων λαμβάνω τὴν ὑπεροχὴν καὶ ἔστι $\overset{\circ}{M}\,\overline{\rho\text{ϙ}\epsilon}$· καὶ ἐκτίθεμαι δύο ἀριθμοὺς ὧν τὸ ὑπό ἐστι $\overset{\circ}{M}\,\overline{\rho\text{ϙ}\epsilon}$, καί εἰσι $\overline{\iota\epsilon}$ καὶ $\overline{\iota\gamma}$· καὶ τῆς τούτων ὑπεροχῆς τὸ ∠′ ἐφ' ἑαυτὸ ἴσον ἐστὶ τῷ ἐλάσσονι $\square^{\omega}$, καὶ γίνεται ὁ ς γ^{ου} $\bar{\bar{\eta}}$.

Ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν α^{ος} $\bar{\epsilon}$, ὁ δὲ β^{ος} $\bar{\epsilon}$, ὁ δὲ γ^{ος} $\bar{\bar{\eta}}$. καὶ ἡ ἀπόδειξις φανερά.

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[a] These are a pair of equations of the form

$$am^2x + a = u^2,$$
$$an^2x + b = v^2.$$

Multiply by n^2, m^2 respectively, getting, say

$$am^2n^2x + an^2 = u'^2,$$
$$am^2n^2x + bm^2 = v'^2.$$

$\therefore$ $$an^2 - bm^2 = u'^2 - v'^2.$$

Let $$an^2 - bm^2 = pq,$$

and put $$u' + v' = p,$$
$$u' - v' = q;$$

$\therefore$ $$u'^2 = \tfrac{1}{4}(p+q)^2,\ v'^2 = \tfrac{1}{4}(p-q)^2,$$

and so $$am^2n^2x + an^2 = \tfrac{1}{4}(p+q)^2,$$
$$am^2n^2x + bm^2 = \tfrac{1}{4}(p-q)^2;$$

whence, from either,

$$x = \frac{\tfrac{1}{4}(p^2+q^2) - \tfrac{1}{2}(an^2+bm^2)}{am^2n^2}.$$

Multiply throughout by 9, getting

$$65-6x=\text{a square}$$

and

$$65-24x=\text{a square.}^{a}$$

Equating the coefficients of x by multiplying the first equation by 4, I get

$$260-24x=\text{a square}$$

and

$$65-24x=\text{a square}$$

Now I take their difference, which is 195, and split it into the two factors 15 and 13. Squaring the half of their difference, and equating the result to the lesser square, I get $x=\frac{8}{3}$.

Returning to the conditions—the first part will be $\frac{5}{3}$, the second $\frac{5}{3}$, and the third $\frac{8}{3}$. And the proof is obvious.

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[a] This is the procedure indicated by Diophantus. In his example,

$$p=15,\ q=13,$$

and $$\{\tfrac{1}{2}(15-13)\}^2=65-24x,$$

whence $$24x=64,\ \text{and}\ x=\frac{8}{3}.$$

(ii.) *Indeterminate Equations of Higher Degree*

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

Εὑρεῖν δύο ἀριθμούς, ὅπως ὁ ἀπὸ τοῦ πρώτου κύβος προσλαβὼν τὸν δεύτερον ποιῇ κύβον, ὁ δὲ ἀπὸ τοῦ δευτέρου τετράγωνος προσλαβὼν τὸν πρῶτον ποιῇ τετράγωνον.

Τετάχθω ὁ α$^{\text{ος}}$ ς ᾱ· ὁ ἄρα β$^{\text{ος}}$ ἔσται Μ̊ κυβικαὶ η̄ ⋔ Κ$^{\text{Υ}}$ ᾱ. καὶ γίνεται ὁ ἀπὸ τοῦ α$^{\text{ου}}$ κύβος, προσλαβὼν τὸν β$^{\text{ον}}$, κύβος.

Λοιπόν ἐστι καὶ τὸν ἀπὸ τοῦ β$^{\text{ου}}$ □$^{\text{ον}}$, προσλαβόντα τὸν α$^{\text{ον}}$, ποιεῖν □$^{\text{ον}}$. ἀλλ' ὁ ἀπὸ τοῦ β$^{\text{ου}}$ □$^{\text{ος}}$, προσλαβὼν τὸν α$^{\text{ον}}$, ποιεῖ Κ$^{\text{Υ}}$Κ ᾱ ς ᾱ Μ̊ ξ̅δ̅ ⋔ Κ$^{\text{Υ}}$ ι̅ϛ̅· ⟨ταῦτα ἴσα □$^{\text{ῳ}}$ τῷ ἀπὸ π$^{\text{λ}}$ Κ$^{\text{Υ}}$ ᾱ Μ̊ η̄, τουτέστι Κ$^{\text{Υ}}$Κ ᾱ Κ$^{\text{Υ}}$ ι̅ϛ̅ Μ̊ ξ̅δ̅·⟩[1] καὶ κοινῶν προστιθεμένων τῶν λειπομένων καὶ ἀφαιρουμένων τῶν ὁμοίων ἀπὸ ὁμοίων, λοιποὶ Κ$^{\text{Υ}}$ λ̅β̅ ἴσοι ς ᾱ· καὶ πάντα παρὰ ς· Δ$^{\text{Υ}}$ λ̅β̅ ἴσαι Μ̊ ᾱ.

Καὶ ἔστιν ἡ Μ̊ □$^{\text{ος}}$, καὶ Δ$^{\text{Υ}}$ λ̅β̅ εἰ ἦσαν □$^{\text{οι}}$, λελυμένη ἄν μοι ἦν ἡ ἴσωσις· ἀλλ' αἱ Δ$^{\text{Υ}}$ λ̅β̅ εἰσὶν ἐκ τῶν δὶς Κ$^{\text{Υ}}$ ι̅ϛ̅· οἱ δὲ Κ$^{\text{Υ}}$ ι̅ϛ̅ εἰσιν ὑπὸ τῶν δὶς Μ̊ η̄

[1] ταῦτα . . . Μ̊ ξ̅δ̅ add. Bachet.

[a] As with equations of the second degree, these may be single or double. Single equations always take the form that an expression in x, of a degree not exceeding the sixth, is to be made equal to a square or cube. The general form is therefore

$$A_0x^6 + A_1x^5 + \ . \ . \ . \ + A_6 = y^2 \text{ or } y^3.$$

Diophantus solves a number of special cases of different degrees.

In double equations, one expression is made equal to a

(ii.) *Indeterminate Equations of Higher Degree* [a]

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

To find two numbers such that the cube of the first added to the second shall make a cube, and the square of the second added to the first shall make a square.

Let the first number be x. Then the second will be a cube number less x^3, say $8-x^3$. And the cube of the first, added to the second, makes a cube.

There remains the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is $x^6+x+64-16x^3$. Let this be equal to $(x^3+8)^2$, that is to x^6+16x^3+64.[b] Then, by adding or subtracting like terms,

$$32x^3=x;$$

and, after dividing by x,

$$32x^2=1.$$

Now 1 is a square, and if $32x^2$ were a square, my equation would be soluble. But $32x^2$ is formed from $2.16x^3$, and $16x^3$ is $(2.8)(x^3)$, that is, it is formed

cube and the other to a square, but only a few simple cases are solved by Diophantus.

[b] The general type of the equation is

$$x^6-Ax^3+Bx+c^2=y^2.$$

Put $y=x^3+c$, then $$x^2=\frac{B}{A+2c},$$

and if the right-hand expression is a square, there is a rational solution.

In the case of the equation $x^6-16x^3+x+64=y^2$ it is not a square, and Diophantus replaces the equation by another, $x^6-128x^3+x+4096=y^2$, in which it is a square.

καὶ τοῦ K^{Y} ᾱ, τουτέστι δὶς τῶν $\mathring{M}$ η̄· ὥστε αἱ $\overline{\lambda\beta}$ Δ^{Y} ἐκ $\delta^{\text{πλ}}$ τῶν η̄ $\mathring{M}$. γέγονεν οὖν μοι εὑρεῖν κύβον ὃς $\delta^{\text{πλ}}$ γενόμενος ποιεῖ $\square^{\text{ον}}$.

Ἔστω ὁ ζητούμενος K^{Y} ᾱ· οὗτος $\delta^{\text{πλ}}$ γενόμενος ποιεῖ K^{Y} δ̄ ἴσ. $\square^{\varphi}$. ἔστω Δ^{Y} $\overline{\iota\varsigma}$· καὶ γίνεται ὁ ϛ $\mathring{M}$ δ̄. ἐπὶ τὰς ὑποστάσεις· ἔσται ὁ K^{Y} $\mathring{M}$ $\overline{\xi\delta}$.

Τάσσω ἄρα τὸν $\beta^{\text{ον}}$ $\mathring{M}$ $\overline{\xi\delta}$ ⋔ K^{Y} ᾱ. καὶ λοιπόν ἐστι τὸν ἀπὸ τοῦ $\beta^{\text{ου}}$ $\square^{\text{ον}}$ προσλαβόντα τὸν $\alpha^{\text{ον}}$ ποιεῖν $\square^{\text{ον}}$. ἀλλὰ ὁ ἀπὸ τοῦ $\beta^{\text{ου}}$ προσλαβὼν τὸν $\alpha^{\text{ον}}$ ποιεῖ $K^{Y}K$ ᾱ $\mathring{M}$ $\overline{,\delta ϟ\varsigma}$ ϛ ᾱ ⋔ K^{Y} $\overline{\rho\kappa\eta}$ ἴσ. $\square^{\varphi}$ τῷ ἀπὸ π^{λ}· K^{Y} ᾱ $\mathring{M}$ $\overline{\xi\delta}$· καὶ γίνεται ὁ $\square^{\text{ος}}$ $K^{Y}K$ ᾱ $\mathring{M}$ $\overline{,\delta ϟ\varsigma}$ K^{Y} $\overline{\rho\kappa\eta}$. καὶ γίνονται λοιποὶ K^{Y} $\overline{\sigma\nu\varsigma}$ ἴσ. ϛ ᾱ. καὶ γίνεται ὁ ϛ ἑνὸς $\iota\varsigma^{\text{ων}}$.

Ἐπὶ τὰς ὑποστάσεις· ἔσται ὁ $\alpha^{\text{ος}}$ ἑνὸς $\iota\varsigma^{\text{ων}}$, ὁ δὲ $\beta^{\text{ος}}$ $\overset{,\delta ϟ\varsigma}{\kappa\varsigma}$. ,βρμγ·

(*e*) Theory of Numbers: Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

Τὸν ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους.

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* It was on this proposition that Fermat wrote a famous note: "On the other hand, it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which, however, the margin is not large enough

from 2 . 8. Therefore $32x^2$ is formed from 4 . 8. My problem therefore becomes to find a cube which, when multiplied by 4, makes a square.

Let the number sought be y^3. Then $4y^3$ = a square $= 16y^2$ say ; whence $y = 4$. Returning to the conditions—the cube will be 64.

I therefore take the second number as $64 - x^3$. There remains the condition that the square on the second added to the first shall make a square. But the square on the second added to the first =

$$\begin{aligned} x^6 + 4096 + x - 128x^3 &= \text{a square} \\ &= (x^3 + 64)^2, \text{ say,} \\ &= x^6 + 4096 + 128x^3. \end{aligned}$$

On taking away the common terms,

$$256x^3 = x,$$

and $$x = \frac{1}{16}.$$

Returning to the conditions—

first number $= \frac{1}{16}$, second number $= \frac{262143}{4096}$.

(e) Theory of Numbers : Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

To divide a given square number into two squares.[a]

to contain." Fermat claimed, in other words, to have proved that $x^m + y^m = z^m$ cannot be solved in rational numbers if $m > 2$. Despite the efforts of many great mathematicians, a proof of this general theorem is still lacking.

Fermat's notes, which established the modern Theory of Numbers, were published in 1670 in Bachet's second edition of the works of Diophantus.

Ἐπιτετάχθω δὴ τὸν ιϛ̅ διελεῖν εἰς δύο τετραγώνους.

Καὶ τετάχθω ὁ α^{ος} Δ^{Y} ᾱ, ὁ ἄρα ἕτερος ἔσται Μ̊ ιϛ̅ ⋔ Δ^{Y} ᾱ· δεήσει ἄρα Μ̊ ιϛ̅ ⋔ Δ^{Y} ᾱ ἴσας εἶναι □^{ῳ}.

Πλάσσω τὸν □^{ον} ἀπὸ ς^{ῶν} ὅσων δήποτε ⋔ τοσούτων Μ̊ ὅσων ἐστὶν ἡ τῶν ιϛ̅ Μ̊ πλευρά· ἔστω ς β̅ ⋔ Μ̊ δ̅. αὐτὸς ἄρα ὁ □^{ος} ἔσται Δ^{Y} δ̅ Μ̊ ιϛ̅ ⋔ ς ιϛ̅· ταῦτα ἴσα Μ̊ ιϛ̅ ⋔ Δ^{Y} ᾱ. κοινὴ προσκείσθω ἡ λεῖψις καὶ ἀπὸ ὁμοίων ὅμοια.

Δ^{Y} ἄρα ε̅ ἴσαι ς ιϛ̅, καὶ γίνεται ὁ ς ιϛ̅ πέμπτων.

Ἔσται ὁ μὲν $\overset{\kappa\epsilon}{\sigma\nu\varsigma}$, ὁ δὲ $\overset{\kappa\epsilon}{\rho\mu\delta}$, καὶ οἱ δύο συντεθέντες ποιοῦσι $\overset{\kappa\epsilon}{\upsilon}$, ἤτοι Μ̊ ιϛ̅, καὶ ἔστιν ἑκάτερος τετράγωνος.

Ibid. v. 11, Dioph. ed. Tannery i. 342. 13–346. 12

Μονάδα διελεῖν εἰς τρεῖς ἀριθμοὺς καὶ προσθεῖναι ἑκάστῳ αὐτῶν πρότερον τὸν αὐτὸν δοθέντα καὶ ποιεῖν ἕκαστον τετράγωνον.

Δεῖ δὴ τὸν διδόμενον ἀριθμὸν μήτε δυάδα εἶναι μήτε τινὰ τῶν ἀπὸ δυάδος ὀκτάδι παραυξανομένων.

Ἐπιτετάχθω δὴ τὴν Μ̊ διελεῖν εἰς τρεῖς ἀριθμοὺς καὶ προσθεῖναι ἑκάστῳ Μ̊ γ̅ καὶ ποιεῖν ἕκαστον □^{ον}.

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[a] Lit. "I take the square from any number of ἀριθμοί *minus* as many units as there are in the side of 16."

[b] *i.e.*, a number of the form $3(8n+2)+1$ or $24n+7$ cannot be the sum of three squares. In fact, a number of the form $8n+7$ cannot be the sum of three squares, but there are other

Let it be required to divide 16 into two squares.

And let the first square $= x^2$; then the other will be $16 - x^2$; it shall be required therefore to make

$$16 - x^2 = \text{a square.}$$

I take a square of the form[a] $(mx - 4)^2$, m being any integer and 4 the root of 16; for example, let the side be $2x - 4$, and the square itself $4x^2 + 16 - 16x$. Then

$$4x^2 + 16 - 16x = 16 - x^2.$$

Add to both sides the negative terms and take like from like. Then

$$5x^2 = 16x,$$

and $$x = \frac{16}{5}.$$

One number will therefore be $\frac{256}{25}$, the other $\frac{144}{25}$, and their sum is $\frac{400}{25}$ or 16, and each is a square.

Ibid. v. 11, Dioph. ed. Tannery i. 342. 13–346. 12

To divide unity into three parts such that, if we add the same number to each of the parts, the results shall all be squares.

It is necessary that the given number be neither 2 nor any multiple of 8 increased by 2.[b]

Let it be required to divide unity into three parts such that, when 3 is added to each, the results shall all be squares.

numbers not of this form which also are not the sum of three squares. Fermat showed that, if $3a + 1$ is the sum of three squares, then it cannot be of the form $4^n(24k + 7)$ or $4^n(8k + 7)$, where $k = 0$ or any integer.

Πάλιν δεῖ τὸν $\bar{\iota}$ διελεῖν εἰς τρεῖς $\square^{\text{ους}}$ ὅπως ἕκαστος αὐτῶν μείζων ᾖ $\overset{\circ}{M}\,\bar{\gamma}$. ἐὰν οὖν πάλιν τὸν $\bar{\iota}$ διέλωμεν εἰς τρεῖς $\square^{\text{ους}}$, τῇ τῆς παρισότητος ἀγωγῇ, ἔσται ἕκαστος αὐτῶν μείζων τριάδος καὶ δυνησόμεθα, ἀφ' ἑκάστου αὐτῶν ἀφελόντες $\overset{\circ}{M}\,\bar{\gamma}$, ἔχειν εἰς οὓς ἡ $\overset{\circ}{M}$ διαιρεῖται.

Λαμβάνομεν ἄρτι τοῦ $\bar{\iota}$ τὸ $\gamma^{\text{ον}}$, γί. $\bar{\gamma}\,\gamma^{\times}$, καὶ ζητοῦμεν τί προστιθέντες μόριον τετραγωνικὸν ταῖς $\overset{\circ}{M}\,\bar{\gamma}\,\gamma^{\times}$, ποιήσομεν $\square^{\text{ον}}$· πάντα $\theta^{\text{κις}}$. δεῖ καὶ τῷ $\bar{\lambda}$ προσθεῖναί τι μόριον τετραγωνικὸν καὶ ποιεῖν τὸν ὅλον $\square^{\text{ον}}$.

[a]Ἔστω τὸ προστιθέμενον μόριον $\Delta^{Y\times}\,\bar{\alpha}$· καὶ πάντα ἐπὶ Δ^{Y}· γίνονται $\Delta^{Y}\,\bar{\lambda}\,\overset{\circ}{M}\,\bar{\alpha}$ ἴσ. $\square^{\varphi}$· τῷ ἀπὸ πλευρᾶς $\varsigma\,\bar{\epsilon}\,\overset{\circ}{M}\,\bar{\alpha}$· γίνεται ὁ $\square^{\text{ος}}$ $\Delta^{Y}\,\overline{\kappa\epsilon}\,\varsigma\,\bar{\iota}\,\overset{\circ}{M}\,\bar{\alpha}$ ἴσ. $\Delta^{Y}\,\bar{\lambda}\,\overset{\circ}{M}\,\bar{\alpha}$· ὅθεν ὁ ς $\overset{\circ}{M}\,\bar{\beta}$, ἡ Δ^{Y} $\overset{\circ}{M}\,\bar{\delta}$, τὸ $\Delta^{Y\times}$ $\overset{\circ}{M}\,\bar{\delta}^{\times}$.

Εἰ οὖν ταῖς $\overset{\circ}{M}\,\bar{\lambda}$ προστίθεται $\overset{\circ}{M}\,\bar{\delta}^{\times}$, ταῖς $\overset{\circ}{M}\,\bar{\gamma}\,\gamma^{\times}$ προστεθήσεται $\lambda\varsigma^{\times}$ καὶ γίνεται $\dfrac{\lambda\varsigma}{\rho\kappa\alpha}$· δεῖ οὖν τὸν $\bar{\iota}$ διελεῖν εἰς τρεῖς $\square^{\text{ους}}$ ὅπως ἑκάστου $\square^{\text{ου}}$ ἡ πλευρὰ πάρισος ᾖ $\overset{\circ}{M}\,\dfrac{\varsigma}{\iota\alpha}$.

Ἀλλὰ καὶ ὁ $\bar{\iota}$ σύγκειται ἐκ δύο $\square^{\text{ων}}$, τοῦ τε $\bar{\theta}$ καὶ τῆς $\overset{\circ}{M}$. διαιροῦμεν τὴν $\overset{\circ}{M}$ εἰς δύο $\square^{\text{ους}}$ τά τε $\dfrac{\kappa\epsilon}{\theta}$ καὶ τὰ $\dfrac{\kappa\epsilon}{\iota\varsigma}$, ὥστε τὸν $\bar{\iota}$ συγκεῖσθαι ἐκ τριῶν $\square^{\text{ων}}$,

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[a] The method has been explained in v. 19, where it is proposed to divide 13 into two squares each$>$6. It will be sufficiently obvious from this example. The method is also used in v. 10, 12, 13, 14.

Then it is required to divide 10 into three squares such that each of them > 3. If then we divide 10 into three squares, according to the method of approximation,[a] each of them will be > 3 and, by taking 3 from each, we shall be able to obtain the parts into which unity is to be divided.

We take, therefore, the third part of 10, which is $3\frac{1}{3}$, and try by adding some square part to $3\frac{1}{3}$ to make a square. On multiplying throughout by 9, it is required to add to 30 some square part which will make the whole a square.

Let the added part be $\frac{1}{x^2}$; multiply throughout by x^2; then

$$30x^2 + 1 = \text{a square}.$$

Let the root be $5x + 1$; then, squaring,

$$25x^2 + 10x + 1 = 30x^2 + 1;$$

whence

$$x = 2,\ x^2 = 4,\ \frac{1}{x^2} = \frac{1}{4}.$$

If, then, to 30 there be added $\frac{1}{4}$, to $3\frac{1}{3}$ there is added $\frac{1}{36}$, and the result is $\frac{121}{36}$. It is therefore required to divide 10 into three squares such that the side of each shall approximate to $\frac{11}{6}$.

But 10 is composed of two squares, 9 and 1. We divide 1 into two squares, $\frac{9}{25}$ and $\frac{16}{25}$, so that 10 is composed of three squares, 9, $\frac{9}{25}$ and $\frac{16}{25}$. It is there-

ἔκ τε τοῦ $\bar{\theta}$ καὶ τοῦ $\overset{\kappa\epsilon}{\iota\varsigma}$ καὶ τοῦ $\overset{\kappa\epsilon}{\theta}$. δεῖ οὖν ἑκάστην τῶν πλ· τούτων παρασκευάσαι πάρισον $\overset{\varsigma}{\iota\alpha}$.

Ἀλλὰ καὶ αἱ πλ· αὐτῶν εἰσιν $\overset{\circ}{M}\,\bar{\gamma}$ καὶ $\overset{\circ}{M}\,\overset{\epsilon}{\delta}$ καὶ $\overset{\circ}{M}\,\overset{\epsilon}{\gamma}$· καὶ πάντα λκις· καὶ γίνονται $\overset{\circ}{M}\,\bar{\varsigma}$ καὶ $\overset{\circ}{M}\,\overline{\kappa\delta}$ καὶ $\overset{\circ}{M}\,\overline{\iota\eta}$. τὰ δὲ $\overline{\iota\alpha}$ ϛα γίνονται $\overset{\circ}{M}\,\overline{\nu\epsilon}$· δεῖ οὖν ἑκάστην πλ· κατασκευάσαι $\overline{\nu\epsilon}$.

Πλάσσομεν ἑνὸς πλευρὰν $\overset{\circ}{M}\,\bar{\gamma} \pitchfork ς\,\overline{\lambda\epsilon}$, ἑτέρου δὴ ς $\overline{\lambda\alpha}$ $\overset{\circ}{M}\,\bar{\delta}$ εων, τοῦ δὲ ἑτέρου ς $\overline{\lambda\zeta}$ $\overset{\circ}{M}\,\bar{\gamma}$ εων. γίνονται οἱ ἀπὸ τῶν εἰρημένων □οι, $\Delta^{Y}\,\overline{,\gamma\phi\nu\epsilon}\ \overset{\circ}{M}\,\bar{\iota} \pitchfork ς\,\overline{\rho\iota\varsigma}$· ταῦτα ἴσα $\overset{\circ}{M}\,\bar{\iota}$. ὅθεν εὑρίσκεται ὁ ς $\overset{,\gamma\phi\nu\epsilon}{\rho\iota\varsigma}$.

Ἐπὶ τὰς ὑποστάσεις· καὶ γίνονται αἱ πλευραὶ τῶν τετραγώνων δοθεῖσαι, ὥστε καὶ αὐτοί. τὰ λοιπὰ δῆλα.

Ibid. iv. 29, Dioph. ed. Tannery i. 258. 19–260. 16

Εὑρεῖν τέσσαρας ἀριθμοὺς ⟨τετραγώνους⟩, οἳ συντεθέντες καὶ προσλαβόντες τὰς ἰδίας πλευρὰς συντεθείσας ποιοῦσι δοθέντα ἀριθμόν.

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[a] The sides are, in fact, $\frac{1321}{711}, \frac{1288}{711}, \frac{1285}{711}$, and the squares are $\frac{1745041}{505521}, \frac{1658944}{505521}, \frac{1651225}{505521}$.

fore required to make each of the sides approximate to $\frac{11}{6}$.

But their sides are 3, $\frac{4}{5}$ and $\frac{3}{5}$. Multiply throughout by 30, getting 90, 24 and 18; and $\frac{11}{6}$ [when multiplied by 30] becomes 55. It is therefore required to make each side approximate to 55.

[Now $3 > \frac{55}{30}$ by $\frac{35}{30}$, $\frac{4}{5} < \frac{55}{30}$ by $\frac{31}{30}$, and $\frac{3}{5} < \frac{55}{30}$ by $\frac{37}{30}$. If, then, we took the sides of the squares as $3 - \frac{35}{30}$, $\frac{4}{5} + \frac{31}{30}$, $\frac{3}{5} + \frac{37}{30}$, the sum of the squares would be $3 . (\frac{11}{6})^2$ or $\frac{363}{36}$, which > 10.

Therefore] we take the side of the first square as $3 - 35x$; of the second as $\frac{4}{5} + 31x$, and of the third as $\frac{3}{5} + 37x$. The sum of the aforesaid squares

$$3555x^2 + 10 - 116x = 10;$$

whence
$$x = \frac{116}{3555}.$$

Returning to the conditions—as the sides of the squares are given, the squares themselves are also given. The rest is obvious.[a]

Ibid. iv. 29, Dioph. ed. Tannery i. 258. 19–260. 16

To find four square numbers such that their sum added to the sum of their sides shall make a given number.

Ἔστω δὴ τὸν $\overline{\iota\beta}$.

Ἐπεὶ πᾶς □$^{\text{ος}}$ προσλαβὼν τὴν ἰδίαν π$^{\lambda}$· καὶ $\mathring{M}$ δ$^{\chi}$, ποιεῖ □$^{\text{ον}}$, οὗ ἡ π$^{\lambda}$· ⋔ $\mathring{M}$ ∠′ ποιεῖ ἀριθμόν τινα, ὅς ἐστι τοῦ ἐξ ἀρχῆς □$^{\text{ου}}$ πλευρά, οἱ τέσσαρες ἀριθμοὶ ἄρα, προσλαβόντες μὲν τὰς ἰδίας π$^{\lambda}$· ποιοῦσι $\mathring{M}$ $\overline{\iota\beta}$, προσλαβόντες δὲ καὶ $\bar{\delta}$ δ$^{\alpha}$, ποιοῦσι τέσσαρας □$^{\text{ους}}$· εἰσὶ δὲ καὶ αἱ $\mathring{M}$ $\overline{\iota\beta}$ μετὰ $\bar{\delta}$ δ$^{\text{ων}}$, ὅ ἐστι $\mathring{M}$ $\bar{\alpha}$, $\mathring{M}$ $\overline{\iota\gamma}$. τὰς $\overline{\iota\gamma}$ ἄρα $\mathring{M}$ διαιρεῖν δεῖ εἰς τέσσαρας □$^{\text{ους}}$, καὶ ἀπὸ τῶν πλευρῶν, ἀφελὼν ἀπὸ ἑκάστης π$^{\lambda}$· $\mathring{M}$ ∠′, ἕξω τῶν $\bar{\delta}$ □$^{\text{ων}}$ τὰς π$^{\lambda}$·

Διαιρεῖται δὲ ὁ $\overline{\iota\gamma}$ εἰς δύο □$^{\text{ους}}$, τόν τε $\bar{\delta}$ καὶ $\bar{\theta}$. καὶ πάλιν ἑκάτερος τούτων διαιρεῖται εἰς δύο □$^{\text{ους}}$, εἰς $\overset{\kappa\epsilon}{\xi\delta}$ καὶ $\overset{\kappa\epsilon}{\lambda\varsigma}$, καὶ $\overset{\kappa\epsilon}{\rho\mu\delta}$ καὶ $\overset{\kappa\epsilon}{\pi\alpha}$. λαβὼν τοίνυν ἑκάστου τὴν πλευράν, $\overset{\epsilon}{\eta}$, $\overset{\epsilon}{\varsigma}$, $\overset{\epsilon}{\iota\beta}$, $\overset{\epsilon}{\theta}$, καὶ αἴρω ἀπὸ ἑκάστου τούτων πλευρᾶς $\mathring{M}$ ∠′, καὶ ἔσονται αἱ π$^{\lambda}$· τῶν ζητουμένων □$^{\text{ων}}$, $\overset{\iota}{\iota\alpha}$, $\overset{\iota}{\zeta}$, $\overset{\iota}{\iota\theta}$, $\overset{\iota}{\iota\gamma}$. αὐτοὶ ἄρα οἱ □$^{\text{οι}}$, ὃς μὲν $\overset{\rho}{\rho\kappa\alpha}$, ὃς δὲ $\overset{\rho}{\mu\theta}$, ὃς δὲ $\overset{\rho}{\tau\xi\alpha}$, ὃς δὲ $\overset{\rho}{\rho\xi\theta}$.

Let it be 12.

Since any square added to its own side and $\frac{1}{4}$ makes a square, whose side *minus* $\frac{1}{2}$ is the number which is the side of the original square,[a] and the four numbers added to their own sides make 12, then if we add $4 \cdot \frac{1}{4}$ they will make four squares. But

$$12 + 4 \cdot \tfrac{1}{4} \text{ (or 1)} = 13.$$

Therefore it is required to divide 13 into four squares, and then, if I subtract $\frac{1}{2}$ from each of their sides, I shall have the sides of the four squares.

Now 13 may be divided into two squares, 4 and 9. And again, each of these may be divided into two squares, $\frac{64}{25}$ and $\frac{36}{25}$, and $\frac{144}{25}$ and $\frac{81}{25}$. I take the side of each $\frac{8}{5}, \frac{6}{5}, \frac{12}{5}, \frac{9}{5}$, and subtract half from each side, and the sides of the required squares will be

$$\frac{11}{10}, \frac{7}{10}, \frac{19}{10}, \frac{13}{10}.$$

The squares themselves are therefore respectively

$$\frac{121}{100}, \frac{49}{100}, \frac{361}{100}, \frac{169}{100}.$$ [b]

[a] *i.e.*, $x^2 + x + \frac{1}{4} = (x + \frac{1}{2})^2$.

[b] In iv. 30 and v. 14 it is also required to divide a number into four squares. As *every number is either a square or the sum of two, three or four squares* (a theorem stated by Fermat and proved by Lagrange), and a square can always be divided into two squares, it follows that any number can be divided into four squares. It is not known whether Diophantus was aware of this.

(*f*) Polygonal Numbers

Dioph. *De polyg. num.*, Praef., Dioph. ed. Tannery i. 450. 3-19

Ἕκαστος τῶν ἀπὸ τῆς τριάδος ἀριθμῶν αὐξομένων μονάδι, πολύγωνός ἐστι πρῶτος[1] ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὅσον ἐστὶν τὸ πλῆθος τῶν ἐν αὐτῷ μονάδων· πλευρά τε αὐτοῦ ἐστιν ὁ ἑξῆς τῆς μονάδος ἀριθμός, ὁ β̄. ἔσται δὲ ὁ μὲν γ̄ τρίγωνος, ὁ δὲ δ̄ τετράγωνος, ὁ δὲ ε̄ πεντάγωνος, καὶ τοῦτο ἑξῆς.

Τῶν δὴ τετραγώνων προδήλων ὄντων ὅτι καθεστήκασι τετράγωνοι διὰ τὸ γεγονέναι αὐτοὺς ἐξ ἀριθμοῦ τινος ἐφ' ἑαυτὸν πολλαπλασιασθέντος, ἐδοκιμάσθη ἕκαστον τῶν πολυγώνων, πολυπλασιαζόμενον ἐπί τινα ἀριθμὸν κατὰ τὴν ἀναλογίαν τοῦ πλήθους τῶν γωνιῶν αὐτοῦ, καὶ προσλαβόντα τετράγωνόν τινα πάλιν κατὰ τὴν ἀναλογίαν τοῦ πλήθους τῶν γωνιῶν αὐτῶν, φαίνεσθαι τετράγωνον· ὃ δὴ παραστήσομεν ὑποδείξαντες πῶς ἀπὸ δοθείσης πλευρᾶς ὁ ἐπιταχθεὶς πολύγωνος εὑρίσκεται, καὶ πῶς δοθέντι πολυγώνῳ ἡ πλευρὰ λαμβάνεται.

[1] πρῶτος Bachet, πρῶτον codd.

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[a] A fragment of the tract *On Polygonal Numbers* is the only work by Diophantus to have survived with the *Arithmetica*. The main fact established in it is that stated in Hypsicles' definition, that the *a*-gonal number of side *n* is

(*f*) Polygonal Numbers[a]

Diophantus, *On Polygonal Numbers*, Preface, Dioph. ed. Tannery i. 450. 3-19

From 3 onwards, every member of the series of natural numbers increasing by unity is the first (after unity) of a particular species of polygon, and it has as many angles as there are units in it; its side is the number next in order after the unit, that is, 2. Thus 3 will be a triangle, 4 a square, 5 a pentagon, and so on in order.[b]

In the case of squares, it is clear that they are squares because they are formed by the multiplication of a number into itself. Similarly it was thought that any polygon, when multiplied by a certain number depending on the number of its angles, with the addition of a certain square also depending on the number of its angles, would also be a square. This we shall establish, showing how any assigned polygonal number may be found from a given side, and how the side may be calculated from a given polygonal number.

$\frac{1}{2}n\{2+(n-1)(a-2)\}$ (*v. supra*, p. 396 n. *a*, and vol. i. p. 98 n. *a*). The method of proof contrasts with that of the *Arithmetica* in being geometrical. For polygonal numbers, *v.* vol. i. pp. 86-99.

[b] The meaning is explained in vol. i. p. 86 n. *a*, especially in the diagram on p. 89. In the example there given, 5 is the first (after unity) of the series of pentagonal numbers 1, 5, 12, 22 . . . It has 5 angles, and each side joins 2 units.

XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, *s.v.* Πάππος

Πάππος, Ἀλεξανδρεύς, φιλόσοφος, γεγονὼς κατὰ τὸν πρεσβύτερον Θεοδόσιον τὸν βασιλέα, ὅτε καὶ Θέων ὁ φιλόσοφος ἤκμαζεν, ὁ γράψας εἰς τὸν Πτολεμαίου Κανόνα. βιβλία δὲ αὐτοῦ Χωρογραφία οἰκουμενική, Εἰς τὰ δ βιβλία τῆς Πτολεμαίου

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[a] Theodosius I reigned from A.D. 379 to 395, but Suidas may have made a mistake over the date. A marginal note opposite the entry *Diocletian* in a Leyden MS. of chronological tables by Theon of Alexandria says, "In his time Pappus wrote"; Diocletian reigned from A.D. 284 to 305. In Rome's edition of Pappus's commentary on Ptolemy's *Syntaxis* (*Studi e Testi*, liv. pp. x-xiii), a cogent argument is given for believing that Pappus actually wrote his *Collection* about A.D. 320.

Suidas obviously had a most imperfect knowledge of Pappus, as he does not mention his greatest work, the *Synagoge* or *Collection*. It is a handbook to the whole of Greek geometry, and is now our sole source for much of the history of that science. The first book and half of the second are missing. The remainder of the second book gives an account of Apollonius's method of working with large numbers (*v. supra*, pp. 352-357). The nature of the remaining books to the eighth will be indicated by the passages here cited. There is some evidence (*v. infra*, p. 607 n. *a*) that the work was originally in twelve books.

The edition of the *Collection* with ancillary material published in three volumes by Friedrich Hultsch (Berlin, 1876–

XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, *s.v. Pappus*

Pappus, an Alexandrian, a philosopher, born in the time of the Emperor Theodosius I, when Theon the philosopher also flourished,[a] who commented on Ptolemy's *Table*. His works include a *Universal Geography*, a *Commentary on the Four Books of*

1878) was a notable event in the revival of Greek mathematical studies. The editor's only major fault is one which he shares with his generation, a tendency to condemn on slender grounds passages as interpolated.

Pappus also wrote a commentary on Euclid's *Elements*; fragments on Book x. are believed to survive in Arabic (*v.* vol. i. p. 456 n. *a*). A commentary by Pappus on Euclid's *Data* is referred to in Marinus's commentary on that work. Pappus (*v.* vol. i. p. 301) himself refers to his commentary on the *Analemma* of Diodorus. The Arabic Fihrist says that he commented on Ptolemy's *Planisphaerium*.

The separate books of the *Collection* were divided by Pappus himself into numbered sections, generally preceded by a preface, and the editors have also divided the books into chapters. References to the *Collection* in the selections here given (*e.g.*, *Coll.* iii. 11. 28, ed. Hultsch 68. 17–70. 8) are first to the book, then to the number or preface in Pappus's division, then to the chapter in the editors' division, and finally to the page and line of Hultsch's edition. In the selections from Book vii. Pappus's own divisions are omitted as they are too complicated, but in the collection of lemmas the numbers of the propositions in Hultsch's edition are added as these are often cited.

Μεγάλης συντάξεως ὑπόμνημα, Ποταμοὺς τοὺς ἐν Λιβύῃ, Ὀνειροκριτικά.

(b) Problems and Theorems

Papp. *Coll.* iii., Praef. 1, ed. Hultsch 30. 3–32. 3

Οἱ τὰ ἐν γεωμετρίᾳ ζητούμενα βουλόμενοι τεχνικώτερον διακρίνειν, ὦ κράτιστε Πανδροσίον, πρόβλημα μὲν ἀξιοῦσι καλεῖν ἐφ' οὗ προβάλλεταί τι ποιῆσαι καὶ κατασκευάσαι, θεώρημα δὲ ἐν ᾧ τινῶν ὑποκειμένων τὸ ἑπόμενον αὐτοῖς καὶ πάντως ἐπισυμβαῖνον θεωρεῖται, τῶν παλαιῶν τῶν μὲν προβλήματα πάντα, τῶν δὲ θεωρήματα εἶναι φασκόντων. ὁ μὲν οὖν τὸ θεώρημα προτείνων, συνιδὼν ὀντινοῦν τρόπον, τὸ ἀκόλουθον τούτῳ ἀξιοῖ ζητεῖν καὶ οὐκ ἂν ἄλλως ὑγιῶς προτείνοι, ὁ δὲ τὸ πρόβλημα προτείνων [ἂν μὲν ἀμαθὴς ᾖ καὶ παντάπασιν ἰδιώτης],[1] κἂν ἀδύνατόν πως κατασκευασθῆναι προστάξῃ, σύγγνωστός ἐστιν καὶ ἀνυπεύθυνος. τοῦ γὰρ ζητοῦντος ἔργον καὶ τοῦτο διορίσαι, τό τε δυνατὸν καὶ τὸ ἀδύνατον, κἂν ᾖ δυνατόν, πότε καὶ πῶς καὶ ποσαχῶς δυνατόν. ἐὰν δὲ προσποιούμενος ᾖ τὰ μαθήματά πως ἀπείρως προβάλλων, οὐκ ἔστιν αἰτίας ἔξω. πρῴην γοῦν τινες τῶν τὰ μαθήματα προσποιουμένων εἰδέναι διὰ σοῦ τὰς τῶν προβλημάτων προτάσεις ἀμαθῶς ἡμῖν ὥρισαν. περὶ ὧν ἔδει καὶ τῶν

[1] ἂν . . . ἰδιώτης om. Hultsch.

[a] Suidas seems to be confusing Ptolemy's Μαθηματικὴ τετράβιβλος σύνταξις (*Tetrabiblos* or *Quadripartitum*) which was in four books but on which Pappus did not comment, with the Μαθηματικὴ σύνταξις (*Syntaxis* or *Almagest*), which *was* the subject of a commentary by Pappus but extended to

Ptolemy's Great Collection,[a] *The Rivers of Libya*, *On the Interpretation of Dreams.*

(*b*) Problems and Theorems

Pappus, *Collection* iii., Preface 1, ed. Hultsch 30. 3-32. 3

Those who favour a more exact terminology in the subjects studied in geometry, most excellent Pandrosion, use the term *problem* to mean an inquiry in which it is proposed to do or to construct something, and the term *theorem* an inquiry in which the consequences and necessary implications of certain hypotheses are investigated, but among the ancients some described them all as problems, some as theorems. Therefore he who propounds a theorem, no matter how he has become aware of it, must set for investigation the conclusion inherent in the premises, and in no other way would he correctly propound the theorem; but he who propounds a problem, even though he may require us to construct something which is in some way impossible, is free from blame and criticism. For it is part of the investigator's task to determine the conditions under which a problem is possible and impossible, and, if possible, when, how and in how many ways it is possible. But when a man professing to know mathematics sets an investigation wrongly he is not free from censure. For example, some persons professing to have learnt mathematics from you lately gave me a wrong enunciation of problems. It is desirable that I should state some of the proofs of

thirteen books. Pappus's commentary now survives only for Books v. and vi., which have been edited by A. Rome, *Studi e Testi*, liv., but it certainly covered the first six books and possibly all thirteen.

παραπλησίων αὐτοῖς ἀποδείξεις τινὰς ἡμᾶς εἰπεῖν εἰς ὠφέλειαν σήν τε καὶ τῶν φιλομαθούντων ἐν τῷ τρίτῳ τούτῳ τῆς Συναγωγῆς βιβλίῳ. τὸ μὲν οὖν πρῶτον τῶν προβλημάτων μέγας τις γεωμέτρης εἶναι δοκῶν ὥρισεν ἀμαθῶς. τὸ γὰρ δύο δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον ἐν συνεχεῖ ἀναλογίᾳ λαβεῖν ἔφασκεν εἰδέναι δι' ἐπιπέδου θεωρίας, ἠξίου δὲ καὶ ἡμᾶς ὁ ἀνὴρ ἐπισκεψαμένους ἀποκρίνασθαι περὶ τῆς ὑπ' αὐτοῦ γενηθείσης κατασκευῆς, ἥτις ἔχει τὸν τρόπον τοῦτον.

(c) The Theory of Means

Ibid. iii. 11. 28, ed. Hultsch 68. 17–70. 8

Τὸ δὲ δεύτερον τῶν προβλημάτων ἦν τόδε·

Ἐν ἡμικυκλίῳ τὰς τρεῖς μεσότητας λαβεῖν ἄλλος τις ἔφασκεν, καὶ ἡμικύκλιον τὸ ΑΒΓ ἐκθέμενος, οὗ κέντρον τὸ Ε, καὶ τυχὸν σημεῖον ἐπὶ τῆς ΑΓ λαβὼν τὸ Δ, καὶ ἀπ' αὐτοῦ πρὸς ὀρθὰς ἀγαγὼν τῇ ΕΓ τὴν ΔΒ, καὶ ἐπιζεύξας τὴν ΕΒ, καὶ αὐτῇ κάθετον ἀγαγὼν ἀπὸ τοῦ Δ τὴν ΔΖ, τὰς τρεῖς μεσότητας ἔλεγεν ἁπλῶς ἐν τῷ ἡμικυκλίῳ ἐκτεθεῖσθαι, τὴν μὲν ΕΓ μέσην ἀριθμητικήν, τὴν δὲ ΔΒ μέσην γεωμετρικήν, τὴν δὲ ΒΖ ἁρμονικήν.

Ὅτι μὲν οὖν ἡ ΒΔ μέση ἐστὶ τῶν ΑΔ, ΔΓ ἐν

[a] The method, as described by Pappus, but not reproduced here, does not actually solve the problem, but it does furnish a series of successive approximations to the solution, and deserves more kindly treatment than it receives from him.

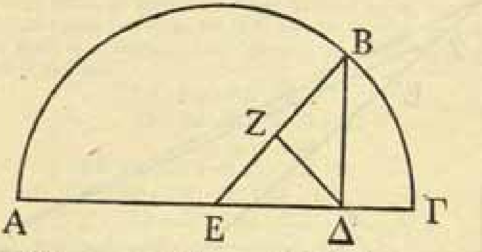
these and of matters akin to them, for the benefit both of yourself and of other lovers of this science, in this third book of the *Collection*. Now the first of these problems was set wrongly by a person who was thought to be a great geometer. For, given two straight lines, he claimed to know how to find by plane methods two means in continuous proportion, and he even asked that I should look into the matter and comment on his construction, which is after this manner.[a]

(c) The Theory of Means

Ibid. iii. 11. 28, ed. Hultsch 68. 17–70. 8

The second of the problems was this :

A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle ΑΒΓ, with centre Ε, and taking any point Δ on ΑΓ, and from it drawing ΔΒ perpendicular to ΕΓ, and joining ΕΒ, and from Δ drawing ΔΖ perpendicular to it, he claimed simply that the three means had been set out in the semicircle, ΕΓ being the arithmetic mean, ΔΒ the geometric mean and ΒΖ the harmonic mean.



That ΒΔ is a mean between ΑΔ, ΔΓ in geometrical

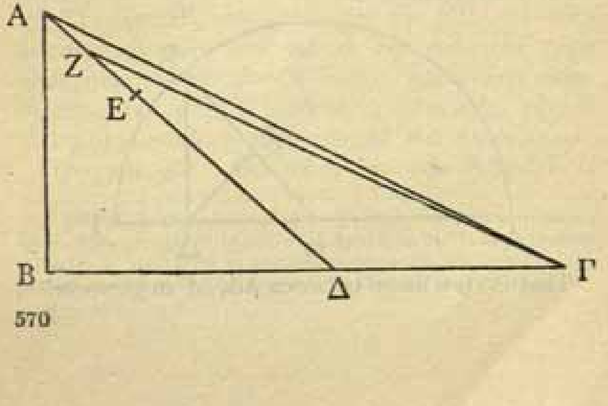
τῇ γεωμετρικῇ ἀναλογίᾳ, ἡ δὲ ΕΓ τῶν ΑΔ, ΔΓ ἐν τῇ ἀριθμητικῇ μεσότητι, φανερόν. ἔστι γὰρ ὡς μὲν ἡ ΑΔ πρὸς ΔΒ, ἡ ΔΒ πρὸς ΔΓ, ὡς δὲ ἡ ΑΔ πρὸς ἑαυτήν, οὕτως ἡ τῶν ΑΔ, ΑΕ ὑπεροχή, τουτέστιν ἡ τῶν ΑΔ, ΕΓ, πρὸς τὴν τῶν ΕΓ, ΓΔ. πῶς δὲ καὶ ἡ ΖΒ μέση ἐστὶν τῆς ἁρμονικῆς μεσότητος, ἢ ποίων εὐθειῶν, οὐκ εἶπεν, μόνον δὲ ὅτι τρίτη ἀνάλογόν ἐστιν τῶν ΕΒ, ΒΔ, ἀγνοῶν ὅτι ἀπὸ τῶν ΕΒ, ΒΔ, ΒΖ ἐν τῇ γεωμετρικῇ ἀναλογίᾳ οὐσῶν πλάσσεται ἡ ἁρμονικὴ μεσότης. δειχθήσεται γὰρ ὑφ᾿ ἡμῶν ὕστερον ὅτι δύο αἱ ΕΒ καὶ τρεῖς αἱ ΔΒ καὶ μία ἡ ΒΖ ὡς μία συντεθεῖσαι ποιοῦσι τὴν μείζονα ἄκραν τῆς ἁρμονικῆς μεσότητος, δύο δὲ αἱ ΒΔ καὶ μία ἡ ΒΖ τὴν μέσην, μία δὲ ἡ ΒΔ καὶ μία ἡ ΒΖ τὴν ἐλαχίστην.

(*d*) The Paradoxes of Erycinus

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

Τὸ δὲ τρίτον τῶν προβλημάτων ἦν τόδε.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν



proportion, and $E\Gamma$ between $A\Delta$, $\Delta\Gamma$ in arithmetical proportion, is clear. For

$$A\Delta : \Delta B = \Delta B : \Delta\Gamma, \quad \text{[Eucl. iii. 31, vi. 8 Por.}$$

and
$$A\Delta : A\Delta = (A\Delta - AE) : (E\Gamma - \Gamma\Delta)$$
$$= (A\Delta - E\Gamma) : (E\Gamma - \Gamma\Delta).$$

But how ZB is a harmonic mean, or between what kind of lines, he did not say, but only that it is a third proportional to EB, BΔ, not knowing that from EB, BΔ, BZ, which are in geometrical proportion, the harmonic mean is formed. For it will be proved by me later that a harmonic proportion can thus be formed—

$$\text{greater extreme} = 2EB + 3\Delta B + BZ,$$
$$\text{mean term} = 2B\Delta + BZ,$$
$$\text{lesser extreme} = B\Delta + BZ.\text{[a]}$$

(*d*) The Paradoxes of Erycinus

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

The third of the problems was this :

Let ABΓ be a right-angled triangle having the

[a] It is Pappus, in fact, who seems to have erred, for BZ is a harmonic mean between AΔ, ΔΓ, as can thus be proved :

Since BΔE is a right-angled triangle in which ΔZ is perpendicular to BE,

$\therefore$ $$BZ : B\Delta = B\Delta : BE,$$
i.e., $$BZ \,.\, BE = B\Delta^2 = A\Delta \,.\, \Delta\Gamma.$$
But $$BE = \tfrac{1}{2}(A\Delta + \Delta\Gamma) ;$$
$\therefore$ $$BZ(A\Delta + \Delta\Gamma) = 2A\Delta \,.\, \Delta\Gamma.$$
$\therefore$ $$A\Delta(BZ - \Delta\Gamma) = \Delta\Gamma(A\Delta - BZ),$$
i.e., $$A\Delta : \Delta\Gamma = (A\Delta - BZ) : (BZ - \Delta\Gamma),$$

and $\therefore$ BZ is a harmonic mean between AΔ, ΔΓ.

The three means and the several extremes have thus been

ἔχον τὴν Β γωνίαν, καὶ διήχθω τις ἡ ΑΔ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ δίχα τμηθείσης τῆς ΕΑ κατὰ τὸ Ζ, καὶ ἐπιζευχθείσης τῆς ΖΓ δεῖξαι συναμφοτέρας τὰς ΔΖΓ δύο πλευρὰς ἐντὸς τοῦ τριγώνου μείζονας τῶν ἐκτὸς συναμφοτέρων τῶν ΒΑΓ πλευρῶν.

Καὶ ἔστι δῆλον. ἐπεὶ γὰρ αἱ ΓΖΑ, τουτέστιν αἱ ΓΖΕ, τῆς ΓΑ μείζονές εἰσιν, ἴση δὲ ἡ ΔΕ τῇ ΑΒ, αἱ ΓΖΔ ἄρα δύο τῶν ΓΑΒ μείζονές εἰσιν. . . .

Ἀλλ' ὅτι τοῦτο μέν, ὅπως ἄν τις ἐθέλοι προτείνειν, ἀπειραχῶς δείκνυται δῆλον, οὐκ ἄκαιρον δὲ καθολικώτερον περὶ τῶν τοιούτων προβλημάτων διαλαβεῖν ἀπὸ τῶν φερομένων παραδόξων Ἐρυκίνου προτείνοντας οὕτως.

(*e*) The Regular Solids

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

Εἰς τὴν δοθεῖσαν σφαῖραν ἐγγράψαι τὰ πέντε πολύεδρα, προγράφεται δὲ τάδε.

Ἔστω ἐν σφαίρᾳ κύκλος ὁ ΑΒΓ, οὗ διάμετρος ἡ ΑΓ καὶ κέντρον τὸ Δ, καὶ προκείσθω εἰς τὸν

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represented by *five* straight lines (ΕΒ, ΒΖ, ΑΔ, ΔΓ, ΒΔ). Pappus takes *six* lines to solve the problem. He proceeds to define the seven other means and to form all ten means as linear functions of three terms in geometrical progression (v. vol. i. pp. 124-129).

angle B right, and let AΔ be drawn, and let ΔE be placed equal to AB, then if EA be bisected at Z, and ZΓ be joined, to show that the sum of the two sides ΔZ, ZΓ within the triangle, is greater than the sum of the two sides BA, AΓ without the triangle.

And it is obvious. For

since $\quad \Gamma Z + ZA > \Gamma A,$ [Eucl. i. 20

i.e., $\quad \Gamma Z + ZE > \Gamma A,$

while $\quad \Delta E = AB,$

$\therefore \quad [\Gamma Z + ZE + E\Delta = \Gamma A + A$

i.e.,] $\quad \Gamma Z + Z\Delta > \Gamma A + AB. \ldots$

But it is clear that this type of proposition, according to the different ways in which one might wish to propound it, can take an infinite number of forms, and it is not out of place to discuss such problems more generally and [first] to propound this from the so-called paradoxes of Erycinus.[a]

(*e*) The Regular Solids[b]

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

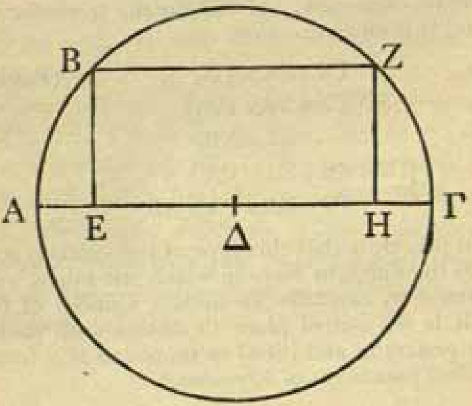
In order to inscribe the five polyhedra in a sphere, these things are premised.

Let ABΓ be a circle in a sphere, with diameter AΓ and centre Δ, and let it be proposed to insert in the

[a] Nothing further is known of Erycinus. The propositions next investigated are more elaborate than the one just solved.

[b] This is the fourth subject dealt with in *Coll.* iii. For the treatment of the subject by earlier geometers, *v.* vol. i. pp. 216-225, 466-479.

κύκλον ἐμβαλεῖν εὐθεῖαν παράλληλον μὲν τῇ ΑΓ διαμέτρῳ, ἴσην δὲ τῇ δοθείσῃ μὴ μείζονι οὔσῃ τῆς ΑΓ διαμέτρου.



Κείσθω τῇ ἡμισείᾳ τῆς δοθείσης ἴση ἡ ΕΔ, καὶ τῇ ΑΓ διαμέτρῳ ἤχθω πρὸς ὀρθὰς ἡ ΕΒ, τῇ δὲ ΑΓ παράλληλος ἡ ΒΖ, ἥτις ἴση ἔσται τῇ δοθείσῃ· διπλῆ γάρ ἐστιν τῆς ΕΔ, ἐπεὶ καὶ ἴση τῇ ΕΗ, παραλλήλου ἀχθείσης τῆς ΖΗ τῇ ΒΕ.

(*f*) Extension of Pythagoras's Theorem

Ibid. iv. 1. 1, ed. Hultsch 176. 9–178. 13

Ἐὰν ᾖ τρίγωνον τὸ ΑΒΓ, καὶ ἀπὸ τῶν ΑΒ, ΒΓ ἀναγραφῇ τυχόντα παραλληλόγραμμα τὰ ΑΒΔΕ, ΒΓΖΗ, καὶ αἱ ΔΕ, ΖΗ ἐκβληθῶσιν ἐπὶ τὸ Θ, καὶ ἐπιζευχθῇ ἡ ΘΒ, γίνεται τὰ ΑΒΔΕ,

circle a chord parallel to the diameter ΑΓ and equal to a given straight line not greater than the diameter ΑΓ.

Let ΕΔ be placed equal to half of the given straight line, and let ΕΒ be drawn perpendicular to the diameter ΑΓ, and let ΒΖ be drawn parallel to ΑΓ; then shall this line be equal to the given straight line. For it is double of ΕΔ, inasmuch as ΖΗ, when drawn, is parallel to ΒΕ, and it is therefore equal to ΕΗ.[a]

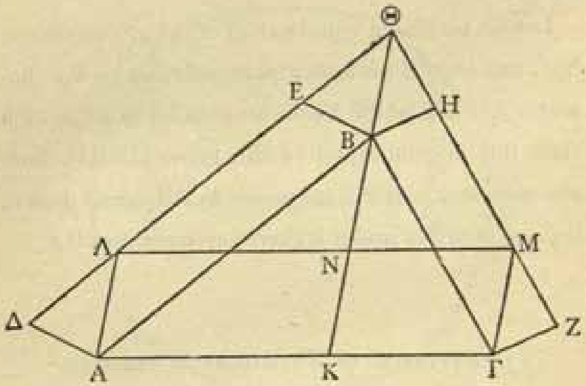
(f) Extension of Pythagoras's Theorem

Ibid. iv. 1. 1, ed. Hultsch 176. 9–178. 13

If ΑΒΓ be a triangle, and on ΑΒ, ΒΓ there be described any parallelograms ΑΒΔΕ, ΒΓΖΗ, and ΔΕ, ΖΗ be produced to Θ, and ΘΒ be joined, then the

[a] This lemma gives the key to Pappus's method of inscribing the regular solids, which is to find in the case of each solid certain parallel circular sections of the sphere. In the case of the cube, for example, he finds two equal and parallel circular sections, the square on whose diameter is two-thirds of the square on the diameter of the sphere. The squares inscribed in these circles are then opposite faces of the cube. In each case the method of analysis and synthesis is followed. The treatment is quite different from Euclid's.

ΒΓΖΗ παραλληλόγραμμα ἴσα τῷ ὑπὸ τῶν ΑΓ, ΘΒ περιεχομένῳ παραλληλογράμμῳ ἐν γωνίᾳ ἥ ἐστιν ἴση συναμφοτέρῳ τῇ ὑπὸ ΒΑΓ, ΔΘΒ.



Ἐκβεβλήσθω γὰρ ἡ ΘΒ ἐπὶ τὸ Κ, καὶ διὰ τῶν Α, Γ τῇ ΘΚ παράλληλοι ἤχθωσαν αἱ ΑΛ, ΓΜ, καὶ ἐπεζεύχθω ἡ ΛΜ. ἐπεὶ παραλληλόγραμμόν ἐστιν τὸ ΑΛΘΒ, αἱ ΑΛ, ΘΒ ἴσαι τέ εἰσιν καὶ παράλληλοι. ὁμοίως καὶ αἱ ΜΓ, ΘΒ ἴσαι τέ εἰσιν καὶ παράλληλοι, ὥστε καὶ αἱ ΛΑ, ΜΓ ἴσαι τέ εἰσιν καὶ παράλληλοι. καὶ αἱ ΛΜ, ΑΓ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν· παραλληλόγραμμον ἄρα ἐστὶν τὸ ΑΛΜΓ ἐν γωνίᾳ τῇ ὑπὸ ΛΑΓ, τουτέστιν συναμφοτέρῳ τῇ τε ὑπὸ ΒΑΓ καὶ ὑπὸ ΔΘΒ· ἴση γάρ ἐστιν ἡ ὑπὸ ΔΘΒ τῇ ὑπὸ ΛΑΒ. καὶ ἐπεὶ τὸ ΔΑΒΕ παραλληλόγραμμον τῷ ΛΑΒΘ ἴσον ἐστίν (ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν

parallelograms ΑΒΔΕ, ΒΓΖΗ are together equal to the parallelogram contained by ΑΓ, ΘΒ in an angle which is equal to the sum of the angles ΒΑΓ, ΔΘΒ.

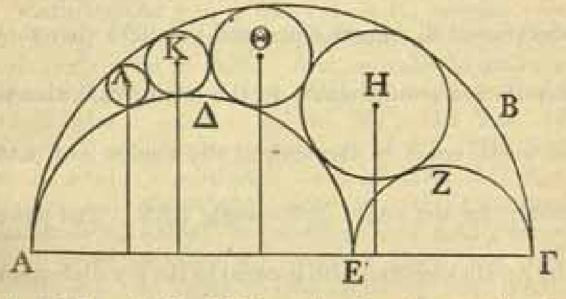
For let ΘΒ be produced to Κ, and through Α, Γ let ΑΛ, ΓΜ be drawn parallel to ΘΚ, and let ΛΜ be joined. Since ΑΛΘΒ is a parallelogram, ΑΛ, ΘΒ are equal and parallel. Similarly ΜΓ, ΘΒ are equal and parallel, so that ΛΑ, ΜΓ are equal and parallel. And therefore ΛΜ, ΑΓ are equal and parallel; therefore ΑΛΜΓ is a parallelogram in the angle ΛΑΓ, that is an angle equal to the sum of the angles ΒΑΓ and ΔΘΒ; for the angle ΔΘΒ = angle ΛΑΒ. And since the parallelogram ΔΑΒΕ is equal to the parallelogram ΛΑΒΘ (for they are upon the same base ΑΒ and in the

τῆς ΑΒ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΒ, ΔΘ), ἀλλὰ τὸ ΛΑΒΘ τῷ ΛΑΚΝ ἴσον ἐστίν (ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν τῆς ΛΑ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΛΑ, ΘΚ), καὶ τὸ ΑΔΕΒ ἄρα τῷ ΛΑΚΝ ἴσον ἐστίν. διὰ τὰ αὐτὰ καὶ τὸ ΒΗΖΓ τῷ ΝΚΓΜ ἴσον ἐστίν· τὰ ἄρα ΔΑΒΕ, ΒΗΖΓ παραλληλόγραμμα τῷ ΛΑΓΜ ἴσα ἐστίν, τουτέστιν τῷ ὑπὸ ΑΓ, ΘΒ ἐν γωνίᾳ τῇ ὑπὸ ΛΑΓ, ἥ ἐστιν ἴση συναμφοτέραις ταῖς ὑπὸ ΒΑΓ, ΒΘΔ. καὶ ἔστι τοῦτο καθολικώτερον πολλῷ τοῦ ἐν τοῖς ὀρθογωνίοις ἐπὶ τῶν τετραγώνων ἐν τοῖς Στοιχείοις δεδειγμένου.

(*g*) Circles Inscribed in the ἄρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

Φέρεται ἔν τισιν ἀρχαία πρότασις τοιαύτη· ὑποκείσθω τρία ἡμικύκλια ἐφαπτόμενα ἀλλήλων



τὰ ΑΒΓ, ΑΔΕ, ΕΖΓ, καὶ εἰς τὸ μεταξὺ τῶν περιφερειῶν αὐτῶν χωρίον, ὃ δὴ καλοῦσιν ἄρβηλον,

same parallels AB, ΔΘ), while ΛΑΒΘ = ΛΑΚΝ (for they are upon the same base ΛΑ and in the same parallels ΛΑ, ΘΚ), therefore ΑΔΕΒ = ΛΑΚΝ. By the same reasoning ΒΗΖΓ = ΝΚΓΜ; therefore the parallelograms ΔΑΒΕ, ΒΗΖΓ are together equal to ΛΑΓΜ, that is, to the parallelogram contained by ΑΓ, ΘΒ in the angle ΛΑΓ, which is equal to the sum of the angles ΒΑΓ, ΒΘΔ. And this is much more general than the theorem proved in the *Elements* about the squares on right-angled triangles.[a]

(g) Circles Inscribed in the ἄρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

There is found in certain [books] an ancient proposition to this effect: Let ΑΒΓ, ΑΔΕ, ΕΖΓ be supposed to be three semicircles touching each other, and in the space between their circumferences, which

[a] Eucl. i. 47, v. vol. i. pp. 178-185. In the case taken by Pappus, the first two parallelograms are drawn outwards and the third, equal to their sum, is drawn inwards. If the areas of parallelograms drawn outwards be regarded as of opposite sign to the areas of those drawn inwards, the theorem may be still further generalized, for the algebraic sum of the three parallelograms is equal to zero.

ἐγγεγράφθωσαν κύκλοι ἐφαπτόμενοι τῶν τε ἡμικυκλίων καὶ ἀλλήλων ὁσοιδηποτοῦν, ὡς οἱ περὶ κέντρα τὰ Η, Θ, Κ, Λ· δεῖξαι τὴν μὲν ἀπὸ τοῦ Η κέντρου κάθετον ἐπὶ τὴν ΑΓ ἴσην τῇ διαμέτρῳ τοῦ περὶ τὸ Η κύκλου, τὴν δ' ἀπὸ τοῦ Θ κάθετον διπλασίαν τῆς διαμέτρου τοῦ περὶ τὸ Θ κύκλου, τὴν δ' ἀπὸ τοῦ Κ κάθετον τριπλασίαν, καὶ τὰς ἑξῆς καθέτους τῶν οἰκείων διαμέτρων πολλαπλασίας κατὰ τοὺς ἑξῆς μονάδι ἀλλήλων ὑπερέχοντας ἀριθμοὺς ἐπ' ἄπειρον γινομένης τῆς τῶν κύκλων ἐγγραφῆς.

(h) Spiral on a Sphere

Ibid. iv. 35. 53-56, ed. Hultsch 264. 3–268. 21

Ὥσπερ ἐν ἐπιπέδῳ νοεῖται γινομένη τις ἕλιξ φερομένου σημείου κατ' εὐθείας κύκλον περιγραφούσης, καὶ ἐπὶ στερεῶν φερομένου σημείου κατὰ μιᾶς πλευρᾶς τιν' ἐπιφάνειαν περιγραφούσης, οὕτως δὴ καὶ ἐπὶ σφαίρας ἕλικα νοεῖν ἀκόλουθόν ἐστι γραφομένην τὸν τρόπον τοῦτον.

Ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΚΛΜ περὶ πόλον τὸ Θ σημεῖον, καὶ ἀπὸ τοῦ Θ μεγίστου

[a] Three propositions (Nos. 4, 5 and 6) about the figure known as the ἄρβηλος from its resemblance to a leather-worker's knife are contained in Archimedes' *Liber Assumptorum*, which has survived in Arabic. They are included as particular cases in Pappus's exposition, which is unfortunately too long for reproduction here. Professor D'Arcy W. Thompson (*The Classical Review*, lvi. (1942), pp. 75-76) gives reasons for thinking that the ἄρβηλος was a saddler's knife rather than a shoemaker's knife, as usually translated.

is called the "leather-worker's knife," let there be inscribed any number whatever of circles touching both the semicircles and one another, as those about the centres H, Θ, K, Λ; to prove that the perpendicular from the centre H to AΓ is equal to the diameter of the circle about H, the perpendicular from Θ is double of the diameter of the circle about Θ, the perpendicular from K is triple, and the [remaining] perpendiculars in order are so many times the diameters of the proper circles according to the numbers in a series increasing by unity, the inscription of the circles proceeding without limit.[a]

(*h*) Spiral on a Sphere [b]

Ibid. iv. 35. 53-56, ed. Hultsch 264. 3–268. 21

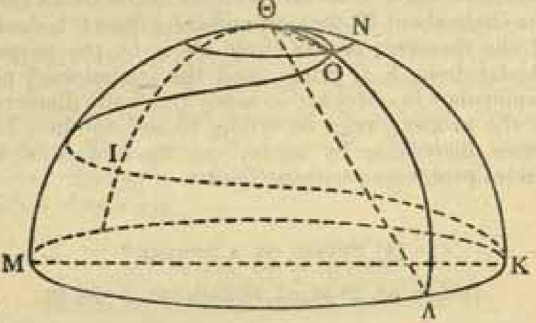
Just as in a plane a spiral is conceived to be generated by the motion of a point along a straight line revolving in a circle, and in solids [, such as the cylinder or cone,][c] by the motion of a point along one straight line describing a certain surface, so also a corresponding spiral can be conceived as described on the sphere after this manner.

Let KΛM be a great circle in a sphere with pole Θ, and from Θ let the quadrant of a great circle ΘNK be

[b] After leaving the ἄρβηλος, Pappus devotes the remainder of Book iv. to solutions of the problems of doubling the cube, squaring the circle and trisecting an angle. This part has been frequently cited already (v. vol. i. pp. 298-309, 336-363). His treatment of the spiral is noteworthy because his method of proof is often markedly different from that of Archimedes: and in the course of it he makes this interesting digression.

[c] Some such addition is necessary, as Commandinus, Chasles and Hultsch realized.

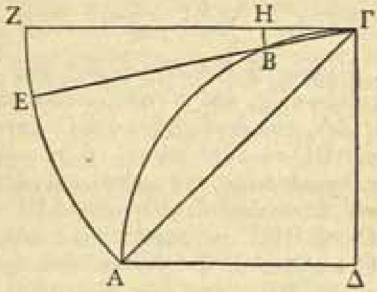
κύκλου τεταρτημόριον γεγράφθω τὸ ΘΝΚ, καὶ ἡ μὲν ΘΝΚ περιφέρεια, περὶ τὸ Θ μένον φερομένη κατὰ τῆς ἐπιφανείας ὡς ἐπὶ τὰ Λ, Μ μέρη,



ἀποκαθιστάσθω πάλιν ἐπὶ τὸ αὐτό, σημεῖον δέ τι φερόμενον ἐπ' αὐτῆς ἀπὸ τοῦ Θ ἐπὶ τὸ Κ παραγινέσθω· γράφει δή τινα ἐπὶ τῆς ἐπιφανείας ἕλικα, οἵα ἐστὶν ἡ ΘΟΙΚ, καὶ ἥτις ἂν ἀπὸ τοῦ Θ γραφῇ μεγίστου κύκλου περιφέρεια, πρὸς τὴν ΚΛ περιφέρειαν λόγον ἔχει ὃν ἡ ΛΘ πρὸς τὴν ΘΟ· λέγω δὴ ὅτι, ἂν ἐκτεθῇ τεταρτημόριον τοῦ μεγίστου ἐν τῇ σφαίρᾳ κύκλου τὸ ΑΒΓ περὶ κέντρον τὸ Δ, καὶ ἐπιζευχθῇ ἡ ΓΑ, γίνεται ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν μεταξὺ τῆς ΘΟΙΚ ἕλικος καὶ τῆς ΚΝΘ περιφερείας ἀπολαμβανομένην ἐπιφάνειαν, οὕτως ὁ ΑΒΓΔ τομεὺς πρὸς τὸ ΑΒΓ τμῆμα.

Ἤχθω γὰρ ἐφαπτομένη τῆς περιφερείας ἡ ΓΖ, καὶ περὶ κέντρον τὸ Γ διὰ τοῦ Α γεγράφθω περιφέρεια ἡ ΑΕΖ· ἴσος ἄρα ὁ ΑΒΓΔ τομεὺς τῷ

described, and, Θ remaining stationary, let the arc ΘNK revolve about the surface in the direction Λ, M



and again return to the same place, and [in the same time] let a point on it move from Θ to K; then it will describe on the surface a certain spiral, such as ΘOIK, and if any arc of a great circle be drawn from Θ [cutting the circle KΛM first in Λ and the spiral first in O], its circumference[a] will bear to the arc KΛ the same ratio as ΛΘ bears to ΘO. I say then that if a quadrant ABΓ of a great circle in the sphere be set out about centre Δ, and ΓA be joined, the surface of the hemisphere will bear to the portion of the surface intercepted between the spiral ΘOIK and the arc KNΘ the same ratio as the sector ABΓΔ bears to the segment ABΓ.

For let ΓZ be drawn to touch the circumference, and with centre Γ let there be described through A the arc AEZ; then the sector ABΓΔ is equal to the

[a] Or, of course, the circumference of the circle KΛM to which it is equal.

ΑΕΖΓ (διπλασία μὲν γὰρ ἡ πρὸς τῷ Δ γωνία τῆς ὑπὸ ΑΓΖ, ἥμισυ δὲ τὸ ἀπὸ ΔΑ τοῦ ἀπὸ ΑΓ)· ὅτι ἄρα καὶ ὡς αἱ εἰρημέναι ἐπιφάνειαι πρὸς ἀλλήλας, οὕτως ὁ ΑΕΖΓ τομεὺς πρὸς τὸ ΑΒΓ τμῆμα.

Ἔστω, ὃ μέρος ἡ ΚΛ περιφέρεια τῆς ὅλης τοῦ κύκλου περιφερείας, καὶ τὸ αὐτὸ μέρος περιφέρεια ἡ ΖΕ τῆς ΖΑ, καὶ ἐπεζεύχθω ἡ ΕΓ· ἔσται δὴ καὶ ἡ ΒΓ τῆς ΑΒΓ τὸ αὐτὸ μέρος. ὃ δὲ μέρος ἡ ΚΛ τῆς ὅλης περιφερείας, τὸ αὐτὸ καὶ ἡ ΘΟ τῆς ΘΟΛ. καὶ ἔστιν ἴση ἡ ΘΟΛ τῇ ΑΒΓ· ἴση ἄρα καὶ ἡ ΘΟ τῇ ΒΓ. γεγράφθω περὶ πόλον τὸν Θ διὰ τοῦ Ο περιφέρεια ἡ ΟΝ, καὶ διὰ τοῦ Β περὶ τὸ Γ κέντρον ἡ ΒΗ. ἐπεὶ οὖν ὡς ἡ ΛΚΘ σφαιρικὴ ἐπιφάνεια πρὸς τὴν ΟΘΝ, ἡ ὅλη τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν τοῦ τμήματος ἐπιφάνειαν οὗ ἡ ἐκ τοῦ πόλου ἐστὶν ἡ ΘΟ, ὡς δ' ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν τοῦ τμήματος ἐπιφάνειαν, οὕτως ἐστὶν τὸ ἀπὸ τῆς τὰ Θ, Λ ἐπιζευγνυούσης εὐθείας τετράγωνον πρὸς τὸ ἀπὸ τῆς ἐπὶ τὰ Θ, Ο, ἢ τὸ ἀπὸ τῆς ΕΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΒΓ, ἔσται ἄρα καὶ ὡς ὁ ΚΛΘ τομεὺς ἐν τῇ ἐπιφανείᾳ πρὸς τὸν ΟΘΝ, οὕτως ὁ ΕΖΓ τομεὺς πρὸς τὸν ΒΗΓ. ὁμοίως δείξομεν ὅτι καὶ ὡς πάντες οἱ ἐν τῷ ἡμισφαιρίῳ τομεῖς οἱ ἴσοι τῷ ΚΛΘ, οἱ

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[a] Pappus's method of proof is, in the Archimedean manner, to circumscribe about the surface to be measured a figure consisting of sectors on the sphere, and to circumscribe about the segment ΑΒΓ a figure consisting of sectors of circles; in the same way figures can be inscribed. The divisions need, therefore, to be as numerous as possible. The conclusion can then be reached by the method of exhaustion.

sector ΑΕΖΓ (for angle ΑΔΓ = 2 . angle ΑΓΖ, and $ΔΑ^2 = \frac{1}{2}ΑΓ^2$) ; I say, then, that the ratio of the aforesaid surfaces one towards the other is the same as the ratio of the sector ΑΕΖΓ to the segment ΑΒΓ.

Let ΖΕ be the same [small] [a] part of ΖΑ as ΚΛ is of the whole circumference of the circle, and let ΕΓ be joined ; then the arc ΒΓ will be the same part of the arc ΑΒΓ.[b] But ΘΟ is the same part of ΘΟΛ as ΚΛ is of the whole circumference [*by the property of the spiral*]. And arc ΘΟΛ = arc ΑΒΓ [*ex constructione*]. Therefore ΘΟ = ΒΓ. Let there be described through Ο about the pole Θ the arc ΟΝ, and through Β about centre Γ the arc ΒΗ. Then since the [sector of the] spherical surface ΛΚΘ bears to the [sector] ΟΘΝ the same ratio as the whole surface of the hemisphere bears to the surface of the segment with pole Θ and circular base ΟΝ,[c] while the surface of the hemisphere bears to the surface of the segment the same ratio as $ΘΛ^2$ to $ΘΟ^2$,[d] or $ΕΓ^2$ to $ΒΓ^2$, therefore the sector ΚΛΘ on the surface [of the sphere] bears to ΟΘΝ the same ratio as the sector ΕΖΓ [in the plane] bears to the sector ΒΗΓ. Similarly we may show that all the sectors [on the surface of] the hemi-

[b] For arc ΖΑ : arc ΖΕ = angle ΖΓΑ : angle ΖΓΕ. But angle ΖΓΑ = $\frac{1}{2}$. angle ΑΔΓ, and angle ΖΓΕ = $\frac{1}{2}$. angle ΒΔΓ [Eucl. iii. 32, 20]. ∴ arc ΖΑ : arc ΖΕ = arc ΑΒΓ : arc ΒΓ.

[c] Because the arc ΛΚ is the same part of the circumference ΚΛΜ as the arc ΟΝ is of its circumference.

[d] The square on ΘΛ is double the square on the radius of the hemisphere, and therefore half the surface of the hemisphere is equal to a circle of radius ΘΛ [Archim. *De sph. et cyl.* i. 33] : and the surface of the segment is equal to a circle of radius ΘΟ [*ibid.* i. 42] : and as circles are to one another as the squares on their radii [Eucl. xii. 2], the surface of the hemisphere bears to the surface of the segment the ratio $ΘΛ^2 : ΘΟ^2$.

εἰσιν ἡ ὅλη τοῦ ἡμισφαιρίου ἐπιφάνεια, πρὸς τοὺς περιγραφομένους περὶ τὴν ἕλικα τομέας ὁμοταγεῖς τῷ ΟΘΝ, οὕτως πάντες οἱ ἐν τῷ ΑΖΓ τομεῖς οἱ ἴσοι τῷ ΕΖΓ, τουτέστιν ὅλος ὁ ΑΖΓ τομεύς, πρὸς τοὺς περιγραφομένους περὶ τὸ ΑΒΓ τμῆμα τοὺς ὁμοταγεῖς τῷ ΓΒΗ. τῷ δ' αὐτῷ τρόπῳ δειχθήσεται καὶ ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τοὺς ἐγγραφομένους τῇ ἕλικι τομέας, οὕτως ὁ ΑΖΓ τομεὺς πρὸς τοὺς ἐγγραφομένους τῷ ΑΒΓ τμήματι τομέας, ὥστε καὶ ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν ὑπὸ τῆς ἕλικος ἀπολαμβανομένην ἐπιφάνειαν, οὕτως ὁ ΑΖΓ τομεύς, τουτέστιν τὸ ΑΒΓΔ τεταρτημόριον, πρὸς τὸ ΑΒΓ τμῆμα. συνάγεται δὲ διὰ τούτου ἡ μὲν ἀπὸ τῆς ἕλικος ἀπολαμβανομένη ἐπιφάνεια πρὸς τὴν ΘΝΚ περιφέρειαν ὀκταπλασία τοῦ ΑΒΓ τμήματος (ἐπεὶ καὶ ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια τοῦ ΑΒΓΔ τομέως), ἡ δὲ μεταξὺ τῆς ἕλικος καὶ τῆς βάσεως τοῦ ἡμισφαιρίου ἐπιφάνεια ὀκταπλασία τοῦ ΑΓΔ τριγώνου, τουτέστιν ἴση τῷ ἀπὸ τῆς διαμέτρου τῆς σφαίρας τετραγώνῳ.

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[a] This would be proved by the method of exhaustion. It is proof of the great part played by this method in Greek geometry that Pappus can take its validity for granted.

[b] For the surface of the hemisphere is double of the circle of radius ΑΔ [Archim. *De sph. et cyl.* i. 33] and the sector ΑΒΓΔ is one-quarter of the circle of radius ΑΔ.

[c] For the surface between the spiral and the base of the hemisphere is equal to the surface of the hemisphere less the surface cut off from the spiral in the direction ΘΝΚ,

i.e. Surface in question = surface of hemisphere − 8 segment ΑΒΓ,

= 8 sector ΑΒΓΔ − 8 segment ΑΒΓ

sphere equal to ΚΛΘ, together making up the whole surface of the hemisphere, bear to the sectors described about the spiral similar to ΟΘΝ the same ratio as the sectors in ΑΖΓ equal to ΕΖΓ, that is the whole sector ΑΖΓ, bear to the sectors described about the segment ΑΒΓ similar to ΓΒΗ. In the same manner it may be shown that the surface of the hemisphere bears to the [sum of the] sectors inscribed in the spiral the same ratio as the sector ΑΖΓ bears to the [sum of the] sectors inscribed in the segment ΑΒΓ, so that the surface of the hemisphere bears to the surface cut off by the spiral the same ratio as the sector ΑΖΓ, that is the quadrant ΑΒΓΔ, bears to the segment ΑΒΓ.[a] From this it may be deduced that the surface cut off from the spiral in the direction of the arc ΘΝΚ is eight times the segment ΑΒΓ (since the surface of the hemisphere is eight times the sector ΑΒΓΔ),[b] while the surface between the spiral and the base of the hemisphere is eight times the triangle ΑΓΔ, that is, it is equal to the square on the diameter of the sphere.[c]

$$\begin{aligned} &= 8 \text{ triangle } \mathrm{A\Gamma\Delta} \\ &= 4\mathrm{A\Delta}^2 \\ &= (2\mathrm{A\Delta})^2, \end{aligned}$$

and 2ΑΔ is the diameter of the sphere.

Heath (*H.G.M.* ii. 384-385) gives for this elegant proposition an analytical equivalent, which I have adapted to the Greek lettering. If ρ, ω are the spherical co-ordinates of Ο with reference to Θ as pole and the arc ΘΝΚ as polar axis, the equation of the spiral is $\omega = 4\rho$. If A is the area of the spiral to be measured, and the radius of the sphere is taken as unity, we have as the element of area

$$dA = d\omega(1 - \cos\rho) = 4d\rho(1 - \cos\rho).$$

(f) Isoperimetric Figures[a]

Ibid. v., Praef. 1-3, ed. Hultsch 304. 5-308. 5

Σοφίας καὶ μαθημάτων ἔννοιαν ἀρίστην μὲν καὶ τελειοτάτην ἀνθρώποις θεὸς ἔδωκεν, ὦ κράτιστε Μεγεθίον, ἐκ μέρους δέ που καὶ τῶν ἀλόγων ζῴων μοῖραν ἀπένειμέν τισιν. ἀνθρώποις μὲν οὖν ἅτε λογικοῖς οὖσι τὸ μετὰ λόγου καὶ ἀποδείξεως παρέσχεν ἕκαστα ποιεῖν, τοῖς δὲ λοιποῖς ζῴοις ἄνευ λόγου τὸ χρήσιμον καὶ βιωφελὲς αὐτὸ μόνον κατά τινα φυσικὴν πρόνοιαν ἑκάστοις ἔχειν ἐδωρήσατο. τοῦτο δὲ μάθοι τις ἂν ὑπάρχον καὶ ἐν ἑτέροις μὲν πλείστοις γένεσιν τῶν ζῴων, οὐχ ἥκιστα δὲ κἀν ταῖς μελίσσαις· ἥ τε γὰρ εὐταξία καὶ πρὸς τὰς ἡγουμένας τῆς ἐν αὐταῖς πολιτείας εὐπείθεια θαυμαστή τις, ἥ τε φιλοτιμία καὶ καθαριότης ἡ περὶ τὴν τοῦ μέλιτος συναγωγὴν καὶ ἡ περὶ τὴν φυλακὴν αὐτοῦ πρόνοια καὶ οἰκονομία πολὺ μᾶλλον θαυμασιωτέρα. πεπιστευμέναι γάρ, ὡς εἰκός, παρὰ θεῶν κομίζειν τοῖς τῶν ἀνθρώπων μουσικοῖς

$$\therefore \qquad A = \int_0^{\frac{1}{2}\pi} 4d\rho(1 - \cos \rho)$$

$$= 2\pi - 4.$$

$$\therefore \qquad \frac{A}{\text{surface of hemisphere}} = \frac{2\pi - 4}{2\pi}$$

$$= \frac{\frac{1}{4}\pi - \frac{1}{2}}{\frac{1}{4}\pi}$$

$$= \frac{\text{segment ΑΒΓ}}{\text{sector ΑΒΓΔ}}.$$

[a] The whole of Book v. in Pappus's *Collection* is devoted to isoperimetry. The first section follows closely the exposition of Zenodorus as given by Theon (*v. supra*, pp. 386-395).

(i) Isoperimetric Figures [a]

Ibid. v., Preface 1-3, ed. Hultsch 304. 5-308. 5

Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of

except that Pappus includes the proposition that *of all circular segments having the same circumference the semicircle is the greatest.* The second section compares the volumes of solids whose surfaces are equal, and is followed by a digression, already quoted (*supra*, pp. 194-197) on the semi-regular solids discovered by Archimedes. After some propositions on the lines of Archimedes' *De sph. et cyl.*, Pappus finally proves that *of regular solids having equal surfaces, that is greatest which has most faces.*

The introduction, here cited, on the sagacity of bees is rightly praised by Heath (*H.G.M.* ii. 389) as an example of the good style of the Greek mathematicians when freed from the restraints of technical language.

τῆς ἀμβροσίας ἀπόμοιράν τινα ταύτην οὐ μάτην ἐκχεῖν εἰς γῆν καὶ ξύλον ἤ τινα ἑτέραν ἀσχήμονα καὶ ἄτακτον ὕλην ἠξίωσαν, ἀλλ' ἐκ τῶν ἡδίστων ἐπὶ γῆς φυομένων ἀνθέων συνάγουσαι τὰ κάλλιστα κατασκευάζουσιν ἐκ τούτων εἰς τὴν τοῦ μέλιτος ὑποδοχὴν ἀγγεῖα τὰ καλούμενα κηρία πάντα μὲν ἀλλήλοις ἴσα καὶ ὅμοια καὶ παρακείμενα, τῷ δὲ σχήματι ἑξάγωνα.

Τοῦτο δ' ὅτι κατά τινα γεωμετρικὴν μηχανῶνται πρόνοιαν οὕτως ἂν μάθοιμεν. πάντως μὲν γὰρ ᾤοντο δεῖν τὰ σχήματα παρακεῖσθαί τε ἀλλήλοις καὶ κοινωνεῖν κατὰ τὰς πλευράς, ἵνα μὴ τοῖς μεταξὺ παραπληρώμασιν ἐμπίπτοντά τινα ἕτερα λυμήνηται αὐτῶν τὰ ἔργα· τρία δὲ σχήματα εὐθύγραμμα τὸ προκείμενον ἐπιτελεῖν ἐδύνατο, λέγω δὲ τεταγμένα τὰ ἰσόπλευρά τε καὶ ἰσογώνια, τὰ δ' ἀνόμοια ταῖς μελίσσαις οὐκ ἤρεσεν. τὰ μὲν οὖν ἰσόπλευρα τρίγωνα καὶ τετράγωνα καὶ τὰ ἑξάγωνα χωρὶς ἀνομοίων παραπληρωμάτων ἀλλήλοις δύναται παρακείμενα τὰς πλευρὰς κοινὰς ἔχειν [ταῦτα[1] γὰρ δύναται συμπληροῦν ἐξ αὐτῶν τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ἑτέρῳ δὲ τεταγμένῳ σχήματι τοῦτο ποιεῖν ἀδύνατον].[2] ὁ γὰρ περὶ τὸ αὐτὸ σημεῖον τόπος ὑπὸ ϛ̄ μὲν τριγώνων ἰσοπλεύρων καὶ διὰ ϛ̄ γωνιῶν, ὧν ἑκάστη διμοίρου ἐστὶν ὀρθῆς, συμπληροῦται, τεσσάρων δὲ τετραγώνων καὶ δ̄ ὀρθῶν γωνιῶν [αὐτοῦ],[3] τριῶν δὲ ἑξαγώνων καὶ ἑξαγώνου γωνιῶν τριῶν, ὧν ἑκάστη ᾱ γʹ ἐστὶν ὀρθῆς. πεντάγωνα δὲ τὰ τρία μὲν οὐ φθάνει συμπληρῶσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ὑπερβάλλει δὲ τὰ τέσσαρα· τρεῖς μὲν γὰρ τοῦ πενταγώνου γωνίαι δ̄ ὀρθῶν ἐλάσσονές εἰσιν

ambrosia in this form, they do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical forethought we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilineal figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. For equilateral triangles and squares and hexagons can lie adjacent to one another and have their sides in common without irregular interstices. For the space about the same point can be filled by six equilateral triangles and six angles, of which each is $\frac{2}{3}$. right angle, or by four squares and four right angles, or by three hexagons and three angles of a hexagon, of which each is $1\frac{1}{3}$. right angle. But three pentagons would not suffice to fill the space about the same point, and four would be more than sufficient ; for three angles of the pentagon are less than four right angles (inasmuch

[1] ταῦτα . . . ἀδύνατον om. Hultsch.
[2] " αὐτοῦ spurium, nisi forte αὐτῶν dedit scriptor "—Hultsch.

(ἑκάστη γὰρ γωνία μιᾶς καὶ ε´ ἐστὶν ὀρθῆς), τέσσαρες δὲ γωνίαι μείζους τῶν τεσσάρων ὀρθῶν. ἑπτάγωνα δὲ οὐδὲ τρία περὶ τὸ αὐτὸ σημεῖον δύναται τίθεσθαι κατὰ τὰς πλευρὰς ἀλλήλοις παρακείμενα· τρεῖς γὰρ ἑπταγώνου γωνίαι τεσσάρων ὀρθῶν μείζονες (ἑκάστη γάρ ἐστιν μιᾶς ὀρθῆς καὶ τριῶν ἑβδόμων). ἔτι δὲ μᾶλλον ἐπὶ τῶν πολυγωνοτέρων ὁ αὐτὸς ἐφαρμόσαι δυνήσεται λόγος. ὄντων δὴ οὖν τριῶν σχημάτων τῶν ἐξ αὐτῶν δυναμένων συμπληρῶσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, τριγώνου τε καὶ τετραγώνου καὶ ἑξαγώνου, τὸ πολυγωνότερον εἵλαντο διὰ τὴν σοφίαν αἱ μέλισσαι πρὸς τὴν παρασκευήν, ἅτε καὶ πλεῖον ἑκατέρου τῶν λοιπῶν αὐτὸ χωρεῖν ὑπολαμβάνουσαι μέλι.

Καὶ αἱ μέλισσαι μὲν τὸ χρήσιμον αὐταῖς ἐπίστανται μόνον τοῦθ᾽ ὅτι τὸ ἑξάγωνον τοῦ τετραγώνου καὶ τοῦ τριγώνου μεῖζόν ἐστιν καὶ χωρῆσαι δύναται πλεῖον μέλι τῆς ἴσης εἰς τὴν ἑκάστου κατασκευὴν ἀναλισκομένης ὕλης, ἡμεῖς δὲ πλέον τῶν μελισσῶν σοφίας μέρος ἔχειν ὑπισχνούμενοι ζητήσομέν τι καὶ περισσότερον. τῶν γὰρ ἴσην ἐχόντων περίμετρον ἰσοπλεύρων τε καὶ ἰσογωνίων ἐπιπέδων σχημάτων μεῖζόν ἐστιν ἀεὶ τὸ πολυγωνότερον, μέγιστος δ᾽ ἐν πᾶσιν ὁ κύκλος, ὅταν ἴσην αὐτοῖς περίμετρον ἔχῃ.

(*j*) Apparent Form of a Circle

Ibid. vi. 48. 90-91, ed. Hultsch 580. 12-27

Ἔστω κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἔστω τῷ τοῦ κύκλου ἐπι-

as each angle is $1\frac{1}{5}$. right angle), and four angles are greater than four right angles. Nor can three heptagons be placed about the same point so as to have their sides adjacent to each other ; for three angles of a heptagon are greater than four right angles (inasmuch as each is $1\frac{3}{7}$. right angle). And the same argument can be applied even more to polygons with a greater number of angles. There being, then, three figures capable by themselves of filling up the space around the same point, the triangle, the square and the hexagon, the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, *of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.*

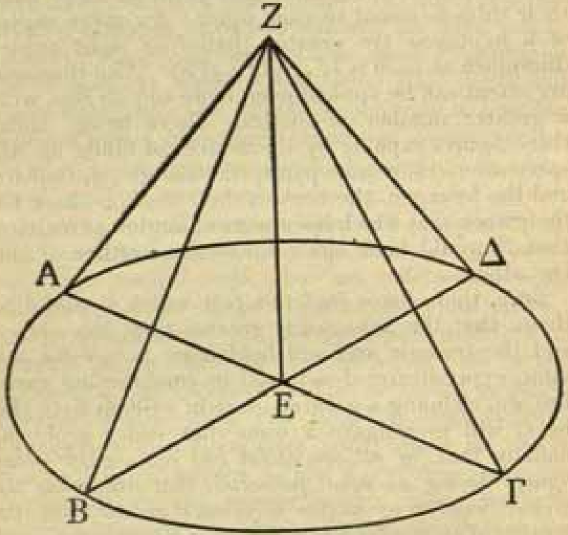
(*j*) Apparent Form of a Circle

Ibid. vi.[a] 48. 90-91, ed. Hultsch 580. 12-27

Let ABΓ be a circle with centre E, and from E let EZ be drawn perpendicular to the plane of the circle ;

[a] Most of Book vi. is astronomical, covering the treatises in the *Little Astronomy* (*v. supra*, p. 408 n. *b*). The proposition here cited comes from a section on Euclid's *Optics*.

πέδῳ ἡ ΕΖ· λέγω, ὅτι ἐὰν ἐπὶ τῆς ΕΖ τὸ ὄμμα τεθῇ ἴσαι αἱ διάμετροι φαίνονται τοῦ κύκλου.

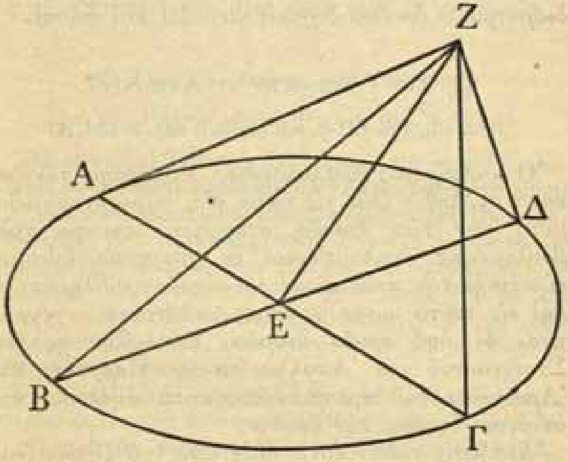


Τοῦτο δὲ δῆλον· ἅπασαι γὰρ αι ἀπὸ τοῦ Ζ πρὸς τὴν τοῦ κύκλου περιφέρειαν προσπίπτουσαι εὐθεῖαι ἴσαι εἰσὶν ἀλλήλαις καὶ ἴσας γωνίας περιέχουσιν.

Μὴ ἔστω δὲ ἡ ΕΖ πρὸς ὀρθὰς τῷ τοῦ κύκλου ἐπιπέδῳ, ἴση δὲ ἔστω τῇ ἐκ τοῦ κέντρου τοῦ κύκλου· λέγω, ὅτι τοῦ ὄμματος ὄντος πρὸς τῷ Ζ σημείῳ καὶ οὕτως αἱ διάμετροι ἴσαι ὁρῶνται.

Ἤχθωσαν γὰρ δύο διάμετροι αἱ ΑΓ, ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ. ἐπεὶ αἱ

I say that, if the eye be placed on EZ, the diameters of the circle appear equal.[a]



This is obvious; for all the straight lines falling from Z on the circumference of the circle are equal one to another and contain equal angles.

Now let EZ be not perpendicular to the plane of the circle, but equal to the radius of the circle; I say that, if the eye be at the point Z, in this case also the diameters appear equal.

For let two diameters ΑΓ, ΒΔ be drawn, and let ZA, ZB, ZΓ, ZΔ be joined. Since the three straight

[a] As they will do if they subtend an equal angle at the eye.

τρεῖς αἱ ΕΑ, ΕΓ, ΕΖ ἴσαι εἰσίν, ὀρθὴ ἄρα ἡ ὑπὸ ΑΖΓ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΖΔ ὀρθή ἐστιν· ἴσαι ἄρα φανήσονται αἱ ΑΓ, ΒΔ διάμετροι. ὁμοίως δὴ δείξομεν ὅτι καὶ πᾶσαι.

(k) The "Treasury of Analysis"

Ibid. vii., Praef. 1-3, ed. Hultsch 634. 3-636. 30

Ὁ καλούμενος ἀναλυόμενος, Ἑρμόδωρε τέκνον, κατὰ σύλληψιν ἰδία τίς ἐστιν ὕλη παρεσκευασμένη μετὰ τὴν τῶν κοινῶν στοιχείων ποίησιν τοῖς βουλομένοις ἀναλαμβάνειν ἐν γραμμαῖς δύναμιν εὑρετικὴν τῶν προτεινομένων αὐτοῖς προβλημάτων, καὶ εἰς τοῦτο μόνον χρησίμη καθεστῶσα. γέγραπται δὲ ὑπὸ τριῶν ἀνδρῶν, Εὐκλείδου τε τοῦ Στοιχειωτοῦ καὶ Ἀπολλωνίου τοῦ Περγαίου καὶ Ἀρισταίου τοῦ πρεσβυτέρου, κατὰ ἀνάλυσιν καὶ σύνθεσιν ἔχουσα τὴν ἔφοδον.

Ἀνάλυσις τοίνυν ἐστὶν ὁδὸς ἀπὸ τοῦ ζητουμένου ὡς ὁμολογουμένου διὰ τῶν ἑξῆς ἀκολούθων ἐπί τι ὁμολογούμενον συνθέσει· ἐν μὲν γὰρ τῇ ἀναλύσει τὸ ζητούμενον ὡς γεγονὸς ὑποθέμενοι τὸ ἐξ οὗ τοῦτο συμβαίνει σκοπούμεθα καὶ πάλιν ἐκείνου τὸ προηγούμενον, ἕως ἂν οὕτως ἀναποδίζοντες καταντήσωμεν εἴς τι τῶν ἤδη γνωριζομένων ἢ τάξιν ἀρχῆς ἐχόντων· καὶ τὴν τοιαύτην ἔφοδον ἀνάλυσιν καλοῦμεν, οἷον ἀνάπαλιν λύσιν.

Ἐν δὲ τῇ συνθέσει ἐξ ὑποστροφῆς τὸ ἐν τῇ ἀναλύσει καταληφθὲν ὕστατον ὑποστησάμενοι γεγονὸς ἤδη, καὶ ἑπόμενα τὰ ἐκεῖ [ἐνταῦθα][1] προ-

[1] ἐνταῦθα om. Hultsch.

lines EA, EΓ, EZ are equal, therefore the angle AZΓ is right. And by the same reasoning the angle BZΔ is right; therefore the diameters AΓ, BΔ appear equal. Similarly we may show that all are equal.

(k) The "Treasury of Analysis"

Ibid. vii., Preface 1-3, ed. Hultsch 634. 3-636. 30

The so-called *Treasury of Analysis*, my dear Hermodorus, is, in short, a special body of doctrine furnished for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves,[a] and for this purpose only is it useful. It is the work of three men, Euclid the writer of the *Elements*, Apollonius of Perga and Aristaeus the elder, and proceeds by the method of analysis and synthesis.

Now *analysis* is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.

But in *synthesis*, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural

[a] Or, perhaps, "to give a complete theoretical solution of problems set to them"; *v. supra*, p. 414 n. *a*.

ἡγούμενα κατὰ φύσιν τάξαντες καὶ ἀλλήλοις ἐπισυνθέντες, εἰς τέλος ἀφικνούμεθα τῆς τοῦ ζητουμένου κατασκευῆς· καὶ τοῦτο καλοῦμεν σύνθεσιν.

Διττὸν δ' ἐστὶν ἀναλύσεως γένος τὸ μὲν ζητητικὸν τἀληθοῦς, ὃ καλεῖται θεωρητικόν, τὸ δὲ ποριστικὸν τοῦ προταθέντος [λέγειν],[1] ὃ καλεῖται προβληματικόν. ἐπὶ μὲν οὖν τοῦ θεωρητικοῦ γένους τὸ ζητούμενον ὡς ὂν ὑποθέμενοι καὶ ὡς ἀληθές, εἶτα διὰ τῶν ἑξῆς ἀκολούθων ὡς ἀληθῶν καὶ ὡς ἔστιν καθ' ὑπόθεσιν προελθόντες ἐπί τι ὁμολογούμενον, ἐὰν μὲν ἀληθὲς ᾖ ἐκεῖνο τὸ ὁμολογούμενον, ἀληθὲς ἔσται καὶ τὸ ζητούμενον, καὶ ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐὰν δὲ ψεύδει ὁμολογουμένῳ ἐντύχομεν, ψεῦδος ἔσται καὶ τὸ ζητούμενον. ἐπὶ δὲ τοῦ προβληματικοῦ γένους τὸ προταθὲν ὡς γνωσθὲν ὑποθέμενοι, εἶτα διὰ τῶν ἑξῆς ἀκολούθων ὡς ἀληθῶν προελθόντες ἐπί τι ὁμολογούμενον, ἐὰν μὲν τὸ ὁμολογούμενον δυνατὸν ᾖ καὶ ποριστόν, ὃ καλοῦσιν οἱ ἀπὸ τῶν μαθημάτων δοθέν, δυνατὸν ἔσται καὶ τὸ προταθέν, καὶ πάλιν ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐὰν δὲ ἀδυνάτῳ ὁμολογουμένῳ ἐντύχομεν, ἀδύνατον ἔσται καὶ τὸ πρόβλημα.

Τοσαῦτα μὲν οὖν περὶ ἀναλύσεως καὶ συνθέσεως.

Τῶν δὲ προειρημένων τοῦ ἀναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη. Εὐκλείδου Δεδομένων βιβλίον ᾱ, Ἀπολλωνίου Λόγου ἀποτομῆς β̄, Χωρίου ἀποτομῆς β̄, Διωρισμένης τομῆς δύο, Ἐπαφῶν δύο, Εὐκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο,

[1] λέγειν om. Hultsch.

order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought ; and this we call synthesis.

Now analysis is of two kinds, one, whose object is to seek the truth, being called *theoretical*, and the other, whose object is to find something set for finding, being called *problematical*. In the theoretical kind we suppose the subject of the inquiry to exist and to be true, and then we pass through its consequences in order, as though they also were true and established by our hypothesis, to something which is admitted ; then, if that which is admitted be true, that which is sought will also be true, and the proof will be the reverse of the analysis, but if we come upon something admitted to be false, that which is sought will also be false. In the problematical kind we suppose that which is set as already known, and then we pass through its consequences in order, as though they were true, up to something admitted ; then, if what is admitted be possible and can be done, that is, if it be what the mathematicians call *given*, what was originally set will also be possible, and the proof will again be the reverse of the analysis, but if we come upon something admitted to be impossible, the problem will also be impossible.

So much for analysis and synthesis.

This is the order of the books in the aforesaid *Treasury of Analysis*. Euclid's *Data*, one book, Apollonius's *Cutting-off of a Ratio*, two books, *Cutting-off of an Area*, two books, *Determinate Section*, two books, *Contacts*, two books, Euclid's *Porisms*, three books, Apollonius's *Vergings*, two books, his *Plane Loci*, two books, *Conics*, eight books, Aristaeus's

Κωνικῶν η̅, Ἀρισταίου Τόπων στερεῶν πέντε, Εὐκλείδου Τόπων τῶν πρὸς ἐπιφανείᾳ δύο, Ἐρατοσθένους Περὶ μεσοτήτων δύο. γίνεται βιβλία λ̅γ̅, ὧν τὰς περιοχὰς μέχρι τῶν Ἀπολλωνίου Κωνικῶν ἐξεθέμην σοι πρὸς ἐπίσκεψιν, καὶ τὸ πλῆθος τῶν τόπων καὶ τῶν διορισμῶν καὶ τῶν πτώσεων καθ' ἕκαστον βιβλίον, ἀλλὰ καὶ τὰ λήμματα τὰ ζητούμενα, καὶ οὐδεμίαν ἐν τῇ πραγματείᾳ τῶν βιβλίων καταλέλοιπα ζήτησιν, ὡς ἐνόμιζον.

(*l*) Locus with Respect to Five or Six Lines

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

Ἐὰν ἀπό τινος σημείου ἐπὶ θέσει δεδομένας εὐθείας πέντε καταχθῶσιν εὐθεῖαι ἐν δεδομέναις γωνίαις, καὶ λόγος ᾖ δεδομένος τοῦ ὑπὸ τριῶν κατηγμένων περιεχομένου στερεοῦ παραλληλεπιπέδου ὀρθογωνίου πρὸς τὸ ὑπὸ τῶν λοιπῶν δύο κατηγμένων καὶ δοθείσης τινὸς περιεχόμενον παραλληλεπίπεδον ὀρθογώνιον, ἅψεται τὸ σημεῖον θέσει δεδομένης γραμμῆς. ἐάν τε ἐπὶ ϛ̅, καὶ λόγος ᾖ δοθεὶς τοῦ ὑπὸ τῶν τριῶν περιεχομένου προειρημένου στερεοῦ πρὸς τὸ ὑπὸ τῶν λοιπῶν τριῶν, πάλιν τὸ σημεῖον ἅψεται θέσει δεδομένης. ἐὰν δὲ ἐπὶ πλείονας τῶν ϛ̅, οὐκέτι μὲν ἔχουσι λέγειν, "ἐὰν λόγος ᾖ δοθεὶς τοῦ ὑπὸ τῶν δ̅ περιεχομένου τινὸς πρὸς τὸ ὑπὸ τῶν λοιπῶν," ἐπεὶ οὐκ ἔστι τι

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[a] These propositions follow a passage on the locus with respect to three or four lines which has already been quoted (*v.* vol. i. pp. 486-489). The passages come from Pappus's

Solid Loci, five books, Euclid's *Surface Loci*, two books, Eratosthenes' *On Means*, two books. In all there are thirty-three books, whose contents as far as Apollonius's *Conics* I have set out for your examination, including not only the number of the propositions, the conditions of possibility and the cases dealt with in each book, but also the lemmas which are required; indeed, I believe that I have not omitted any inquiry arising in the study of these books.

(*l*) Locus with Respect to Five or Six Lines [a]

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say "if the ratio be given between some figure contained by four of them to some figure contained by the remainder," since no figure can be contained in more

account of the *Conics* of Apollonius, who had worked out the locus with respect to three or four lines. It was by reflection on this passage that Descartes evolved the system of co-ordinates described in his *Géométrie*.

περιεχόμενον ὑπὸ πλειόνων ἢ τριῶν διαστάσεων. συγκεχωρήκασι δὲ ἑαυτοῖς οἱ βραχὺ πρὸ ἡμῶν ἑρμηνεύειν τὰ τοιαῦτα, μηδὲ ἓν μηδαμῶς διάληπτον σημαίνοντες, τὸ ὑπὸ τῶνδε περιεχόμενον λέγοντες ἐπὶ τὸ ἀπὸ τῆσδε τετράγωνον ἢ ἐπὶ τὸ ὑπὸ τῶνδε. παρῆν δὲ διὰ τῶν συνημμένων λόγων ταῦτα καὶ λέγειν καὶ δεικνύναι καθόλου καὶ ἐπὶ τῶν προειρημένων προτάσεων καὶ ἐπὶ τούτων τὸν τρόπον τοῦτον· ἐὰν ἀπό τινος σημείου ἐπὶ θέσει δεδομένας εὐθείας καταχθῶσιν εὐθεῖαι ἐν δεδομέναις γωνίαις, καὶ δεδομένος ᾖ λόγος ὁ συνημμένος ἐξ οὗ ἔχει μία κατηγμένη πρὸς μίαν καὶ ἑτέρα πρὸς ἑτέραν, καὶ ἄλλη πρὸς ἄλλην, καὶ ἡ λοιπὴ πρὸς δοθεῖσαν, ἐὰν ὦσιν ζ, ἐὰν δὲ η, καὶ ἡ λοιπὴ πρὸς λοιπήν, τὸ σημεῖον ἅψεται θέσει δεδομένης γραμμῆς· καὶ ὁμοίως ὅσαι ἂν ὦσιν περισσαὶ ἢ ἄρτιαι τὸ πλῆθος. τούτων, ὡς ἔφην, ἑπομένων τῷ ἐπὶ τέσσαρας τόπῳ οὐδὲ ἓν συντεθείκασιν, ὥστε τὴν γραμμὴν εἰδέναι.

than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions,[a] but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner : *If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line—if there be seven, or, if there be eight, that which the fourth bears to the fourth—the point will lie on a curve given in position* ; and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.[b]

[a] As Heron in his formula for the area of a triangle, given the sides (*supra*, pp. 476-477).

[b] The general proposition can thus be stated : If $p_1, p_2, p_3 \ldots p_n$ be the lengths of straight lines drawn to meet n given straight lines at given angles (where n is odd), and a be a given straight line, then if

$$\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \ldots \frac{p_n}{a} = \lambda,$$

where λ is a constant, the point will lie on a curve given in position. This will also be true if n is even and

$$\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \ldots \frac{p_{n-1}}{p_n} = \lambda.$$

(*m*) Anticipation of Guldin's Theorem

Ibid. vii. 41-42, ed. Hultsch 680. 30-682. 20

Ταῦθ' οἱ βλέποντες ἥκιστα ἐπαίρονται, καθάπερ οἱ πάλαι καὶ τῶν τὰ κρείττονα γραψάντων ἕκαστοι· ἐγὼ δὲ καὶ πρὸς ἀρχαῖς ἔτι τῶν μαθημάτων καὶ τῆς ὑπὸ φύσεως προκειμένης ζητημάτων ὕλης κινουμένους ὁρῶν ἅπαντας, αἰδούμενος ἐγὼ καὶ δείξας γε πολλῷ κρείσσονα καὶ πολλὴν προφερόμενα ὠφέλειαν . . . ἵνα δὲ μὴ κεναῖς χερσὶ τοῦτο φθεγξάμενος ὧδε χωρισθῶ τοῦ λόγου, ταῦτα δώσω ταῖς ἀναγνοῦσιν· ὁ μὲν τῶν τελείων ἀμφοιστικῶν λόγος συνῆπται ἔκ τε τῶν ἀμφοισμάτων καὶ τῶν ἐπὶ τοὺς ἄξονας ὁμοίως κατηγμένων εὐθειῶν ἀπὸ τῶν ἐν αὐτοῖς κεντροβαρικῶν σημείων, ὁ δὲ τῶν ἀτελῶν ἔκ τε τῶν ἀμφοισμάτων καὶ τῶν περιφερειῶν, ὅσας ἐποίησεν τὰ ἐν τούτοις κεντροβαρικὰ σημεῖα, ὁ δὲ τούτων τῶν περιφερειῶν λόγος συνῆπται δῆλον ὡς ἔκ τε τῶν κατηγμένων καὶ ὧν περιέχουσιν αἱ τούτων ἄκραι, εἰ καὶ εἶεν πρὸς τοῖς ἄξοσιν ἀμφοιστικῶν, γωνιῶν. περι-

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[a] Paul Guldin (1577–1643), or Guldinus, is generally credited with the discovery of the celebrated theorem here enunciated by Pappus. It may be stated: *If any plane figure revolve about an external axis in its plane, the volume of the solid figure so generated is equal to the product of the area of the figure and the distance travelled by the centre of gravity of the figure.* There is a corresponding theorem for the area.

[b] The whole passage is ascribed to an interpolator by Hultsch, but without justice; and, as Heath observes (*H.G.M.* ii. 403), it is difficult to think of any Greek mathematician after Pappus's time who could have discovered such an advanced proposition.

Though the meaning is clear enough, an exact translation

(m) Anticipation of Guldin's Theorem [a]

Ibid. vii. 41-42, ed. Hultsch 680. 30-682. 20 [b]

The men who study these matters are not of the same quality as the ancients and the best writers. Seeing that all geometers are occupied with the first principles of mathematics and the natural origin of the subject matter of investigation, and being ashamed to pursue such topics myself, I have proved propositions of much greater importance and utility . . . and in order not to make such a statement with empty hands, before leaving the argument I will give these enunciations to my readers. *Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded* (a) *of the ratio* [*of the areas*] *of the figures, and* (b) *of the ratio of the straight lines similarly drawn to*[c] *the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in a ratio compounded* (a) *of the ratio* [*of the areas*] *of the figures, and* (b) *of the ratio of the arcs described by the centres of gravity of the respective figures, the ratio of the arcs being itself compounded* (1) *of the ratio of the straight lines similarly drawn* [*from the respective centres of gravity to the axes of rotation*] *and* (2) *of the ratio of the angles contained about the axes of revolution by the extremities of these straight lines.*[d] These propositions, which are practi-

is impossible; I have drawn on the translations made by Halley (v. Papp. *Coll.*, ed. Hultsch 683 n. 2) and Heath (*H.G.M.* ii. 402-403). The obscurity of the language is presumably the only reason why Hultsch brackets the passage, as he says: "exciderunt autem in eodem loco pauciora plurave genuina Pappi verba."

[c] *i.e.*, drawn to meet at the same angles.

[d] The extremities are the centres of gravity.

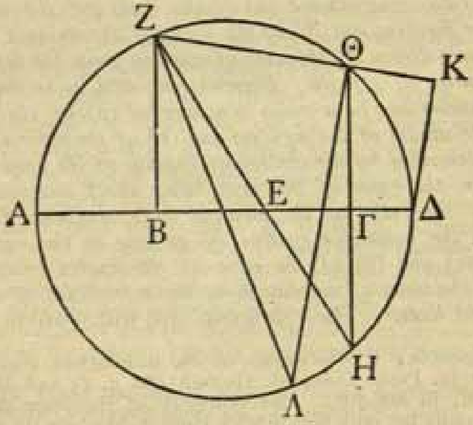
ἔχουσι δὲ αὗται αἱ προτάσεις, σχεδὸν οὖσαι μία, πλεῖστα ὅσα καὶ παντοῖα θεωρήματα γραμμῶν τε καὶ ἐπιφανειῶν καὶ στερεῶν, πάνθ' ἅμα καὶ μιᾷ δείξει καὶ τὰ μήπω δεδειγμένα καὶ τὰ ἤδη ὡς καὶ τὰ ἐν τῷ δωδεκάτῳ τῶνδε τῶν στοιχείων.

(*n*) Lemmas to the Treatises

(i.) *To the "Determinate Section" of Apollonius*

Ibid. vii. 115, ed. Hultsch, Prop. 61, 756. 28–760. 4

Τριῶν δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ, ΓΔ, ἐὰν γένηται ὡς τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ,



οὕτως τὸ ἀπὸ ΒΕ πρὸς τὸ ἀπὸ ΕΓ, μοναχὸς λόγος καὶ ἐλάχιστός ἐστιν ὁ τοῦ ὑπὸ ΑΕΔ πρὸς τὸ ὑπὸ

cally one, include a large number of theorems of all sorts about curves, surfaces and solids, all of which are proved simultaneously by one demonstration, and include propositions never before proved as well as those already proved, such as those in the twelfth book of these elements.[a]

(n) Lemmas to the Treatises [b]

(i.) *To the "Determinate Section" of Apollonius*

Ibid. vii. 115, ed. Hultsch, Prop. 61, 756. 28–760. 4

Given three straight lines AB, BΓ, ΓΔ,[c] *if* AB . BΔ : AΓ . ΓΔ = $BE^2 : E\Gamma^2$, *then the ratio* AE . EΔ : BE . EΓ

[a] If the passage be genuine, which there seems little reason to doubt, this is evidence that Pappus's work ran to twelve books at least.

[b] The greater part of Book vii. is devoted to lemmas required for the books in the *Treasury of Analysis* as far as Apollonius's *Conics*, with the exception of Euclid's *Data* and with the addition of two isolated lemmas to Euclid's *Surface-Loci*. The lemmas are numerous and often highly interesting from the mathematical point of view. The two here cited are given only as samples of this important collection: the first lemma to the *Surface-Loci*, one of the two passages in Greek referring to the focus-directrix property of a conic, has already been given (vol. i. pp. 492-503).

[c] It is left to be understood that they are in one straight line AΔ.

ΒΕΓ· λέγω δὴ ὅτι ὁ αὐτός ἐστιν τῷ τοῦ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ᾗ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ.

Γεγράφθω περὶ τὴν ΑΔ κύκλος, καὶ ἤχθωσαν ὀρθαὶ αἱ ΒΖ, ΓΗ. ἐπεὶ οὖν ἐστιν ὡς τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ, τουτέστιν ὡς τὸ ἀπὸ ΒΖ πρὸς τὸ ἀπὸ ΓΗ, οὕτως τὸ ἀπὸ ΒΕ πρὸς τὸ ἀπὸ ΕΓ, καὶ μήκει ἄρα ἐστὶν ὡς ἡ ΒΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΕ πρὸς τὴν ΕΓ· εὐθεῖα ἄρα ἐστὶν ἡ διὰ τῶν Ζ, Ε, Η. ἔστω ἡ ΖΕΗ, καὶ ἐκβεβλήσθω ἡ μὲν ΗΓ ἐπὶ τὸ Θ, ἐπιζευχθεῖσα δὲ ἡ ΖΘ ἐκβεβλήσθω ἐπὶ τὸ Κ, καὶ ἐπ' αὐτὴν κάθετος ἤχθω ἡ ΔΚ. καὶ διὰ δὴ τὸ προγεγραμμένον λῆμμα γίνεται τὸ μὲν ὑπὸ ΑΓ, ΒΔ ἴσον τῷ ἀπὸ ΖΚ, τὸ δὲ ὑπὸ ΑΒ, ΓΔ τῷ ἀπὸ ΘΚ· λοιπὴ ἄρα ἡ ΖΘ ἐστὶν ἡ ὑπεροχὴ ᾗ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ. ἤχθω οὖν διὰ τοῦ κέντρου ἡ ΖΛ, καὶ ἐπεζεύχθω ἡ ΘΛ. ἐπεὶ οὖν ὀρθὴ ἡ ὑπὸ ΖΘΛ ὀρθῇ τῇ ὑπὸ ΕΓΗ ἐστὶν ἴση, ἔστιν δὲ καὶ ἡ πρὸς τῷ Λ τῇ πρὸς τῷ Η γωνίᾳ ἴση, ἰσογώνια ἄρα τὰ τρίγωνα· ἔστιν ἄρα ὡς ἡ ΛΖ πρὸς τὴν ΘΖ, τουτέστιν ὡς ἡ ΑΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΕΗ πρὸς τὴν ΕΓ· καὶ ὡς ἄρα τὸ ἀπὸ ΑΔ πρὸς τὸ ἀπὸ ΖΘ, οὕτως τὸ ἀπὸ ΕΗ πρὸς τὸ ἀπὸ ΕΓ, καὶ τὸ ὑπὸ ΗΕ, ΕΖ, τουτέστιν τὸ ὑπὸ ΑΕ, ΕΔ, πρὸς τὸ ὑπὸ ΒΕ, ΕΓ. καὶ ἔστιν ὁ μὲν τοῦ ὑπὸ ΑΕ, ΕΔ πρὸς τὸ ὑπὸ ΒΕ,

* For, because ΒΖ : ΓΗ = ΒΕ : ΕΓ, the triangles ΖΕΒ, ΗΕΓ are similar, and angle ΖΕΒ = angle ΗΕΓ; ∴ Γ is in the same straight line with Β, Ε [Eucl. I. 13, Conv.].

is singular and a minimum; and I say that this ratio is equal to $A\Delta^2 : (\sqrt{A\Gamma \cdot B\Delta} - \sqrt{AB \cdot \Gamma\Delta})^2$.

Let a circle be described about $A\Delta$, and let BZ, ΓH be drawn perpendicular [to $A\Delta$]. Then since

$$AB \cdot B\Delta : A\Gamma \cdot \Gamma\Delta = BE^2 : E\Gamma^2, \qquad [\textit{ex hyp.}$$

i.e.,

$$BZ^2 : \Gamma H^2 = BE^2 : E\Gamma^2, \qquad [\text{Eucl. x. 33, Lemma}$$

$$\therefore \qquad BZ : \Gamma H = BE : E\Gamma.$$

Therefore Z, E, H lie on a straight line.[a] Let it be ZEH, and let $H\Gamma$ be produced to Θ, and let $Z\Theta$ be joined and produced to K, and let ΔK be drawn perpendicular to it. Then by the lemma just proved [Lemma 19]

$$A\Gamma \cdot B\Delta = ZK^2,$$
$$AB \cdot \Gamma\Delta = \Theta K^2;$$

[on taking the roots and] subtracting,

$$[ZK - \Theta K =]Z\Theta = \sqrt{A\Gamma \cdot B\Delta} - \sqrt{AB \cdot \Gamma\Delta}.$$

Let $Z\Lambda$ be drawn through the centre, and let $\Theta\Lambda$ be joined. Then since the right angle $Z\Theta\Lambda$ = the right angle $E\Gamma H$, and the angle at Λ = the angle at H, therefore the triangles [$Z\Theta\Lambda$, $E\Gamma H$] are equiangular;

$$\therefore \qquad \Lambda Z : \Theta Z = EH : E\Gamma,$$

i.e.,

$$A\Delta : Z\Theta = EH : E\Gamma;$$

$$\therefore \qquad A\Delta^2 : Z\Theta^2 = EH^2 : E\Gamma^2$$
$$= HE \cdot EZ : BE \cdot E\Gamma\ ^{b}$$
$$= AE \cdot E\Delta : BE \cdot E\Gamma. \qquad [\text{Eucl. iii. 35}$$

And [therefore] the ratio $AE \cdot E\Delta : BE \cdot E\Gamma$ is

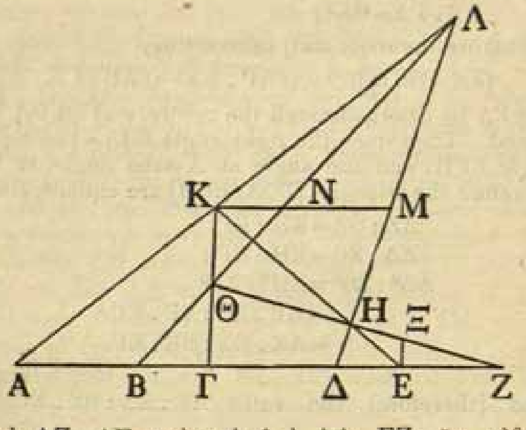
[b] Because, on account of the similarity of the triangles $H\Gamma E$, ZBE, we have $HE : E\Gamma = EZ : EB$.

ΕΓ μοναχὸς καὶ ἐλάσσων λόγος, ἡ δὲ ΖΘ ἡ ὑπεροχὴ ᾗ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ τῶν ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ [τουτέστιν τὸ ἀπὸ τῆς ΖΚ τοῦ ἀπὸ τῆς ΘΚ],[1] ὥστε ὁ μοναχὸς καὶ ἐλάσσων λόγος ὁ αὐτός ἐστιν τῷ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ᾗ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ, ὅπερ :~

(ii.) *To the "Porisms" of Euclid*

Ibid. vii. 198, ed. Hultsch, Prop. 130, 872. 23–874. 27

Καταγραφὴ ἡ ΑΒΓΔΕΖΗΘΚΛ, ἔστω δὲ ὡς τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΒ, ΓΖ, οὕτως τὸ



ὑπὸ ΑΖ, ΔΕ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ· ὅτι εὐθεῖά ἐστιν ἡ διὰ τῶν Θ, Η, Ζ σημείων.

singular and a minimum, while [, as proved above,] $Z\Theta = \sqrt{A\Gamma . B\Delta} - \sqrt{AB . \Gamma\Delta}$, so that the same singular and minimum ratio =

$$A\Delta^2 : (\sqrt{A\Gamma . B\Delta} - \sqrt{AB . \Gamma\Delta})^2. \qquad \text{Q.E.D.}^a$$

(ii.) *To the "Porisms" of Euclid* [b]

Ibid. vii. 198, ed. Hultsch, Prop. 130, 872. 23–874. 27

Let ΑΒΓΔΕΖΗΘΚΛ be a figure,[c] and let $AZ . B\Gamma : AB . \Gamma Z = AZ . \Delta E : A\Delta . EZ$; [I say] that the line through the points Θ, Η, Ζ is a straight line.

[a] Notice the sign :∼ used in the Greek for ὅπερ ἔδει δεῖξαι. In all Pappus proves this property for three different positions of the points, and it supports the view (*v. supra*, p. 341 n. *a*) that Apollonius's work formed a complete treatise on involution.

[b] *v.* vol. i. pp. 478–485.

[c] Following Breton de Champ and Hultsch I reproduce the second of the eight figures in the MSS., which vary according to the disposition of the points.

[1] τουτέστιν . . . τῆς ΘΚ om. Hultsch.

Ἐπεί ἐστιν ὡς τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΒ, ΓΖ, οὕτως τὸ ὑπὸ ΑΖ, ΔΕ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ ἐναλλάξ ἐστιν ὡς τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΖ, ΔΕ, τουτέστιν ὡς ἡ ΒΓ πρὸς τὴν ΔΕ, οὕτως τὸ ὑπὸ ΑΒ, ΓΖ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ. ἀλλ' ὁ μὲν τῆς ΒΓ πρὸς τὴν ΔΕ συνῆπται λόγος, ἐὰν διὰ τοῦ Κ τῇ ΑΖ παράλληλος ἀχθῇ ἡ ΚΜ, ἔκ τε τοῦ τῆς ΒΓ πρὸς ΚΝ καὶ τῆς ΚΝ πρὸς ΚΜ καὶ ἔτι τοῦ τῆς ΚΜ πρὸς ΔΕ, ὁ δὲ τοῦ ὑπὸ ΑΒ, ΓΖ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ συνῆπται ἔκ τε τοῦ τῆς ΒΑ πρὸς ΑΔ καὶ τοῦ τῆς ΓΖ πρὸς τὴν ΖΕ. κοινὸς ἐκκεκρούσθω ὁ τῆς ΒΑ πρὸς ΑΔ ὁ αὐτὸς ὢν τῷ τῆς ΝΚ πρὸς ΚΜ· λοιπὸν ἄρα ὁ τῆς ΓΖ πρὸς τὴν ΖΕ συνῆπται ἔκ τε τοῦ τῆς ΒΓ πρὸς τὴν ΚΝ, τουτέστιν τοῦ τῆς ΘΓ πρὸς τὴν ΚΘ, καὶ τοῦ τῆς ΚΜ πρὸς τὴν ΔΕ, τουτέστιν τοῦ τῆς ΚΗ πρὸς τὴν ΗΕ· εὐθεῖα ἄρα ἡ διὰ τῶν Θ, Η, Ζ.

Ἐὰν γὰρ διὰ τοῦ Ε τῇ ΘΓ παράλληλον ἀγάγω τὴν ΕΞ, καὶ ἐπιζευχθεῖσα ἡ ΘΗ ἐκβληθῇ ἐπὶ τὸ Ξ, ὁ μὲν τῆς ΚΗ πρὸς τὴν ΗΕ λόγος ὁ αὐτός ἐστιν τῷ τῆς ΚΘ πρὸς τὴν ΕΞ, ὁ δὲ συνημμένος ἔκ τε τοῦ τῆς ΓΘ πρὸς τὴν ΘΚ καὶ τοῦ τῆς ΘΚ πρὸς τὴν ΕΞ μεταβάλλεται εἰς τὸν τῆς ΘΓ πρὸς ΕΞ λόγον, καὶ ὁ τῆς ΓΖ πρὸς ΖΕ λόγος ὁ αὐτὸς τῷ τῆς ΓΘ πρὸς τὴν ΕΞ· παραλλήλου οὔσης τῆς ΓΘ τῇ ΕΞ, εὐθεῖα ἄρα ἐστὶν ἡ διὰ τῶν Θ, Ξ, Ζ (τοῦτο γὰρ φανερόν), ὥστε καὶ ἡ διὰ τῶν Θ, Η, Ζ εὐθεῖά ἐστιν.

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[a] It is not perhaps obvious, but is easily proved, and is in fact proved by Pappus in the course of iv. 21, ed. Hultsch 212. 4-13, by drawing an auxiliary parallelogram.

[b] Conversely, if ΗΘΚΛ be any quadrilateral, and any

Since $AZ \cdot B\Gamma : AB \cdot \Gamma Z = AZ \cdot \Delta E : A\Delta \cdot EZ$,
permutando

$$AZ \cdot B\Gamma : AZ \cdot \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ,$$

i.e.,
$$B\Gamma : \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ.$$

But, if KM be drawn through K parallel to AZ,

$$B\Gamma : \Delta E = (B\Gamma : KN) \cdot (KN : KM) \cdot (KM : \Delta E),$$

and

$$AB \cdot \Gamma Z : A\Delta \cdot EZ = (BA : A\Delta) \cdot (\Gamma Z : ZE).$$

Let the equal ratios $BA : A\Delta$ and $NK : KM$ be eliminated ;

then the remaining ratio

$$\Gamma Z : ZE = (B\Gamma : KN) \cdot (KM : \Delta E),$$

i.e.,
$$\Gamma Z : ZE = (\Theta\Gamma : K\Theta) \cdot (KH : HE) ;$$

then shall the line through Θ, H, Z be a straight line.

For if through E I draw $E\Xi$ parallel to $\Theta\Gamma$, and if ΘH be joined and produced to Ξ,

$$KH : HE = K\Theta : E\Xi,$$

and
$$(\Gamma\Theta : \Theta K) \cdot (\Theta K : E\Xi) = \Theta\Gamma : E\Xi,$$

$\therefore$
$$\Gamma Z : ZE = \Gamma\Theta : E\Xi ;$$

and since $\Gamma\Theta$ is parallel to $E\Xi$, the line through Θ, Ξ, Z is a straight line (for this is obvious[a]), and therefore the line through Θ, H, Z is a straight line.[b]

transversal cut pairs of opposite sides and the diagonals in the points A, Z, Δ, Γ, B, E, then $B\Gamma : \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ$. This is one of the ways of expressing the proposition enunciated by Desargues : *The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution* (v. L. Cremona, *Elements of Projective Geometry*, tr. by C. Leudesdorf, 1885, pp. 106-108). A number of special cases are also proved by Pappus.

(o) Mechanics

Ibid. viii., Praef. 1-3, ed. Hultsch 1022. 3-1028. 3

Ἡ μηχανικὴ θεωρία, τέκνον Ἑρμόδωρε, πρὸς πολλὰ καὶ μεγάλα τῶν ἐν τῷ βίῳ χρήσιμος ὑπάρχουσα πλείστης εἰκότως ἀποδοχῆς ἠξίωται πρὸς τῶν φιλοσόφων καὶ πᾶσι τοῖς ἀπὸ τῶν μαθημάτων περισπούδαστός ἐστιν, ἐπειδὴ σχεδὸν πρώτη τῆς περὶ τὴν ὕλην τῶν ἐν τῷ κόσμῳ στοιχείων φυσιολογίας ἅπτεται. στάσεως γὰρ καὶ φορᾶς σωμάτων καὶ τῆς κατὰ τόπον κινήσεως ἐν τοῖς ὅλοις θεωρηματικὴ τυγχάνουσα τὰ μὲν κινούμενα κατὰ φύσιν αἰτιολογεῖ, τὰ δ' ἀναγκάζουσα παρὰ φύσιν ἔξω τῶν οἰκείων τόπων εἰς ἐναντίας κινήσεις μεθίστησιν ἐπιμηχανωμένη διὰ τῶν ἐξ αὐτῆς τῆς ὕλης ὑποπιπτόντων αὐτῇ θεωρημάτων. τῆς δὲ μηχανικῆς τὸ μὲν εἶναι λογικὸν τὸ δὲ χειρουργικὸν οἱ περὶ τὸν Ἥρωνα μηχανικοὶ λέγουσιν· καὶ τὸ μὲν λογικὸν συνεστάναι μέρος ἔκ τε γεωμετρίας καὶ ἀριθμητικῆς καὶ ἀστρονομίας καὶ τῶν φυσικῶν λόγων, τὸ δὲ χειρουργικὸν ἔκ τε χαλκευτικῆς καὶ οἰκοδομικῆς καὶ τεκτονικῆς καὶ ζωγραφικῆς καὶ τῆς ἐν τούτοις κατὰ χεῖρα ἀσκήσεως· τὸν μὲν οὖν ἐν ταῖς προειρημέναις ἐπιστήμαις ἐκ παιδὸς γενόμενον κἀν ταῖς προειρημέναις τέχναις ἕξιν εἰληφότα πρὸς δὲ τούτοις φύσιν εὐκίνητον ἔχοντα, κράτιστον ἔσεσθαι μηχανικῶν ἔργων εὑρετὴν καὶ ἀρχιτέκτονά φασιν. μὴ δυνατοῦ δ' ὄντος τὸν αὐτὸν μαθημάτων

[a] After the historical preface here quoted, much of Book viii. is devoted to arrangements of toothed wheels, already encountered in the section on Heron (*supra*, pp. 488-497). A

(o) Mechanics [a]

Ibid. viii., Preface 1-3, ed. Hultsch 1022. 3–1028. 3

The science of mechanics, my dear Hermodorus, has many important uses in practical life, and is held by philosophers to be worthy of the highest esteem, and is zealously studied by mathematicians, because it takes almost first place in dealing with the nature of the material elements of the universe. For it deals generally with the stability and movement of bodies [about their centres of gravity],[b] and their motions in space, inquiring not only into the causes of those that move in virtue of their nature, but forcibly transferring [others] from their own places in a motion contrary to their nature; and it contrives to do this by using theorems appropriate to the subject matter. The mechanicians of Heron's school [c] say that mechanics can be divided into a *theoretical* and a *manual* part; the theoretical part is composed of geometry, arithmetic, astronomy and physics, the manual of work in metals, architecture, carpentering and painting and anything involving skill with the hands. The man who had been trained from his youth in the aforesaid sciences as well as practised in the aforesaid arts, and in addition has a versatile mind, would be, they say, the best architect and inventor of mechanical devices. But as it is impossible for the same person to familiarize himself with such

number of interesting theoretical problems are solved in the course of the book, including the construction of a conic through five points (viii. 13-17, ed. Hultsch 1072. 30–1084. 2).

[b] It is made clear by Pappus later (viii., Praef. 5, ed. Hultsch 1030. 1-17) that φορά has this meaning.

[c] With Pappus, this is practically equivalent to Heron himself: *cf.* vol. i. p. 184 n. *b*.

τε τοσούτων περιγενέσθαι καὶ μαθεῖν ἅμα τὰς προειρημένας τέχνας παραγγέλλουσι τῷ τὰ μηχανικὰ ἔργα μεταχειρίζεσθαι βουλομένῳ χρῆσθαι ταῖς οἰκείαις τέχναις ὑποχειρίοις ἐν ταῖς παρ' ἕκαστα χρείαις.

Μάλιστα δὲ πάντων ἀναγκαιόταται τέχναι τυγχάνουσιν πρὸς τὴν τοῦ βίου χρείαν [μηχανικὴ προηγουμένη τῆς ἀρχιτεκτονῆς][1] ἥ τε τῶν μαγγαναρίων, μηχανικῶν καὶ αὐτῶν κατὰ τοὺς ἀρχαίους λεγομένων (μεγάλα γὰρ οὗτοι βάρη διὰ μηχανῶν παρὰ φύσιν εἰς ὕψος ἀνάγουσιν ἐλάττονι δυνάμει κινοῦντες), καὶ ἡ τῶν ὀργανοποιῶν τῶν πρὸς τὸν πόλεμον ἀναγκαίων, καλουμένων δὲ καὶ αὐτῶν μηχανικῶν (βέλη γὰρ καὶ λίθινα καὶ σιδηρᾶ καὶ τὰ παραπλήσια τούτοις ἐξαποστέλλεται εἰς μακρὸν ὁδοῦ μῆκος τοῖς ὑπ' αὐτῶν γινομένοις ὀργάνοις καταπαλτικοῖς), πρὸς δὲ ταύταις ἡ τῶν ἰδίως πάλιν καλουμένων μηχανοποιῶν (ἐκ βάθους γὰρ πολλοῦ ὕδωρ εὐκολώτερον ἀνάγεται διὰ τῶν ἀντληματικῶν ὀργάνων ὧν αὐτοὶ κατασκευάζουσιν). καλοῦσι δὲ μηχανικοὺς οἱ παλαιοὶ καὶ τοὺς θαυμασιουργούς, ὧν οἱ μὲν διὰ πνευμάτων φιλοτεχνοῦσιν, ὡς Ἥρων Πνευματικοῖς, οἱ δὲ διὰ νευρίων καὶ σπάρτων ἐμψύχων κινήσεις δοκοῦσι μιμεῖσθαι, ὡς Ἥρων Αὐτομάτοις καὶ Ζυγίοις, ἄλλοι δὲ διὰ τῶν ἐφ' ὕδατος ὀχουμένων, ὡς Ἀρχιμήδης Ὀχουμένοις, ἢ τῶν δι' ὕδατος ὡρολογίων, ὡς Ἥρων Ὑδρείοις, ἃ δὴ καὶ τῇ γνωμονικῇ

[1] μηχανικὴ . . . ἀρχιτεκτονῆς om. Hultsch.

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[a] μάγγανον is properly the *block of a pulley*, as in Heron's

mathematical studies and at the same time to learn the above-mentioned arts, they instruct a person wishing to undertake practical tasks in mechanics to use the resources given to him by actual experience in his special art.

Of all the [mechanical] arts the most necessary for the purposes of practical life are : (1) that of the *makers of mechanical powers*,[a] they themselves being called mechanicians by the ancients—for they lift great weights by mechanical means to a height contrary to nature, moving them by a lesser force ; (2) that of the *makers of engines of war*, they also being called mechanicians—for they hurl to a great distance weapons made of stone and iron and suchlike objects, by means of the instruments, known as catapults, constructed by them ; (3) in addition, that of the men who are properly called *makers of engines*—for by means of instruments for drawing water which they construct water is more easily raised from a great depth ; (4) the ancients also describe as mechanicians the *wonder-workers*, of whom some work by means of pneumatics, as Heron in his *Pneumatica*,[b] some by using strings and ropes, thinking to imitate the movements of living things, as Heron in his *Automata* and *Balancings*,[b] some by means of floating bodies, as Archimedes in his book *On Floating Bodies*,[c] or by using water to tell the time, as Heron in his *Hydria*,[d] which appears to have affinities with the

Belopoeica, ed. Schneider 84. 12, *Greek Papyri in the British Museum* iii. (ed. Kenyon and Bell) 1164 n. 8.

[b] *v. supra*, p. 466 n. *a*. [c] *v. supra*, pp. 242-257.

[d] This work is mentioned in the *Pneumatica*, under the title Περὶ ὑδρίων ὡροσκοπείων, as having been in four books. Fragments are preserved in Proclus (*Hypotyposis* 4) and in Pappus's commentary on Book v. of Ptolemy's *Syntaxis*.

θεωρίᾳ κοινωνοῦντα φαίνεται. μηχανικοὺς δὲ καλοῦσιν καὶ τοὺς τὰς σφαιροποιΐας [ποιεῖν][1] ἐπισταμένους, ὑφ' ὧν εἰκὼν τοῦ οὐρανοῦ κατασκευάζεται δι' ὁμαλῆς καὶ ἐγκυκλίου κινήσεως ὕδατος.

Πάντων δὲ τούτων τὴν αἰτίαν καὶ τὸν λόγον ἐπεγνωκέναι φασίν τινες τὸν Συρακόσιον Ἀρχιμήδη· μόνος γὰρ οὗτος ἐν τῷ καθ' ἡμᾶς βίῳ ποικίλῃ πρὸς πάντα κέχρηται τῇ φύσει καὶ τῇ ἐπινοίᾳ, καθὼς καὶ Γέμινος ὁ μαθηματικὸς ἐν τῷ Περὶ τῆς τῶν μαθημάτων τάξεώς φησιν. Κάρπος δέ πού φησιν ὁ Ἀντιοχεὺς Ἀρχιμήδη τὸν Συρακόσιον ἓν μόνον βιβλίον συντεταχέναι μηχανικὸν τὸ κατὰ τὴν σφαιροποιΐαν, τῶν δὲ ἄλλων οὐδὲν ἠξιωκέναι συντάξαι. καίτοι παρὰ τοῖς πολλοῖς ἐπὶ μηχανικῇ δοξασθεὶς καὶ μεγαλοφυής τις γενόμενος ὁ θαυμαστὸς ἐκεῖνος, ὥστε διαμεῖναι παρὰ πᾶσιν ἀνθρώποις ὑπερβαλλόντως ὑμνούμενος, τῶν τε προηγουμένων γεωμετρικῆς καὶ ἀριθμητικῆς ἐχομένων θεωρίας τὰ βραχύτατα δοκοῦντα εἶναι σπουδαίως συνέγραφεν· ὃς φαίνεται τὰς εἰρημένας ἐπιστήμας οὕτως ἀγαπήσας ὡς μηδὲν ἔξωθεν ὑπομένειν αὐταῖς ἐπεισάγειν. αὐτὸς δὲ Κάρπος καὶ ἄλλοι τινὲς συνεχρήσαντο γεωμετρίᾳ καὶ εἰς τέχνας τινὰς εὐλόγως· γεωμετρία γὰρ οὐδὲν βλάπτεται, σωματοποιεῖν πεφυκυῖα πολλὰς τέχνας, διὰ τοῦ συνεῖναι αὐταῖς [μήτηρ οὖν ὥσπερ οὖσα τεχνῶν οὐ βλάπτεται διὰ τοῦ φροντίζειν ὀργανικῆς καὶ ἀρχιτεκτονικῆς· οὐδὲ γὰρ διὰ τὸ συνεῖναι γεωμορίᾳ καὶ γνωμονικῇ καὶ μηχανικῇ καὶ σκηνογραφίᾳ βλάπτεταί τι],[2] τοὐναντίον δὲ προάγουσα

[1] ποιεῖν om. Hultsch. [2] μήτηρ . . . τι om. Hultsch.

science of sun-dials; (5) they also describe as mechanicians the *makers of spheres*, who know how to make models of the heavens, using the uniform circular motion of water.

Archimedes of Syracuse is acknowledged by some to have understood the cause and reason of all these arts; for he alone applied his versatile mind and inventive genius to all the purposes of ordinary life, as Geminus the mathematician says in his book *On the Classification of Mathematics*.[a] Carpus of Antioch[b] says somewhere that Archimedes of Syracuse wrote only one book on mechanics, that on the construction of spheres,[c] not regarding any other matters of this sort as worth describing. Yet that remarkable man is universally honoured and held in esteem, so that his praises are still loudly sung by all men, but he himself on purpose took care to write as briefly as seemed possible on the most advanced parts of geometry and subjects connected with arithmetic; and he obviously had so much affection for these sciences that he allowed nothing extraneous to mingle with them. Carpus himself and certain others also applied geometry to some arts, and with reason; for geometry is in no way injured, but is capable of giving content to many arts by being associated with them, and, so far from being injured, it is obviously, while itself

[a] For Geminus and this work, *v. supra*, p. 370 n. *c*.

[b] Carpus has already been encountered (vol. i. p. 334) as the discoverer (according to Iamblichus) of a *curve arising from a double motion* which can be used for squaring the circle. He is several times mentioned by Proclus, but his date is uncertain.

[c] This work is not otherwise known.

μὲν ταύτας φαίνεται, τιμωμένη δὲ καὶ κοσμουμένη δεόντως ὑπ' αὐτῶν.

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[a] With the great figure of Pappus, these selections illustrating the history of Greek mathematics may appropriately come to an end. Mathematical works continued to be written in Greek almost to the dawn of the Renaissance, and

advancing those arts, appropriately honoured and adorned by them.[a]

they serve to illustrate the continuity of Greek influence in the intellectual life of Europe. But, after Pappus, these works mainly take the form of comment on the classical treatises. Some, such as those of Proclus, Theon of Alexandria, and Eutocius of Ascalon have often been cited already, and others have been mentioned in the notes.

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